



Direct and Inverse Eddy Current Problems

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Motivation

Inverse Electromagnetics & Eddy currents

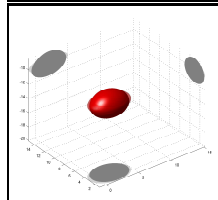
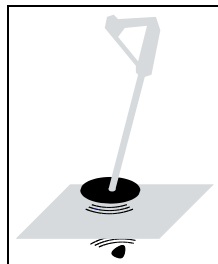
Inverse Electromagnetics

Inverse Electromagnetics:

- ▶ Generate EM field
(*drive excitation current through coil*)
- ▶ Measure EM field
(*induced voltages in meas. coil*)
- ▶ Gain information from measurements

Applications:

- ▶ Metal detection (*buried conductor*)
- ▶ Non-destructive testing
(*crack in metal, metal in concrete*)
- ▶ ...



Maxwell's equations

Classical Electromagnetics: **Maxwell's equations**

$$\begin{aligned}\operatorname{curl} H &= \epsilon \partial_t E + \sigma E + J && \text{in } \mathbb{R}^3 \times]0, T[\\ \operatorname{curl} E &= -\mu \partial_t H && \text{in } \mathbb{R}^3 \times]0, T[\end{aligned}$$

$E(x, t)$:	Electric field	$\epsilon(x)$:	Permittivity
$H(x, t)$:	Magnetic field	$\mu(x)$:	Permeability
$J(x, t)$:	Excitation current	$\sigma(x)$:	Conductivity

Knowing $J, \sigma, \mu, \epsilon + \text{init. cond.}$ determines E and H .



Eddy currents

Maxwell's equations

$$\begin{aligned}\operatorname{curl} H &= \epsilon \partial_t E + \sigma E + J && \text{in } \mathbb{R}^3 \times]0, T[\\ \operatorname{curl} E &= -\mu \partial_t H && \text{in } \mathbb{R}^3 \times]0, T[\end{aligned}$$

Eddy current approximation: Neglect displacement currents $\epsilon \partial_t E$

- Justified for low-frequency excitations

(Alonso 1999, Ammari/Bufa/Nédélec 2000)

$$\rightsquigarrow \quad \partial_t(\sigma E) + \operatorname{curl} \left(\frac{1}{\mu} \operatorname{curl} E \right) = -\partial_t J \quad \text{in } \mathbb{R}^3 \times]0, T[$$



Where's Eddy?

- ▶ $\sigma = 0$: (Quasi-)Magnetostatics

$$\text{curl} \left(\frac{1}{\mu} \text{curl} E \right) = -\partial_t J$$

Excitation $\partial_t J$ instantly generates magn. field $\frac{1}{\mu} \text{curl} E = -\partial_t H$.

- ▶ $\sigma \neq 0$: Eddy currents

$$\partial_t(\sigma E) + \text{curl} \left(\frac{1}{\mu} \text{curl} E \right) = -\partial_t J$$

$\partial_t J$ generates changing magn. field + currents inside conductor

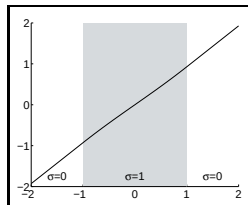
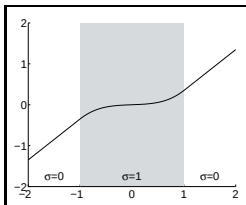
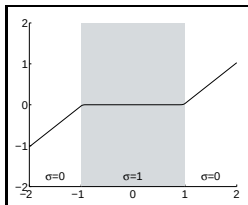
Induced currents oppose what created them (Lenz law)

Parabolic-elliptic equations

$$\partial_t(\sigma E) + \operatorname{curl} \left(\frac{1}{\mu} \operatorname{curl} E \right) = -\partial_t J \quad \text{in } \mathbb{R}^3 \times]0, T[$$

- ▶ **parabolic** inside conductor $\Omega = \operatorname{supp}(\sigma)$
- ▶ **elliptic** outside conductor

Scalar example: $(\sigma u)_t = u_{xx}$, $u(\cdot, 0) = 0$, $u_x(-2, \cdot) = u_x(2, \cdot) = 1$.





The direct problem

Unified variational formulation
for the parabolic-elliptic eddy current problem

Standard approach

$$\partial_t(\sigma E) + \operatorname{curl} \left(\frac{1}{\mu} \operatorname{curl} E \right) = -\partial_t J \quad \text{in } \mathbb{R}^3 \times]0, T[$$

Standard approach: *Decouple elliptic and parabolic part*
(e.g. Bossavit 1999, Acevedo/Meddahi/Rodriguez 2009)

Find $(E_{\mathbb{R}^3 \setminus \Omega}, E_{\Omega}) \in H_{\mathbb{R}^3 \setminus \Omega} \times H_{\Omega}$ s.t.

- ▶ E_{Ω} solves parabolic equation + init. cond.
 - ▶ $E_{\mathbb{R}^3 \setminus \Omega}$ solves elliptic equation
 - ▶ interface conditions are satisfied
-

Problem: Theory (solution spaces, coercivity constants, etc.) depends on $\Omega = \operatorname{supp} \sigma$ and on lower bounds of $\sigma|_{\Omega}$.

Unified approach?

Parabolic-elliptic eddy current equation

$$\partial_t(\sigma E) + \operatorname{curl} \left(\frac{1}{\mu} \operatorname{curl} E \right) = -\partial_t J \quad \text{in } \mathbb{R}^3 \times]0, T[$$

Inverse problem: Find σ (or $\Omega = \operatorname{supp} \sigma$) from measurements of E

- requires **unified** solution theory

Test for unified theory: Can we linearize E w.r.t. σ ?

*How does the solution of an elliptic equation change
if the equation becomes a little bit parabolic?*

(For scalar analogue: Frühauf/**H.**/Scherzer 2007, **H.** 2007)

Rigorous formulation

Rigorous formulation: Let $\mu \in L_+^\infty$, $\sigma \in L^\infty$, $\sigma \geq 0$,

$$\begin{aligned} J_t &\in L^2(0, T, W(\operatorname{curl})') && \text{with } \operatorname{div} J_t = 0 \\ E_0 &\in L^2(\mathbb{R}^3)^3 && \text{with } \operatorname{div}(\sigma E_0) = 0. \end{aligned}$$

For $E \in L^2(0, T, W(\operatorname{curl}))$ the eddy current equations

$$\begin{aligned} \partial_t(\sigma E) + \operatorname{curl} \left(\frac{1}{\mu} \operatorname{curl} E \right) &= -J_t && \text{in } \mathbb{R}^3 \times]0, T[\\ \sqrt{\sigma} E(x, 0) &= \sqrt{\sigma(x)} E_0(x) && \text{in } \mathbb{R}^3 \end{aligned}$$

are well-defined and (if solvable) uniquely determine $\operatorname{curl} E$, $\sqrt{\sigma} E$.

Natural variational formulation

Natural unified variational formulation ($E_0 = 0$ for simplicity):

Find $E \in L^2(0, T, W(\text{curl}))$ that solves

$$\int_0^T \int_{\mathbb{R}^3} \left(\sigma E \cdot \partial_t \Phi - \frac{1}{\mu} \text{curl } E \cdot \text{curl } \Phi \right) = \int_0^T \int_{\mathbb{R}^3} J_t \cdot \Phi.$$

for all smooth Φ with $\Phi(\cdot, T) = 0$.

- ▶ **equivalent** to eddy current equation
- ▶ **not coercive**, does not yield existence results

Gauged formulation

Gauged unified variational formulation ($E_0 = 0$ for simplicity)

Find **divergence-free** $E \in L^2(0, T, W^1(\mathbb{R}^3))$ that solves

$$\int_0^T \int_{\mathbb{R}^3} \left(\sigma E \cdot \partial_t \Phi - \frac{1}{\mu} \operatorname{curl} E \cdot \operatorname{curl} \Phi \right) = \int_0^T \int_{\mathbb{R}^3} J_t \cdot \Phi.$$

for all smooth **divergence-free** Φ with $\Phi(\cdot, T) = 0$.

- ▶ coercive, yields existence and continuity results
- ▶ **not equivalent** to eddy current equation
($\sigma \neq \text{const.} \rightsquigarrow \operatorname{div} \sigma E \neq \sigma \operatorname{div} E$)
- ▶ does **not determine true solution** up to gauge (curl-free) field

Coercive unified formulation

How to obtain **coercive + equivalent** unified formulation?

- ▶ Ansatz $E = A + \nabla\varphi$ with divergence-free A .
(almost the standard (A, φ) -formulation with Coulomb gauge)

- ▶ Consider $\nabla\varphi = \nabla\varphi_A$ as function of A by solving

$$\operatorname{div} \sigma \nabla \varphi_A = -\operatorname{div} \sigma A.$$

($\rightsquigarrow \operatorname{div} \sigma E = 0$).

- ▶ Obtain coercive formulation for A
(Lions-Lax-Milgram Theorem $\rightsquigarrow$ Solvability and continuity results)
- ▶ A determines E
(more precisely: $\operatorname{curl} E$ and $\sqrt{\sigma}E$)

Unified variational formulation

Unified variational formulation (*Arnold/H., SIAP, 2012*)

Find **divergence-free** $A \in L^2(0, T, W^1(\mathbb{R}^3))$ that solves

$$\int_0^T \int_{\mathbb{R}^3} \left(\sigma(A + \nabla \varphi_A) \cdot \partial_t \Phi - \frac{1}{\mu} \operatorname{curl} A \cdot \operatorname{curl} \Phi \right) = \int_0^T \int_{\mathbb{R}^3} J_t \cdot \Phi.$$

for all smooth **divergence-free** Φ with $\Phi(\cdot, T) = 0$.

- ▶ coercive, uniquely solvable
- ▶ $E := A + \nabla \varphi_A$ is **one** solution of the eddy current equation
- ↪ $\operatorname{curl} E, \sqrt{\sigma} E$ depend continuously on J_t (uniformly w.r.t. σ)
(for **all** solutions of the eddy current equation)

Asymptotic results

Unified variational formulation

- ▶ allows to rigorously linearize E w.r.t. σ around $\sigma_0 = 0$ (elliptic equation becoming a little bit parabolic in some region...)
- ▶ easily extends from $\mathbb{R}^3$ to bounded domain O (O simply conn. with Lipschitz-boundary, $\nu \wedge E|_{\partial O} = 0$)
- ▶ justifies **parabolic regularization**: If E_ϵ solves

$$\partial_t(\sigma_\epsilon E_\epsilon) + \operatorname{curl} \left(\frac{1}{\mu} \operatorname{curl} E_\epsilon \right) = -\partial_t J \quad \text{in } O \times]0, T[,$$

with $\sigma_\epsilon(x) = \max\{\sigma(x), \epsilon\}$ then

$$\sigma_\epsilon E_\epsilon \rightarrow \sigma E, \quad \operatorname{curl} E_\epsilon \rightarrow \operatorname{curl} E$$

(Arnold/H., submitted to proceedings of IPDO 2013)

Open problems

- ▶ Theory requires some regularity of $\Omega = \text{supp } \sigma$ and $\sigma \in L^{\infty}_{+}(\Omega)$ in order to determine φ from A .
- ▶ Solution theory for

$$\text{div } \sigma \nabla \varphi = -\text{div } \sigma A$$

for general $\sigma \in L^{\infty}$, $\sigma \geq 0$?

- ▶ Elliptic regularization of the variational formulation
(i.e., adding $\epsilon \int_0^T \int_{\mathbb{R}^3} A \cdot \Phi \, dx$)

is justified, but relation to elliptic regularization of the PDE

$$\partial_t(\sigma E) + \text{curl} \left(\frac{1}{\mu} \text{curl } E \right) + \epsilon E = -\partial_t J \quad \text{in } \mathbb{R}^3 \times]0, T[,$$

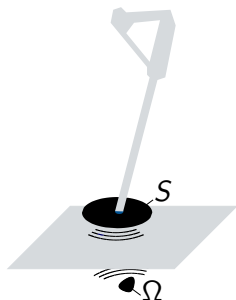
is not clear.



The inverse problem



Setup



Locate $\Omega = \text{supp } \sigma$ in

Detecting conductors:

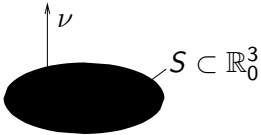
- ▶ Apply surface currents J on S
(divergence-free, no electrostatic effects)
- ▶ Measure electric field E on S
(tangential component, up to grad. fields)
- ▶ Measurement operator

$$\Lambda_\sigma : J_t \mapsto \gamma_\tau E := (\nu \wedge E|_S) \wedge \nu$$

$$\partial_t(\sigma E) + \text{curl} \left(\frac{1}{\mu} \text{curl } E \right) = -J_t \quad \text{in } \mathbb{R}^3 \times]0, T[$$

(+ zero IC) from all possible surface currents and measured values.

Measurement operator



A diagram showing a black elliptical surface S in $\mathbb{R}^3$. A vertical arrow labeled ν points upwards from the center of the ellipse, representing the normal vector.

$$\begin{aligned}
 TL^2 &:= \{u \in L^2(S)^3 \mid u \cdot \nu = 0\} \\
 TL_\diamond^2 &:= \{u \in TL^2 \mid \int_S u \cdot \nabla \psi = 0 \\
 &\quad \forall \text{ smooth } \psi\}
 \end{aligned}$$

Measurement operator

$$\Lambda_\sigma : L^2(0, T, TL_\diamond^2) \rightarrow L^2(0, T, TL_\diamond^{2'}), \quad J_t \mapsto \gamma_\tau E,$$

where E solves eddy current eq. with $[\nu \times \text{curl } E]_S = J_t$ on S .

Remark

$$TL_\diamond^{2'} \cong TL^2 / TL_\diamond^{2\perp} \rightsquigarrow E \text{ not unique, but } \Lambda_\sigma \text{ well-defined.}$$



Sampling methods

Non-iterative shape detection methods:

- ▶ Linear Sampling Method (Colton/Kirsch 1996)
 - ▶ characterizes subset of scatterer by range test
 - ▶ allows fast numerical implementation
- ▶ Factorization Method (Kirsch 1998)
 - ▶ characterizes scatterer by range test
 - ▶ yields uniqueness under definiteness assumptions
 - ▶ allows fast numerical implementation
- ▶ Beyond LSM/FM?



Sampling ingredients

Ingredients for LSM and FM:

- ▶ **Reference measurements:** $\Lambda := \Lambda_\sigma - \Lambda_0$,
 $\Lambda_0 : J_t \mapsto \gamma_\tau F$, F solves $\operatorname{curl} \operatorname{curl} F = -J_t$ in $\mathbb{R}^3 \times]0, T[$.
- ▶ **Time-integration:** Consider $I\Lambda$,
with $I : E(\cdot, \cdot) \mapsto \int_0^T E(\cdot, t) \, dt$
- ▶ **Singular test functions**

$$G_{z,d}(x) := \operatorname{curl} \frac{d}{4\pi|x-z|}, \quad x \in \mathbb{R}^3 \setminus \{z\}$$

LSM and FM

Arnold/H. (submitted):

For every z below S , $z \notin \Omega$ and direction $d \in \mathbb{R}^3$.

Theorem (LSM)

$$\gamma_{\tau} G_{z,d} \in \mathcal{R}(I\Lambda) \quad \Rightarrow \quad z \in \Omega$$

Theorem (FM)

If, additionally, $\sup \mu|_{\Omega} < 1$ (diamagnetic scatterer)

$$\gamma_{\tau} G_{z,d} \in \mathcal{R}(I(\Lambda + \Lambda')^{1/2}) \quad \Leftrightarrow \quad z \in \Omega$$

Beyond LSM/FM?

- ▶ Beyond LSM/FM?: [Monotony methods](#)
- ▶ For EIT: Λ_σ NtD-operator for conductivity $\sigma = 1 + \chi_D$
 $D = \text{Union of all balls } B \text{ where } \Lambda_{1+\chi_B} \leq \Lambda_\sigma$ (**H.**/Ullrich)
(under the assumptions of the FM)
- ▶ stable test criterion (no infinity tests)
- ▶ allows fast numerical implementation
- ▶ allows extensions to indefinite cases

Conclusions

Inverse transient eddy current problems

- ▶ require unified parabolic-elliptic theory
- ▶ can be approached by sampling methods (LSM/FM)

Open problems

- ▶ Solution theory for

$$\operatorname{div} \sigma \nabla \varphi = -\operatorname{div} \sigma A$$

for general $\sigma \in L^\infty$, $\sigma \geq 0$?

- ▶ Monotony based methods beyond EIT?
Monotony for parabolic-elliptic problems?