Electric Field Associated with an infinite wire carrying current I PhL 4.15.14

I looked at this again on May 5, 2014, seems OK. My appendix P never talks about fields *outside* a round wire. Inside the wire things seem to agree with Appendix P and Chapter 2.

For very low ω, for a round wire, and for $\mu_1 = \mu_2 = \mu$, the conclusion is that there is a small E field E_z both inside and outside the wire which is shown in this summary quoted from the doc end. The E field inside agrees with the low ω limit of Chapter 2. [reviewed 4/30/14]

Magnetic field:

inside: $B_\theta(r) = (\mu I/2\pi a^2)r$

outside: $B_\theta(r) = (\mu I/2\pi)(1/r)$

Electric field:

inside: $E_z(r) = I/(\sigma\pi a^2) + j\omega\mu\, I\, (1/8\pi)\, (2r^2/a^2 - 1)$

outside: $E_z(r) = I/(\sigma\pi a^2) + j\omega\, (\mu I/8\pi)\, [\, 4\ln(r/a) + 1]$

Current:

inside: $J_z(r) = I/(\pi a^2) + j\omega\sigma\mu I\, (1/8\pi)\, (2r^2/a^2 - 1)$

outside: $J_z(r) = 0$

Here σ is always the conductivity inside the wire, and we assume outside is an insulator.

I am assuming low ω in this entire doc so $J_r \approx 0$, there is no surface charge pumping, wavelength is very long, etc. I also assume that μ applies to the wire and to its surroundings.

We know the *magnetic* field for this problem, it is this:

inside: $2\pi r\, H_\theta = I\, (r/a)^2$ $\Rightarrow B_\theta = (\mu I/2\pi a^2)r$

outside: $2\pi r\, H_\theta = I$ $\Rightarrow B_\theta = (\mu I/2\pi)(1/r)$

But what is going on with *electric* fields here? Well, we know that

$$\text{curl } \mathbf{E} = -j\omega \mathbf{B}$$

or

$$\text{curl } \mathbf{E} = \hat{\mathbf{r}}\, [\, r^{-1}\partial_\theta E_z - \partial_z E_\theta] + \hat{\boldsymbol{\theta}}\, [\partial_z E_r - \partial_r E_z] + \hat{\mathbf{z}}\, [\, r^{-1}\partial_r(rE_\theta) - r^{-1}\partial_\theta E_r\,]$$

Assuming no z dependence, because we assume a very long wavelength, because we assume a very low ω, this says

$$\text{curl } \mathbf{E} = \hat{\mathbf{r}}\,[\, r^{-1}\partial_\theta E_z] + \hat{\boldsymbol{\theta}}\,[-\partial_r E_z] + \hat{\mathbf{z}}\,[\, r^{-1}\partial_r(rE_\theta) - r^{-1}\partial_\theta E_r\,]$$

We then have these three equations

$$r^{-1}\partial_\theta E_z = 0$$

$$-\partial_r E_z = -j\omega B_\theta$$

$$r^{-1}\partial_r(rE_\theta) - r^{-1}\partial_\theta E_r = 0$$

The first says that E_z is independent of θ, so we have just $E_z(r)$. The third I let be satisfied with both E_θ and $E_r = 0$, call it an ansatz. The second equation then says

$$\partial_r E_z = j\omega B_\theta$$

Inside the wire this says

$$\partial_r E_z = j\omega B_\theta = j\omega\,(\mu I/2\pi a^2)r$$

$$\Rightarrow\ E_z(r) = j\omega\,(\mu I/2\pi a^2)(1/2)r^2 + C_1 \qquad (*)$$

This seems to be the skin effect in the low ω limit, I will have to check that momentarily.

[Compare this to (P.4.6) with $J_{ext} = I/(\pi a^2)$ so here we would have $E_z(r) = j\omega\,\mu J_{ext}(1/4)r^2 + C_1$]

Outside the wire we have instead

$$\partial_r E_z = j\omega B_\theta = j\omega\,(\mu I/2\pi)(1/r)$$

$$\Rightarrow\ E_z(r) = j\omega\,(\mu I/2\pi)\ln(r) + C_2$$

We know that E_z is continuous at the wire surface, so we can match these two at that point

$$j\omega\,(\mu I/2\pi a^2)(1/2)a^2 + C_1 = j\omega\,(\mu I/2\pi)\ln(a) + C_2$$

or

$$j\omega\,(\mu I/4\pi) + C_1 = j\omega\,(\mu I/4\pi)\,2\ln(a) + C_2$$

so that

$$C_2 = C_1 + j\omega\,(\mu I/4\pi)\,(1 - 2\ln a)$$

The total current in the wire is given by

$$I = \int r dr d\theta\, J_z = \sigma \int r dr d\theta\, E_z = \sigma\, 2\pi \int r dr\, [j\omega (\mu I/2\pi a^2)(1/2) r^2 + C_1]$$

$$= 2\pi\sigma \{ j\omega (\mu I/2\pi a^2)(1/2) \int r dr\, r^2 + C_1 \int r dr \}$$

$$= 2\pi\sigma \{ j\omega (\mu I/2\pi a^2)(1/2)(1/4) a^4 + C_1 (1/2) a^2 \}$$

$$= \sigma\pi a^2 \{ j\omega (\mu I/8\pi) + C_1 \}$$

where I checked dimensions on the first term. Then we have

$$I/(\sigma\pi a^2) = j\omega (\mu I/8\pi) + C_1 \quad \Rightarrow \quad C_1 = I/(\sigma\pi a^2) - j\omega (\mu I/8\pi)$$

Then inside the wire we must have

$$E_z(r) = j\omega (\mu I/2\pi a^2)(1/2) r^2 + C_1$$

$$= j\omega (\mu I/4\pi a^2) r^2 + I/(\sigma\pi a^2) - j\omega (\mu I/8\pi)$$

$$= j\omega (I\, \mu/4\pi a^2) r^2 + I/(\sigma\pi a^2) - j\omega (\mu\, I/8\pi)$$

$$= I/(\sigma\pi a^2) + j\omega (\mu\, I/8\pi a^2) 2r^2 - j\omega (\mu\, I/8\pi)$$

$$= I/(\sigma\pi a^2) + j\omega\mu\, I (1/8\pi) (2r^2/a^2 - 1) \qquad (*)$$

Notice that

$$E_z(a) = I/(\sigma\pi a^2) + j\omega\mu\, I (1/8\pi)$$

For small ω which is our regime, the second term in (*) is a small correction term which is negative near the wire center and positive near the surface, expressing the skin effect. The first term is just the DC current we would expect. So this seems good.

What about outside the wire? We find that

$$C_2 = C_1 + j\omega (\mu I/4\pi) (1 - 2 \ln a)$$

$$= I/(\sigma\pi a^2) - j\omega (\mu I/8\pi) + j\omega (\mu I/4\pi) (1 - 2 \ln a)$$

$$= I/(\sigma\pi a^2) - j\omega (\mu I/8\pi) + j\omega (\mu I/8\pi) (2 - 4 \ln a)$$

$$= I/(\sigma\pi a^2) + j\omega (\mu I/8\pi) (2 - 4 \ln a - 1)$$

$= I/(\sigma\pi a^2) + j\omega (\mu I/8\pi) (1 - 4 \ln a)$

Then outside the wire we get

$E_z(r) = j\omega (\mu I/2\pi) \ln(r) + C_2$

$= j\omega (\mu I/2\pi) \ln(r) + I/(\sigma\pi a^2) + j\omega (\mu I/8\pi) (1 - 4 \ln a)$

$= I/(\sigma\pi a^2) + j\omega (\mu I/8\pi) 4\ln(r) + j\omega (\mu I/8\pi) - j\omega (\mu I/8\pi) 4 \ln(a)$

$= I/(\sigma\pi a^2) + j\omega (\mu I/8\pi) 4\ln(r/a) + j\omega (\mu I/8\pi)$

$= I/(\sigma\pi a^2) + j\omega (\mu I/8\pi) [4\ln(r/a) + 1]$

Notice that

$E_z(a) = I/(\sigma\pi a^2) + j\omega (\mu I/8\pi)$

which matches the inside result shown above.

Does my inside result agree with Chapter 2 of lines doc?

$E_z(r) = I/(\sigma\pi a^2) + j\omega\mu I (1/8\pi) (2r^2/a^2 - 1)$

Chapter 2 says this

$$E_z(r) = \frac{\mu I}{2\pi a} \frac{J_0(\beta r)}{J_1(\beta a)} (-j\omega/\beta)$$

But for small ω we have small β and then

$J_0(x) \approx 1 - x^2/4$

$J_1(x) \approx (x/2)(1 - x^2/8) \quad \Rightarrow \quad 1/J_1(x) \approx (2/x) (1 + x^2/8)$

so this gives

$$E_z(r) = \frac{\mu I}{2\pi a} (1 - \beta^2 r^2/4) (2/\beta a) (1 + \beta^2 a^2/8) (-j\omega/\beta)$$

$$= \frac{\mu I}{2\pi a} (1 - \beta^2 r^2/4) (1 + \beta^2 a^2/8) (-2j\omega/\beta^2 a)$$

Now $\beta^2 = \omega^2\mu \xi = \omega^2\mu (- j\sigma/\omega) = -j\sigma\mu\omega$ so let's install that in three places to get

$$= \frac{\mu I}{2\pi a}\left(1 + j\sigma\mu\omega\, r^2/4 \right)(1 - j\sigma\mu\omega\, a^2/8)(-2j\omega/[-j\sigma\mu\omega]a)$$

$$= \frac{\mu I}{2\pi a}\left(1 + j\sigma\mu\omega\, r^2/4 \right)(1 - j\sigma\mu\omega\, a^2/8)(2/[\sigma\mu]a)$$

$$= \frac{\mu I}{2\pi a}\left[1 + (j\sigma\mu\omega/8)(2r^2 - a^2) \right](2/[\sigma\mu]a)$$

$$= \frac{I}{\sigma\pi a^2}\left[1 + (j\sigma\mu\omega/8)(2r^2 - a^2) \right]$$

$$= \frac{I}{\sigma\pi a^2} + (j\omega\mu I/8\pi a^2)(2r^2 - a^2) \qquad \text{// Chapter 2 small } \omega \text{ limit}$$

The limit I just got in this document was

$$= I/(\sigma\pi a^2) + j\omega\mu\, I\,(1/8\pi)\,(2r^2/a^2 - 1)$$

and this does agree exactly, hurray.

Summary of Fields Associated with a Wire

Low ω only, and $\mu_1 = \mu_2 = \mu$ only.

Magnetic field:

inside: $B_\theta(r) = (\mu I/2\pi a^2)r$

outside: $B_\theta(r) = (\mu I/2\pi)(1/r)$

Electric field:

inside: $E_z(r) = I/(\sigma\pi a^2) + j\omega\mu\, I\,(1/8\pi)\,(2r^2/a^2 - 1)$

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Current:

inside: $J_z(r) = I/(\pi a^2) + j\omega\sigma\mu I\,(1/8\pi)\,(2r^2/a^2 - 1)$

outside: $J_z(r) = 0$

Here σ is always the conductivity inside the wire, and we assume outside is an insulator.

Comment: At low ω, the E_z both inside and "near" the wire is very small. For example, the adder term has this size at 100 Hz :

$\omega\mu I \approx 2\pi$ 100 Hz * 4π x 10^{-7} * 1 amp $\approx$ 0.8 mV/m

```
evalf(8*Pi^2*1e-5);
                .0007895683523
```

while the main term for a 1 mm radius wire carrying 1 amp is

$I/(\pi a^2\sigma) \approx$ 5.5 mV/m

```
evalf(1/(Pi*1e-6*5.81e7))
                .005478655527
```

What about this curl equation:

curl **E** = -jω**B**

I could apply this (integral form) to a rectangular loop parallel to the wire which captures the flux. I then get

line integral of E = dz [$E_z(r+dr)-E_z(r)$] = -jω B dz dr

so that

$\partial_r E_z$ = -jωB

which we already saw above. If we think of B = large and ω = small,