

A JUSTIFICATION OF EDDY CURRENTS MODEL FOR THE MAXWELL EQUATIONS*

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Abstract. This paper is concerned with the approximation of the Maxwell equations by the eddy currents model, which appears as a correction of the quasi-static model. The eddy currents model is obtained by neglecting the displacement currents in the Maxwell equations and exhibits an elliptic character in the time-harmonic formulation. Our main concern in this paper is to show that the eddy currents model approximates the full Maxwell system up to the *second order* with respect to the frequency *if and only if* an additional condition on the current source is fulfilled. Otherwise, it is a *first-order* approximation to the Maxwell equations. We also study the well-posedness of the eddy currents model and investigate the time-dependent case. All our results strongly depend on the *topology properties* of the domains under consideration. This dependence which is specific to Maxwell's equations *does not appear* for the two- or the three-dimensional Helmholtz operator.

Key words. eddy current problems, Maxwell's equations, low-frequency analysis, validity of approximation

AMS subject classifications. 35C20, 35B40, 35Q60

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1. Introduction. In many problems encountered in electromagnetism, the numerical resolution of the full system of Maxwell's equations can be extremely expensive in terms of computer time. In some cases, however, it is possible to use a simplified model which approximates Maxwell's equations in some sense and which can be solved in a more economic way. Such a situation happens, for instance, in problems in connection with machines working at power frequencies. In that particular case, one can use the so-called eddy currents model which differs from Maxwell's equations by neglecting the displacement currents.

Significant mathematical and numerical results on eddy current problems have been obtained by Hariharan [11], Hariharan and MacCamy [12], and MacCamy and Stephan [16] for a two-dimensional problem. The boundary integral equations methods are successfully applied in these papers to solve two- and three-dimensional eddy current problems.

It is now allowed that in many situations the topology of a conductor strongly influences the surface current density in the presence of an exciting electromagnetic field. This is not the case for a dielectric material. The topology of the surface of a dielectric material does not have any significant effect. Our main concern in this paper is to show that the topology of the conductor affects the use of the eddy currents model for approximating Maxwell's equations. We will prove that the eddy currents model approximates the full Maxwell system up to the *second order* with respect to the frequency *if and only if* the limit of the source current as the frequency goes to zero is divergence free in the conductor and is orthogonal to all the Neumann fields in

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the conductor. Otherwise, it is a *first-order* approximation to the Maxwell equations. This influence of the topology is very specific to full Maxwell's equations and does not appear in the models studied by MacCamy and Stephan [16], [17]. Note that if the conductor is connected, and simply connected, then the only necessary condition is that the limit of the source current as the frequency goes to zero is divergence free.

In this paper we also prove the well-posedness of the eddy currents model and investigate the time-dependent case. These results also depend on the *topology properties* of the domain under consideration. Note that this dependence is specific to Maxwell's equations. It does not appear for the Helmholtz equation or for the models investigated in [16] and [17]. To the best of our knowledge, the results established in this paper are new and the topological considerations discussed here are significant in some applications related to complex electronic systems and the design of subshields and grounding networks as shown in the paper by Baum [6].

The paper is organized as follows. Section 2 is concerned with the formulation of the model problem and some auxiliary results. The time-harmonic Maxwell system and the eddy currents model are presented. In section 3, we study the well-posedness of the eddy currents model. The variational formulation of the eddy currents model relies on a classical Hodge decomposition lemma (Lemma 2.1) and Proposition 3.1, which asserts that the electric and magnetic fields, solutions of the eddy currents model, are square integrable on all $\mathbb{R}^3$. In section 4, we focus our attention on the Neumann fields in the conductor. Section 5 is devoted to the asymptotic analysis of the eddy currents model at low frequencies. The analysis is based on an asymptotic expansion of the solution in powers of the frequency. Recalling the low-frequency analysis of Maxwell's equations in section 6, we prove in section 7 the main result of this paper which is concerned with the approximation of the Maxwell equations by the eddy currents model. Finally in section 8 we deal with the time-dependent case. We prove that the same results are valid under suitable hypothesis on the spectrum of the source current.

This justifies the use of the eddy currents model as a second-order approximation if and only if the additional condition on the source current is fulfilled. Otherwise, the eddy currents model gives a first-order approximation to the Maxwell equations.

2. Notations and problem formulation. In this section we give a precise definition of the type of geometry in which we are working and we make some assumptions on the material functions: the electric permittivity, the magnetic permeability, and the conductivity. Next, we present the dissipative Maxwell equations and the eddy currents model and discuss compatibility conditions on the source current.

Let the conductor Ω_C be a bounded open subset of $\mathbb{R}^3$ with a Lipschitz-continuous boundary Γ . We denote by Γ_i with $i = 1, \dots, p$ the connected components of Γ and by $\mathbf{n}$ the outer normal to Γ . Assume that the conductor Ω_C is not simply connected, there exists $\Sigma_j \subset \Omega_C$, with $j = 1, \dots, q$, q regular disjoint manifolds such that $\Omega_C \setminus \bigcup_{j=1}^q \Sigma_j$ is simply connected. We set $\Omega_e = \mathbb{R}^3 \setminus \overline{\Omega_C}$. Let the material functions ε, μ , and σ be the electric permittivity, the magnetic permeability, and the conductivity, respectively. We make the following general assumptions:

- $\varepsilon(\mathbf{x})$, $\mu(\mathbf{x})$, and $\sigma(\mathbf{x})$ are all real valued bounded functions. Furthermore, $\varepsilon(\mathbf{x}) \geq \varepsilon_0 > 0$, $\mu(\mathbf{x}) \geq \mu_0 > 0$, and $\sigma(\mathbf{x}) \geq \sigma_0 > 0$ in Ω_C , where ε_0, μ_0 , and σ_0 are positive constants.
- There exists $R > 0$ such that outside the sphere $S_R = \{|\mathbf{x}| = R\}$ the material parameters ε and μ are constants. We denote by $B_R = \{|\mathbf{x}| < R\}$ the corresponding ball and by $\Omega_R = \Omega_e \cap B_R$.

- $\sigma(\mathbf{x}) \equiv 0$ in Ω_e .
- The source current $\mathbf{J}$ is such that its support is included in the ball B_R . Furthermore, it is square integrable on B_R and it may depend on the angular frequency.

In order to shorten our notations we set $\varepsilon(\mathbf{x}) \equiv \mu(\mathbf{x}) \equiv 1$ for $\mathbf{x} \in \mathbb{R}^3 \setminus \overline{B_R}$.

Let $\mathcal{L}^2(\Omega_e)$ (resp., $\mathcal{L}^2(\Omega_C)$) be the space of square integrable complex functions on Ω_e (resp., Ω_C) and $\mathcal{H}^{-1/2}(\Gamma)$ the classical Sobolev space on Γ of order $-1/2$. We also introduce the classical Sobolev spaces

$$\mathcal{H}^1(\Omega_e) = \{\varphi \in \mathcal{L}^2(\Omega_e); \text{grad } \varphi \in \mathcal{L}^2(\Omega_e)^3\},$$

$$\mathcal{H}^1(\Omega_C) = \{\varphi \in \mathcal{L}^2(\Omega_C); \text{grad } \varphi \in \mathcal{L}^2(\Omega_C)^3\},$$

provided with the norm

$$\|\varphi\|_{\mathcal{H}^1(\Omega_e)} = (\|\varphi\|_{\mathcal{L}^2(\Omega_e)}^2 + \|\text{grad } \varphi\|_{\mathcal{L}^2(\Omega_e)^3}^2)^{1/2},$$

$$\|\varphi\|_{\mathcal{H}^1(\Omega_C)} = (\|\varphi\|_{\mathcal{L}^2(\Omega_C)}^2 + \|\text{grad } \varphi\|_{\mathcal{L}^2(\Omega_C)^3}^2)^{1/2},$$

respectively, and the following functional spaces

$$\begin{aligned} \mathcal{H}(\text{curl}, \Omega_e) &:= \{\mathbf{u} \in \mathcal{L}^2(\Omega_e)^3; \text{curl } \mathbf{u} \in \mathcal{L}^2(\Omega_e)^3\}, \\ \mathcal{H}(\text{curl}, \Omega_C) &:= \{\mathbf{u} \in \mathcal{L}^2(\Omega_C)^3; \text{curl } \mathbf{u} \in \mathcal{L}^2(\Omega_C)^3\}, \\ \mathcal{H}(\text{div}_\varepsilon^0, \Omega_e) &:= \{\mathbf{u} \in \mathcal{L}^2(\Omega_e)^3; \text{div}(\varepsilon \mathbf{u}) = 0 \text{ in } \Omega_e\}, \\ \mathcal{H}(\text{div}_\sigma^0, \Omega_C) &:= \{\mathbf{u} \in \mathcal{L}^2(\Omega_C)^3; \text{div}(\sigma \mathbf{u}) = 0 \text{ in } \Omega_C\} \end{aligned}$$

normed by

$$\|\mathbf{u}\|_{\mathcal{H}(\text{curl}, \Omega_e)} = (\|\mathbf{u}\|_{\mathcal{L}^2(\Omega_e)^3}^2 + \|\text{curl } \mathbf{u}\|_{\mathcal{L}^2(\Omega_e)^3}^2)^{1/2},$$

$$\|\mathbf{u}\|_{\mathcal{H}(\text{curl}, \Omega_C)} = (\|\mathbf{u}\|_{\mathcal{L}^2(\Omega_C)^3}^2 + \|\text{curl } \mathbf{u}\|_{\mathcal{L}^2(\Omega_C)^3}^2)^{1/2},$$

$$\|\mathbf{u}\|_{\mathcal{H}(\text{div}_\varepsilon^0, \Omega_e)} = (\|\mathbf{u}\|_{\mathcal{L}^2(\Omega_e)^3}^2 + \|\varepsilon \mathbf{u} \cdot \mathbf{n}\|_{\mathcal{H}^{-1/2}(\Gamma)}^2)^{1/2},$$

$$\|\mathbf{u}\|_{\mathcal{H}(\text{div}_\sigma^0, \Omega_C)} = (\|\mathbf{u}\|_{\mathcal{L}^2(\Omega_C)^3}^2 + \|\sigma \mathbf{u} \cdot \mathbf{n}\|_{\mathcal{H}^{-1/2}(\Gamma)}^2)^{1/2}.$$

The system of time-harmonic dissipative Maxwell equations is

$$(2.1) \quad \text{curl } \mathbf{H}^m = +i \varepsilon \omega \mathbf{E}^m + \sigma \mathbf{E}^m + \mathbf{J} \quad \text{in } \mathbb{R}^3,$$

$$(2.2) \quad \text{curl } \mathbf{E}^m = -i \mu \omega \mathbf{H}^m \quad \text{in } \mathbb{R}^3,$$

$$(2.3) \quad \lim \left(\mathbf{H}^m \times \frac{\mathbf{x}}{|\mathbf{x}|} - \mathbf{E}^m \right) = 0, \quad \text{uniformly for } |\mathbf{x}| \rightarrow \infty,$$

where ω is the angular frequency and the sup-index m stands for Maxwell. The dissipation in the system of equations (2.1)–(2.3) is due to the conductivity function $\sigma(\mathbf{x})$, which satisfies $\sigma(\mathbf{x}) \geq \sigma_0 > 0$ in the conductor Ω_C . Note that the system of equations (2.1)–(2.2) is written in the sense of distributions. Since the problem

will be to find the fields $\mathbf{E}^m$ and $\mathbf{H}^m$ locally in $\mathcal{L}^2(\mathbb{R}^3)$, then $\text{curl } \mathbf{E}^m$ and $\text{curl } \mathbf{H}^m$ will be locally in $\mathcal{L}^2(\mathbb{R}^3)$, so there is no jump of the tangential components of the fields $\mathbf{E}^m$ and $\mathbf{H}^m$ across any interface [14]. However, the normal components are not continuous. In fact, there is no jump of $(i\varepsilon\omega\mathbf{E}^m + \sigma\mathbf{E}^m + \mathbf{J}) \cdot \mathbf{n}$ and $\mu\mathbf{H}^m \cdot \mathbf{n}$, where $\mathbf{n}$ is a unit normal to the interface. By this skew an interface separating two materials with different parameters plays a key role in the formulation of the system of Maxwell's equations.

Dealing with problems in connection with machines working at power frequencies, the full system of the Maxwell equations (2.1)–(2.3) is usually simplified. Since the angular frequency ω is very small, displacement currents are usually neglected, which means that $\omega\varepsilon$ is set to zero in Ω_e . Moreover, the Silver–Müller radiation condition (2.3) is replaced by (2.7)–(2.8) below, which holds uniformly for $|\mathbf{x}| \rightarrow +\infty$. All these modifications together yield the eddy currents model, which appears as a correction of the quasi-static model. We refer the reader to Bossavit [8] and Kettunen, Forsman, and Bossavit [15] for a derivation of the eddy currents model and its numerical implementation.

Note that the eddy currents model has the advantage to exhibit an elliptic character. In the time-dependent case, the transient eddy currents model yields parabolic-elliptic interface problems; mainly, the Laplace equation in one domain coupled with the heat equation in the other domain. On these problems, significant mathematical and numerical results have been obtained by MacCamy and Suri [18], Costabel, Ervin, and Stephan [10], and Lee, Hariharan, and Ida [13].

Assume that $\text{div } (\mathbf{J}) = 0$ in Ω_e . The mathematical model of the eddy currents has then the following form:

$$(2.4) \quad \text{curl } \mathbf{H} = \sigma \mathbf{E} + \mathbf{J} \quad \text{in } \mathbb{R}^3,$$

$$(2.5) \quad \text{curl } \mathbf{E} = -i\mu\omega\mathbf{H} \quad \text{in } \mathbb{R}^3,$$

$$(2.6) \quad \text{div } (\varepsilon\mathbf{E}) = 0 \quad \text{in } \Omega_e,$$

$$(2.7) \quad \mathbf{H}(\mathbf{x}) = O\left(\frac{1}{|\mathbf{x}|}\right), \quad \text{uniformly for } |\mathbf{x}| \rightarrow \infty,$$

$$(2.8) \quad \mathbf{E}(\mathbf{x}) = O\left(\frac{1}{|\mathbf{x}|}\right), \quad \text{uniformly for } |\mathbf{x}| \rightarrow \infty.$$

From the above discussion on interface conditions, standard jump conditions may be deduced from the above system. In fact, the tangential parts of the electric and magnetic fields are continuous across the interface Γ . Then we have

$$[\sigma\mathbf{E} \cdot \mathbf{n}]_\Gamma = \sigma\mathbf{E}_{\text{int}} \cdot \mathbf{n} = -[\mathbf{J} \cdot \mathbf{n}]_\Gamma,$$

where $[f]_\Gamma$ denotes the jump of the function f across the interface Γ and $\mathbf{E}_{\text{int}} \equiv \mathbf{E}|_{\Omega_C}$.

Due to (2.4) the current source $\mathbf{J}$ has to satisfy some compatibility conditions. In particular, outside the conductor Ω_C , the term $\sigma\mathbf{E}$ is identically zero and the function $\mathbf{J}$ is then the curl of the magnetic field $\mathbf{H}$. In order to characterize this constraint on $\mathbf{J}$, we introduce three other functional spaces:

$$\begin{aligned} \text{grad } \mathcal{H}_0^1(\Omega_e) &:= \{\text{grad } \varphi; \varphi \in \mathcal{H}^1(\Omega_e), \varphi|_{\partial\Omega_C} = 0\}, \\ D_{\text{ext}}^\varepsilon &:= \{\mathbf{d} \in \mathcal{L}^2(\Omega_e)^3; \text{curl } \mathbf{d} = 0 \text{ in } \Omega_e\}, \end{aligned}$$

$$\begin{aligned} \operatorname{div}(\varepsilon \mathbf{d}) &= 0 \text{ in } \Omega_e; \mathbf{d} \times \mathbf{n}|_{\Gamma} = 0\}, \\ \operatorname{curl} \mathcal{H}(\operatorname{curl}, \Omega_e) &:= \{\operatorname{curl} \mathbf{u}; \mathbf{u} \in \mathcal{H}(\operatorname{curl}, \Omega_e)\}. \end{aligned}$$

Note that the space $D_{\text{ext}}^\varepsilon$ is formed only by the *Dirichlet fields*; it is well known that it is finite-dimensional and its dimension is equal the number of connected components of the boundary Γ . A complete characterization of this space is given in [20], [21], and [23], and, in particular,

$$\begin{aligned} D_{\text{ext}}^\varepsilon &\equiv \operatorname{span}\{\operatorname{grad} \varphi_j; \varphi_j \in \mathcal{H}^1(\Omega_e), \operatorname{div}(\varepsilon \operatorname{grad} \varphi_j) = 0 \text{ in } \Omega_e, \\ &\quad \varphi_j|_{\Gamma_i} = \delta_{i,j}; \text{ for } i, j = 1, \dots, p\}, \end{aligned}$$

where $\delta_{i,j}$ denotes the Kronecker index.

The following lemma is a simple consequence of the Hodge decomposition presented in [20] and of some results on the Dirichlet fields obtained in [23].

LEMMA 2.1. *The following Hodge decomposition holds:*

$$(2.9) \quad \mathcal{L}^2(\Omega_e)^3 = \operatorname{curl} \mathcal{H}(\operatorname{curl}, \Omega_e) \overset{\perp}{\oplus} \operatorname{grad} \mathcal{H}_0^1(\Omega_e) \overset{\perp}{\oplus} D_{\text{ext}}^\varepsilon,$$

where $\overset{\perp}{\oplus}$ denotes the direct sum of spaces which are orthogonal with respect to the $\mathcal{L}^2$ -scalar product.

From the previous Lemma and (2.4), it follows that

$$(2.10) \quad \begin{aligned} \mathbf{J} \perp \operatorname{grad} \mathcal{H}_0^1(\Omega_e) \overset{\perp}{\oplus} D_{\text{ext}}^\varepsilon &\iff \operatorname{div}(\mathbf{J})|_{\Omega_e} = 0 \\ &\text{and } \int_{\Gamma_j} \mathbf{J} \cdot \mathbf{n} \, d\Gamma = 0 \, \forall j = 1, \dots, p. \end{aligned}$$

From now on, we assume that $\mathbf{J}$ satisfies the right-hand side (or the left one) of (2.10).

Our concern is with the low-frequency solutions of the Maxwell system (2.1)–(2.3) and the eddy currents model (2.4)–(2.8). Our aim is to show that $(\mathbf{E}, \mathbf{H})$ is a second-order approximation of $(\mathbf{E}^m, \mathbf{H}^m)$ when the frequency goes to zero if and only if the following additional condition on the source current $\mathbf{J}$ is fulfilled: the zero-order term in the asymptotic expansion of the source current with respect to the frequency ω is divergence free in the conductor Ω_C and it is orthogonal to all the Neumann fields in Ω_C .

3. Existence and uniqueness for the eddy currents model. In this section, we investigate questions on existence and uniqueness for the eddy currents model (2.4)–(2.8). Our main result is that problem (2.4)–(2.8), together with the additional condition on the electric field

$$(3.11) \quad \int_{\Gamma_j} \varepsilon \mathbf{E} \cdot \mathbf{n} = 0 \quad \text{for } j = 1, \dots, p,$$

is well posed.

Let us state a useful result on the behavior of the electric and magnetic fields as $|\mathbf{x}|$ goes to $+\infty$.

PROPOSITION 3.1. *Let $\mathbf{H}$ and $\mathbf{E}$ be two vector fields satisfying the system (2.4)–(2.8). Then they have the following behavior at the infinity:*

$$(3.12) \quad \mathbf{H}(\mathbf{x}) = O\left(\frac{1}{|\mathbf{x}|^2}\right), \quad \mathbf{E}(\mathbf{x}) = O\left(\frac{1}{|\mathbf{x}|^2}\right), \quad \text{uniformly for } |\mathbf{x}| \rightarrow \infty.$$

Proof. It suffices to prove the result for the electric field $\mathbf{E}$. From (2.4)–(2.8) we have, by making use of the identity $\Delta = \text{grad div} - \text{curl curl}$ and recalling that $\text{div} \varepsilon \mathbf{E} = 0$ in Ω_e ,

$$(3.13) \quad \begin{aligned} \Delta \mathbf{E} &= 0 && \text{in } \mathbb{R}^3 \setminus B_R, \\ \text{div } \mathbf{E} &= 0 && \text{in } \mathbb{R}^3 \setminus B_R, \\ \mathbf{E}(\mathbf{x}) &= O\left(\frac{1}{|\mathbf{x}|}\right), && \text{uniformly for } |\mathbf{x}| \rightarrow \infty. \end{aligned}$$

Let $\{Y_l^m\}_{-l \leq m \leq l}$ be an orthonormal sequence of spherical harmonics of order l on the unit sphere S_1 , normalized such that

$$\int_{S_1} Y_l^m \bar{Y}_{l'}^{m'} = \delta_{l,l'} \delta_{m,m'}.$$

According to [9] it follows from (3.13) that the vector field $\mathbf{E}$ has the following expansion:

$$(3.14) \quad \mathbf{E} = \mathbf{E}(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{1}{r^{l+1}} \mathbf{E}_l^m Y_l^m(\theta, \varphi),$$

where $r > R$, $\theta \in [0, 2\pi)$ and $\varphi \in [-\pi/2, \pi/2)$ are the spherical coordinates in $\mathbb{R}^3$ and $\{\mathbf{E}_l^m\}_{l,m}$ are constant vectors in $\mathbb{R}^3$.

By imposing (3.13), we obtain

$$(3.15) \quad \text{div } \mathbf{E}(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \text{grad} \left(\frac{1}{r^{l+1}} Y_l^m(\theta, \varphi) \right) \cdot \mathbf{E}_l^m = 0 \quad \forall r > R, \theta, \varphi.$$

Recalling that $\text{grad } Y_l^m = \frac{1}{r} \text{grad}_{S_1} Y_l^m$ and $\text{grad}_{S_1} Y_0^0 = 0$, we arrive at

$$(3.16) \quad 0 = \text{div } \mathbf{E} = \frac{1}{r^2} \mathbf{E}_0^0 \cdot \mathbf{n}_r + O\left(\frac{1}{r^3}\right),$$

where $\mathbf{n}_r = \mathbf{n}_r(\theta, \varphi)$ denotes the unit radial vector outer to the ball B_R . Since the first term on the right-hand side of (3.16) needs to be zero for any value of θ and φ , we necessarily have $\mathbf{E}_0^0 = 0$ and this, together with (3.14), yields (3.12). $\square$

One important consequence of Proposition 3.1 is that the vector fields $\mathbf{E}$ and $\mathbf{H}$ are in $\mathcal{L}^2(\mathbb{R}^3)$. This is a key property while writing the variational formulation of the eddy currents problem.

THEOREM 3.2. *Let the functional space $\mathcal{V}$ be defined by*

$$(3.17) \quad \mathcal{H}(\text{curl}, \mathbb{R}^3) \cap \mathcal{H}(\text{div}_\varepsilon^0, \Omega_e) = \mathcal{V} \oplus D_{\text{ext}}^\varepsilon.$$

The variational formulation: find $\mathbf{u} \in \mathcal{V}$ such that

$$(3.18) \quad \int_{\mathbb{R}^3} \mu^{-1} \text{curl } \mathbf{u} \cdot \text{curl } \bar{\mathbf{u}}^t + i\omega \int_{\Omega_C} \sigma \mathbf{u} \cdot \bar{\mathbf{u}}^t = -i\omega \int_{\Omega_R} \mathbf{J} \cdot \bar{\mathbf{u}}^t \quad \forall \mathbf{u}^t \in \mathcal{V}$$

has a unique solution in $\mathcal{V}$. Moreover, the vector fields

$$\mathbf{E} \equiv \mathbf{u} \quad \text{and} \quad \mathbf{H} \equiv -i(\omega \mu)^{-1} \text{curl } \mathbf{u} \quad \text{in } \mathbb{R}^3$$

are the unique solutions of (2.4)–(2.8) together with the additional condition (3.11).

Proof. First, we note that the bilinear form $\int_{\mathbb{R}^3} \mu^{-1} \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \bar{\mathbf{u}}^t$ is not coercive on $\mathcal{H}(\operatorname{curl}, \mathbb{R}^3)$. Moreover, any field in $D_{\text{ext}}^\varepsilon$ yields a solution of the problem

$$\int_{\mathbb{R}^3} \mu^{-1} \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \bar{\mathbf{u}}^t + i\omega \int_{\Omega_C} \sigma \mathbf{u} \cdot \bar{\mathbf{u}}^t = 0 \quad \forall \mathbf{u}^t \in \mathcal{H}(\operatorname{curl}, \mathbb{R}^3).$$

To avoid these drawbacks, we introduce the identities (2.6) and (3.11) in the variational space $\mathcal{V}$ and consider the variational problem (3.18) on $\mathcal{V} \times \mathcal{V}$.

We set

$$a(\mathbf{u}, \mathbf{u}^t) = \int_{\mathbb{R}^3} \mu^{-1} \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \bar{\mathbf{u}}^t + i\omega \int_{\Omega_C} \sigma \mathbf{u} \cdot \bar{\mathbf{u}}^t.$$

By taking $\mathbf{u}^t = \mathbf{u}$ in (3.18) it is easy to see that

$$(3.19) \quad a(\mathbf{u}, \mathbf{u}) \geq C \|\operatorname{curl} \mathbf{u}\|_{\mathcal{L}^2(\mathbb{R}^3)^3}^2 + C' \|\mathbf{u}\|_{\mathcal{L}^2(\Omega_C)^3}^2,$$

where C and C' are two positive constants.

Let

$$|\mathbf{u}|_\star = (\|\operatorname{curl} \mathbf{u}\|_{\mathcal{L}^2(\mathbb{R}^3)^3}^2 + \|\mathbf{u}\|_{\mathcal{L}^2(\Omega_C)^3}^2)^{1/2}.$$

It is obvious that the functional $|\cdot|_\star$ is a seminorm on $\mathcal{V}$.

Let us first prove that

$$|\mathbf{v}|_\star = 0 \implies \mathbf{v} \equiv 0.$$

Let $\mathbf{v} \in \mathcal{V}$ be such that $|\mathbf{v}|_\star = 0$, then $\mathbf{v}$ is a solution of

$$(3.20) \quad \begin{cases} \operatorname{div}(\varepsilon \mathbf{v}) = 0 & \text{in } \Omega_e, \\ \operatorname{curl} \mathbf{v} = 0 & \text{in } \Omega_e, \\ \mathbf{v} \times \mathbf{n}|_\Gamma = 0 & \text{on } \Gamma. \end{cases}$$

Therefore, a function $\mathbf{v}$ satisfying (3.20) is in $D_{\text{ext}}^\varepsilon$; since $\mathbf{v}$ is also in $\mathcal{V}$ we may conclude that $\mathbf{v} = 0$. This proves uniqueness of solutions to (3.18). Further, for any sequence $(v_n)_{n \in \mathbb{N}} \in \mathcal{V}$ such that $|v_n|_\star$ is bounded it is also easy to show that we can extract a subsequence which converges to an element $v \in \mathcal{V}$, so we have existence of a solution to (3.18).

Thus, we obtain the existence and uniqueness of solution of the variational formulation (3.18) in $\mathcal{V}$.

Let now $\mathbf{E}$ be a solution of the problem (2.4)–(2.8) together with the additional condition (3.11). We shall prove that $\mathbf{E}$ is the solution of the variational formulation (3.18). It is obvious that $\mathbf{E} \in \mathcal{V}$. By eliminating the magnetic field $\mathbf{H}$ in the system of equations (2.4)–(2.8), we arrive at

$$\operatorname{curl} \mu^{-1} \operatorname{curl} \mathbf{E} = -i\omega \sigma \mathbf{E} - i\omega \mathbf{J} \quad \text{in } \mathcal{D}'(\mathbb{R}^3),$$

i.e., in the sense of distribution in $\mathbb{R}^3$. Note that since $\operatorname{curl} \mathbf{u} \in \mathcal{L}^2(\mathbb{R}^3)$, then automatically there is no jump of the tangential component of $\mathbf{u}$, i.e., $\mathbf{E}$, across interfaces.

Now, if we multiply the above equation by a test function $\mathbf{u}^t \in \mathcal{V}$ and use Proposition 3.1, which asserts that the integration by parts over $\mathbb{R}^3$ is allowed, we get that $\mathbf{E} \in \mathcal{V}$ is the solution of the variational formulation (3.18). This ends the proof of Theorem 3.2. $\square$

4. Neumann fields. Let $(\mathbf{E}, \mathbf{H})$ be the unique solution of the eddy current problem (2.4)–(2.8) together with the additional condition (3.11). We shall analyze in this section some of the properties of this solution. In particular, we focus our attention on the presence of Neumann fields in the conductor Ω_C .

Let us start by recalling the definition of *Neumann fields*. According to [7], the space of Neumann fields in the conductor Ω_C is defined by

$$(4.21) \quad N_{\text{int}}^\sigma := \{\mathbf{u} \in \mathcal{H}(\text{curl}, \Omega_C) \cap \mathcal{H}(\text{div}_\sigma^0, \Omega_C); \text{curl } \mathbf{u} = 0 \text{ in } \Omega_C; \mathbf{u} \cdot \mathbf{n}|_\Gamma = 0\}.$$

Following [5], [20], and [7], we know that this space is finite-dimensional and its dimension is equal to q , the number of regular cuts contained in Ω_C which are necessary to reduce Ω_C to a simply connected region (see section 2). A basis for this space is characterized as the set $\{\text{grad } q_i\}_{i=1, \dots, q}$, where $\{q_i\}_{i=1, \dots, q}$ are solutions of the following problem:

$$(4.22) \quad \begin{array}{llll} \text{div}(\sigma \text{grad } q_i) & = 0 & \text{in } \Omega_C \setminus \bigcup_{j=1}^q \Sigma_j \\ \forall j = 1, \dots, q & [q_i]_{\Sigma_j} & = \delta_{i,j} & \text{across } \Sigma_j \\ \forall j = 1, \dots, q & [\text{grad } q_i \cdot \mathbf{n}]_{\Sigma_j} & = 0 & \text{across } \Sigma_j, \\ & \partial_n q_i & = 0 & \text{on } \Gamma, \end{array}$$

where $[f]_{\Sigma_j}$ denotes the jump of the function f across the manifold Σ_j . It is important to underline that $q_i \in \mathcal{H}^1(\Omega_C \setminus \bigcup_{j=1}^q \Sigma_j)$ but in general $q_i \notin \mathcal{H}^1(\Omega_C)$. However, $\text{grad } q_i \in \mathcal{L}^2(\Omega_C \setminus \bigcup_{j=1}^q \Sigma_j)^3$ and therefore can be extended to $\mathcal{L}^2(\Omega_C)^3$. From now on, by $\text{grad } q$, $q \in \mathcal{H}^1(\Omega_C \setminus \bigcup_{j=1}^q \Sigma_j)$, we mean the extension of $\text{grad } q$ to $\mathcal{L}^2(\Omega_C)^3$.

The Neumann fields coefficients of the electric field $\mathbf{E}$, solution of (2.4)–(2.8), and (3.11) are the integrals

$$\int_{\Omega_C} \sigma \mathbf{E} \cdot \text{grad } q_i \quad \text{for } i = 1, \dots, q.$$

We shall evaluate them as a function of the source density current $\mathbf{J}$.

Using (2.4), we obtain, integrating by parts,

$$\begin{aligned} \int_{\Omega_C} \sigma \mathbf{E} \cdot \text{grad } q_i &= \int_{\Omega_C} (\text{curl } \mathbf{H} - \mathbf{J}) \cdot \text{grad } q_i \\ &= \int_{\Omega_C} \mathbf{H} \cdot \text{curl}(\text{grad } q_i) - \int_\Gamma \mathbf{H} \times \mathbf{n} \cdot \text{grad } q_i - \int_{\Omega_C} \mathbf{J} \cdot \text{grad } q_i. \end{aligned}$$

From the jump condition $[\mathbf{H} \times \mathbf{n}]_\Gamma = 0$ it follows that $\text{div}_\Gamma(\mathbf{H} \times \mathbf{n}) = \mathbf{J}_{\text{ext}} \cdot \mathbf{n}$, where we set $\mathbf{J}_{\text{ext}} := \mathbf{J}|_{\Omega_e}$. This yields

$$\int_\Gamma \mathbf{H} \times \mathbf{n} \cdot \text{grad } q_i = - \int_\Gamma \Delta_\Gamma^{-1}(\mathbf{J}_{\text{ext}} \cdot \mathbf{n}) \Delta_\Gamma q_i.$$

Therefore

$$\int_{\Omega_C} \sigma \mathbf{E} \cdot \text{grad } q_i = \alpha_i(\mathbf{J}),$$

where

$$(4.23) \quad \alpha_i(\mathbf{J}) = \int_\Gamma \Delta_\Gamma^{-1}(\mathbf{J}_{\text{ext}} \cdot \mathbf{n}) \Delta_\Gamma q_i - \int_{\Omega_C} \mathbf{J} \cdot \text{grad } q_i.$$

Note that the first term in the expression of $\alpha_i(\mathbf{J})$ takes into account the exterior of the conductor Ω_C while the second one takes into account the internal part. We also note that for a function $\varphi \in \mathcal{H}^1(\Omega_C)$, we have

$$\int_{\Gamma} \Delta_{\Gamma}^{-1}(\mathbf{J}_{\text{ext}} \cdot \mathbf{n}) \Delta_{\Gamma} \varphi = \int_{\Gamma} \mathbf{J}_{\text{ext}} \cdot \mathbf{n} \varphi.$$

The following holds.

LEMMA 4.1. *Let $(\mathbf{E}, \mathbf{H})$ be the solution of the eddy currents model (2.4)–(2.8) together with the additional condition (3.11). Then*

$$\int_{\Omega_C} \sigma \mathbf{E} \cdot \text{grad} q_i = \alpha_i(\mathbf{J}) \quad \text{for } i = 1, \dots, q,$$

where $\{\alpha_i(\mathbf{J})\}_{i=1, \dots, q}$ are defined by (4.23).

Now, if we assume that $\mathbf{J}$ is orthogonal to N_{int}^{σ} and there is no source current that goes from the exterior to the interior of the conductor Ω_C , then the vector field $\mathbf{E}$ is orthogonal to the space of Neumann fields N_{int}^{σ} .

LEMMA 4.2. *The following holds:*

$$(4.24) \quad \mathbf{J} \perp N_{\text{int}}^{\sigma} \text{ and } \mathbf{J}_{\text{ext}|_{\Gamma}} \cdot \mathbf{n} = 0 \Rightarrow \int_{\Omega_C} \sigma \mathbf{E} \cdot \text{grad} q_i = 0 \quad \forall i = 1, \dots, q.$$

5. Low-frequency analysis of the eddy currents model. From now on, we assume that the density current $\mathbf{J}$ admits the following series expansion:

$$(5.25) \quad \mathbf{J} = \sum_{j \geq 0} \omega^j \mathbf{J}_j; \quad \forall j \geq 0, \quad \text{div}(\mathbf{J}_j)|_{\Omega_e} = 0, \quad \int_{\Gamma_i} \mathbf{J}_j \cdot \mathbf{n} = 0 \quad \forall i = 1, \dots, p,$$

where the last two equalities come from the fact that $\mathbf{J}$ verifies one side of (2.10). As standard in asymptotic analysis we proceed formally and we suppose that the solution of (2.4)–(2.8) can also be expanded in power series, namely,

$$(5.26) \quad \mathbf{E} = \sum_{j=0}^{\infty} \omega^j \mathbf{E}_j; \quad \mathbf{H} = \sum_{j=0}^{\infty} \omega^j \mathbf{H}_j.$$

In the following, we show first that the terms $(\mathbf{E}_j, \mathbf{H}_j)$ can be uniquely determined and, afterward, we study the convergence of the series in (5.26).

Using together (2.4)–(2.8) and (3.18), we write $\mathbf{E}$ as the unique solution of the following problem:

$$(5.27) \quad \mathbf{E} \in \mathcal{V},$$

$$(5.28) \quad \int_{\mathbb{R}^3} \mu^{-1} \text{curl} \mathbf{E} \cdot \text{curl} \bar{\mathbf{E}}^t + i\omega \int_{\Omega_C} \sigma \mathbf{E} \cdot \bar{\mathbf{E}}^t = -i\omega \int_{\Omega_R} \mathbf{J} \cdot \bar{\mathbf{E}}^t \quad \forall \mathbf{E}^t \in \mathcal{V},$$

$$(5.29) \quad \text{div}(\sigma \mathbf{E} + \mathbf{J}) = 0 \quad \text{in } \mathbb{R}^3,$$

$$(5.30) \quad \int_{\Omega_C} \sigma \mathbf{E} \cdot \text{grad} q_i = \alpha_i(\mathbf{J}) \quad \text{for } i = 1, \dots, q.$$

This system is not minimal due to (5.29)–(5.30), but it turns out to be useful when writing the system of equations satisfied by the asymptotics $\{\mathbf{E}_j\}_{j \geq 0}$.

Using then (5.26) in (5.27)–(5.30) and writing separately the contribution coming from different powers of the frequency ω in (5.27)–(5.30), we obtain the following systems of equations for the coefficients. For the first one we have

$$(5.31) \quad \mathbf{E}_0 \in \mathcal{V},$$

$$(5.32) \quad \int_{\mathbb{R}^3} \mu^{-1} \operatorname{curl} \mathbf{E}_0 \cdot \operatorname{curl} \bar{\mathbf{E}}^t = 0 \quad \forall \mathbf{E}^t \in \mathcal{V},$$

$$(5.33) \quad \operatorname{div}(\sigma \mathbf{E}_0 + \mathbf{J}_0) = 0 \quad \text{in } \mathbb{R}^3,$$

$$(5.34) \quad \int_{\Omega_C} \sigma \mathbf{E}_0 \cdot \operatorname{grad} q_i = \alpha_i(\mathbf{J}_0) \quad \text{for } i = 1, \dots, q,$$

while for the other ones,

$$(5.35) \quad \mathbf{E}_j \in \mathcal{V},$$

$$(5.36) \quad \int_{\mathbb{R}^3} \mu^{-1} \operatorname{curl} \mathbf{E}_j \cdot \operatorname{curl} \bar{\mathbf{E}}^t = -i \int_{\Omega_C} \sigma \mathbf{E}_{j-1} \cdot \bar{\mathbf{E}}^t - i \int_{\Omega_R} \mathbf{J}_{j-1} \cdot \bar{\mathbf{E}}^t \quad \forall \mathbf{E}^t \in \mathcal{V},$$

$$(5.37) \quad \operatorname{div}(\sigma \mathbf{E}_j + \mathbf{J}_j) = 0 \quad \text{in } \mathbb{R}^3,$$

$$(5.38) \quad \int_{\Omega_C} \sigma \mathbf{E}_j \cdot \operatorname{grad} q_i = \alpha_i(\mathbf{J}_j) \quad \text{for } i = 1, \dots, q.$$

The following theorem holds.

THEOREM 5.1. *The problems (5.31)–(5.34) and (5.35)–(5.38) are well posed.*

Proof. We focus our attention only on the well-posedness of the system of equations (5.31)–(5.34), since the well-posedness of the system (5.35)–(5.38) may be proved by the same arguments.

It is quite natural to split the problem (5.31)–(5.34) into an *interior problem* defined on the conductor Ω_C and an *exterior problem* defined on Ω_e . We then split $\mathbf{E}_0$ into $(\mathbf{E}_0^C, \mathbf{E}_0^{NC})$, where $\mathbf{E}_0^C = \mathbf{E}|_{\Omega_C}$ and $\mathbf{E}_0^{NC} = \mathbf{E}|_{\Omega_e}$. From (5.31)–(5.34) and (5.25) we obtain the following two problems:

$$(5.39) \quad \begin{array}{cc} \text{Interior problem} & \text{Exterior problem} \\ \left\{ \begin{array}{ll} \operatorname{curl} \mathbf{E}_0^C = 0 & \text{in } \Omega_C, \\ \operatorname{div}(\sigma \mathbf{E}_0^C + \mathbf{J}_0) = 0 & \text{in } \Omega_C, \\ \sigma \mathbf{E}_0^C \cdot \mathbf{n} = [\mathbf{J}_0 \cdot \mathbf{n}]_{|\Gamma} & \text{on } \Gamma, \\ \int_{\Omega_C} \sigma \mathbf{E}_0^C \cdot \operatorname{grad} q_i = \alpha_i(\mathbf{J}_0) & \text{for } i = 1, \dots, q; \end{array} \right. & \left\{ \begin{array}{ll} \operatorname{curl} \mathbf{E}_0^{NC} = 0 & \text{in } \Omega_C, \\ \operatorname{div}(\varepsilon \mathbf{E}_0^{NC}) = 0 & \text{in } \Omega_C, \\ \mathbf{E}_0^{NC}|_{\Gamma} \times \mathbf{n} = \mathbf{E}_0^C|_{\Gamma} \times \mathbf{n} & \text{on } \Gamma, \\ \mathbf{E}_0^{NC} \perp D_{\text{ext}}^\varepsilon. \end{array} \right. \end{array}$$

First note that we have to solve the interior problem and then the exterior one. The exterior problem is coupled only to the interior one by the boundary condition on Γ . The study of the interior problem does not require any data from the exterior problem. Once the interior problem is solved the problem becomes classical. We claim that the interior problem is well posed. The existence of at least one solution to the interior problem (separately, the exterior problem) has been proved by Picard [20] and Alonso and Valli [1]. Here, we prove only the uniqueness.

If we set $\mathbf{J}_0 = 0$, then we obtain a homogeneous system since $\operatorname{div}(\mathbf{J}_0) = 0$, $[\mathbf{J}_0 \cdot \mathbf{n}]_{|\Gamma} = 0$, $\alpha_i(\mathbf{J}_0) = 0$. Thus, from the first three equations of the interior problem

in (5.39), we have that $\mathbf{E}_0^C \in N_{\text{int}}^\sigma$, while the last one imposes that it is orthogonal to the space of Neumann fields N_{int}^σ .

The well-posedness of the exterior problem in (5.39) is classical. $\square$

For what concerns the terms $\{\mathbf{H}_j\}_{j \geq 0}$ in the asymptotics expansion of the magnetic field $\mathbf{H}$, we simply have to use (2.6) and (5.26) in order to derive the following identity:

$$(5.40) \quad \mathbf{H}_j = i\mu^{-1} \text{curl } \mathbf{E}_{j+1} \quad \forall j \geq 0.$$

The complete variational formulation (3.18) could also be decomposed into an interior and an exterior problem. Also in this case, the interior field can be found before and does not depend directly on the exterior one. For the exterior problem the solution depends continuously on the boundary data.

PROPOSITION 5.2. *Let $\mathbf{E} \in \mathcal{V}$ be the unique solution of the eddy currents model (2.4)–(2.8) together with the additional condition (3.11). There exists $\omega_0 > 0$ such that for $\omega \in (0, \omega_0)$ the following inequality holds:*

$$(5.41) \quad \|\mathbf{E}\|_{\mathcal{L}^2(\mathbb{R}^3)^3} + \|\text{curl } \mathbf{E}\|_{\mathcal{L}^2(\Omega_e)^3} \leq C \left(\omega \|\mathbf{J}\|_{\mathcal{L}^2(\Omega_R)^3} + \|\mathbf{E}\|_{\mathcal{H}(\text{curl}, \Omega_C)} \right),$$

where C is a positive constant independent of ω .

Proof. It is easy to see that $\mathbf{E} \in \mathcal{V}$ satisfies

$$(5.42) \quad \text{curl } \frac{1}{\mu} \text{curl } \mathbf{E} = -i\omega \mathbf{J} \quad \text{in } \Omega_e,$$

$$(5.43) \quad \mathbf{E} \times \mathbf{n} = \mathbf{E}_{\text{int}} \times \mathbf{n} \quad \text{on } \Gamma,$$

where $\mathbf{E}_{\text{int}} \equiv \mathbf{E}|_{\Omega_C}$. From the boundary value problem (5.42)–(5.43) using $\text{div}(\varepsilon \mathbf{E}) = 0$, it is classical to deduce that the following estimate holds:

$$\|\mathbf{E}\|_{\mathcal{H}(\text{curl}, \Omega_e)} \leq C \left(\omega \|\mathbf{J}\|_{\mathcal{L}^2(\Omega_R)^3} + \|\mathbf{n} \times (\mathbf{n} \times \mathbf{E})\|_{\mathcal{TH}^{-1/2}(\text{curl}, \Gamma)} \right),$$

where the constant C does not depend on the frequency ω . Here, $\mathcal{TH}^{-1/2}(\text{curl}, \Gamma)$ is the space of tangential traces of $\mathcal{H}(\text{curl}, \Omega_e)$.

Since $[\mathbf{n} \times (\mathbf{n} \times \mathbf{E})]_\Gamma = 0$, it follows from the classical trace theorem (see Paquet [19] and Alonso and Valli [1]) that there exists a constant C' , independent of the frequency ω , such that

$$\|\mathbf{n} \times (\mathbf{n} \times \mathbf{E})\|_{\mathcal{TH}^{-1/2}(\text{curl}, \Gamma)} \leq C' \|\mathbf{E}\|_{\mathcal{H}(\text{curl}, \Omega_C)},$$

and then we have the result. $\square$

THEOREM 5.3. *Let $(\mathbf{E}_j, \mathbf{H}_j)$, for $j \geq 0$, be the unique solutions of the problems (5.31)–(5.34), (5.35)–(5.38) and (5.40). Let $(\mathbf{E}, \mathbf{H})$ be the solution of (2.4)–(2.8) and (3.11). Then, there exists $\omega_0 > 0$ such that for $M \in \mathbb{N}$ and $\omega \in (0, \omega_0)$, the following estimates hold:*

$$(5.44) \quad \left\| \mathbf{E} - \sum_{j=0}^M \omega^j \mathbf{E}_j \right\|_{\mathcal{L}^2(\mathbb{R}^3)^3} \leq C \omega^{M+1},$$

$$(5.45) \quad \left\| \mathbf{H} - \sum_{j=0}^M \omega^j \mathbf{H}_j \right\|_{\mathcal{L}^2(\mathbb{R}^3)^3} \leq C \omega^{M+1},$$

where the constant C does not depend on ω but only on the conductor Ω_C .

Proof. The quantity $\mathbf{E} - \sum_{j=0}^M \omega^j \mathbf{E}_j$ satisfies the variational equation (3.18) when the source density current is replaced by $\mathbf{J} - \sum_{j=0}^M \omega^j \mathbf{J}_j$. By standard manipulations, we have

$$(5.46) \quad \left\| \operatorname{curl} \left(\mathbf{E} - \sum_{j=0}^M \omega^j \mathbf{E}_j \right) \right\|_{\mathcal{L}^2(\mathbb{R}^3)^3} + \sigma \omega \left\| \mathbf{E} - \sum_{j=0}^M \omega^j \mathbf{E}_j \right\|_{\mathcal{L}^2(\Omega_C)^3} \\ \leq C \omega \left\| \mathbf{J} - \sum_{j=0}^M \omega^j \mathbf{J}_j \right\|_{\mathcal{L}^2(\Omega_R)^3} \left\| \mathbf{E} - \sum_{j=0}^M \omega^j \mathbf{E}_j \right\|_{\mathcal{L}^2(\Omega_R)^3}.$$

Using then the Proposition 5.2, we obtain

$$(5.47) \quad \left\| \mathbf{E} - \sum_{j=0}^M \omega^j \mathbf{E}_j \right\|_{\mathcal{L}^2(\mathbb{R}^3)^3} \leq C \left\| \mathbf{J} - \sum_{j=0}^M \omega^j \mathbf{J}_j \right\|_{\mathcal{L}^2(\Omega_R)^3} \leq C' \omega^{M+1}, \\ \left\| \operatorname{curl} \left(\mathbf{E} - \sum_{j=0}^M \omega^j \mathbf{E}_j \right) \right\|_{\mathcal{L}^2(\mathbb{R}^3)^3} \leq C' \omega^{M+2}.$$

From (5.47) and (5.40) we obtain the estimates (5.44) and (5.45). $\square$

6. Low-frequency analysis of the Maxwell system. In this section, we briefly collect some results on the solutions $(\mathbf{E}^m, \mathbf{H}^m)$ of the Maxwell equations (2.1)–(2.3) which are useful while comparing the solutions of (2.1)–(2.3) and (2.4)–(2.8).

We begin with some general considerations.

Due to the compatibility condition (2.10) that we have imposed on $\mathbf{J}$, from (2.1) we also have that $\varepsilon \mathbf{E}^m$ has to verify the same orthogonality condition, namely,

$$(6.48) \quad \varepsilon \mathbf{E}^m|_{\Omega_e} \perp \operatorname{grad} \mathcal{H}_0^1(\Omega_e) \stackrel{\perp}{\oplus} D_{\text{ext}}^\varepsilon$$

or equivalently

$$(6.49) \quad \operatorname{div}(\varepsilon \mathbf{E}^m)|_{\Omega_e} = 0 \quad \text{and} \quad \int_{\Gamma_i} \varepsilon \mathbf{E}^m \cdot \mathbf{n} = 0 \quad \forall i = 1, \dots, p.$$

It is now classical to prove the following results on the existence and uniqueness for the Maxwell system. The material coefficients ε, μ , and σ may only be assumed bounded and measurable, i.e., jumps are allowed [22], [2].

PROPOSITION 6.1. *The Maxwell system (2.1)–(2.3) has a unique solution.*

We pass now to a brief presentation of some results concerning the low-frequency asymptotic analysis of the Maxwell system (2.1)–(2.3).

According to [2], [3], [4] we know that the same asymptotic analysis we carried out in the previous sections for the eddy currents model is valid and well posed in the case of the Maxwell model (2.1)–(2.3) also. Moreover for (2.1)–(2.3) the proofs of the same results are more technical since both the electric and the magnetic fields do not belong to $\mathcal{L}^2(\mathbb{R}^3)$. The following theorem is just a collection of the results that we need for our purposes and a complete proof of it can be easily deduced from the several situations analyzed in [2], [3], [4].

THEOREM 6.2. *Let $(\mathbf{E}^m, \mathbf{H}^m)$ be the solution of (2.1)–(2.3) and let the source current $\mathbf{J}$ verify the assumption (5.25). Both the electric and the magnetic fields can be expanded in power series with respect to ω , that is, there exists ω_0 such that for any $\omega \in (0, \omega_0)$ and $M \in \mathbb{N}$ the following estimates hold:*

$$(6.50) \quad \left\| \mathbf{E}^m - \sum_{j=0}^M \omega^j \mathbf{E}_j^m \right\|_{\mathcal{L}^2(B_R)^3} \leq C_m \omega^{M+1},$$

$$(6.51) \quad \left\| \mathbf{H}^m - \sum_{j=0}^M \omega^j \mathbf{H}_j^m \right\|_{\mathcal{L}^2(B_R)^3} \leq C_m \omega^{M+1},$$

where the constant C_m does not depend on the frequency ω . Moreover, the terms $(\mathbf{E}_j^m, \mathbf{H}_j^m)$ are determined by

$$(6.52) \quad \begin{cases} \varepsilon \mathbf{E}_0^m|_{\Omega_e} \perp \text{grad } \mathcal{H}_0^1(\Omega_e) \oplus D_{\text{ext}}^\varepsilon, \\ \text{curl } \mathbf{E}_0^m = 0 & \text{in } \mathbb{R}^3, \\ \text{div}(\sigma \mathbf{E}_0^m + \mathbf{J}_0) = 0 & \text{in } \mathbb{R}^3, \\ \int_{\Omega_C} \sigma \mathbf{E}_0^m \cdot \text{grad } q_i = \alpha_i(\mathbf{J}_0) & \text{for } i = 1, \dots, q, \end{cases}$$

$$(6.53) \quad \begin{cases} \varepsilon \mathbf{E}_1^m|_{\Omega_e} \perp \text{grad } \mathcal{H}_0^1(\Omega_e) \oplus D_{\text{ext}}^\varepsilon, \\ \text{curl } \mu^{-1} \text{curl } \mathbf{E}_1^m = -i\sigma \mathbf{E}_0^m - i\mathbf{J}_0 & \text{in } \mathbb{R}^3, \\ \text{div}(i\varepsilon \mathbf{E}_0^m + \sigma \mathbf{E}_1^m + \mathbf{J}_1) = 0 & \text{in } \mathbb{R}^3, \\ \int_{\Omega_C} \sigma \mathbf{E}_1^m \cdot \text{grad } q_i = \alpha_i(\mathbf{J}_1) - i\alpha_i(\varepsilon \mathbf{E}_0^m) & \text{for } i = 1, \dots, q, \end{cases}$$

and more for $j \geq 2$

$$(6.54) \quad \begin{cases} \varepsilon \mathbf{E}_{j+1}^m|_{\Omega_e} \perp \text{grad } \mathcal{H}_0^1(\Omega_e) \oplus D_{\text{ext}}^\varepsilon, \\ \text{curl } \mu^{-1} \text{curl } \mathbf{E}_{j+1}^m = \varepsilon \mathbf{E}_{j-1}^m - i\sigma \mathbf{E}_j^m - i\mathbf{J}_j & \text{in } \mathbb{R}^3, \\ \text{div}(i\varepsilon \mathbf{E}_j^m + \sigma \mathbf{E}_{j+1}^m + \mathbf{J}_{j+1}) = 0 & \text{in } \mathbb{R}^3, \\ \int_{\Omega_C} \sigma \mathbf{E}_{j+1}^m \cdot \text{grad } q_i = \alpha_i(\mathbf{J}_{j+1}) - i\alpha_i(\varepsilon \mathbf{E}_j^m) & \text{for } i = 1, \dots, q. \end{cases}$$

Finally, the terms $(\mathbf{H}_j^m)_{j \geq 0}$ in the asymptotic expansion of the magnetic field $\mathbf{H}^m$ are determined by

$$(6.55) \quad \mathbf{H}_j^m = i\mu^{-1} \text{curl } \mathbf{E}_{j+1}^m \quad \text{in } \mathbb{R}^3 \quad \forall j \geq 0.$$

A complete characterization of all these coefficients should involve some radiation condition or exact external operators as detailed in [2]. Further, from (6.52)–(6.53) and by applying Proposition 3.1 the following Lemma holds.

LEMMA 6.3. *The vector fields $\mathbf{E}_0^m$ and $\mathbf{E}_1^m$ satisfy*

$$(6.56) \quad \begin{cases} \Delta \mathbf{E}_0^m = 0 & \text{in } \mathbb{R}^3 \setminus \overline{B_R}, \\ \text{div } \mathbf{E}_0^m = 0 & \text{in } \mathbb{R}^3 \setminus \overline{B_R}, \end{cases} \quad \begin{cases} \Delta \mathbf{E}_1^m = 0 & \text{in } \mathbb{R}^3 \setminus \overline{B_R}, \\ \text{div } \mathbf{E}_1^m = 0 & \text{in } \mathbb{R}^3 \setminus \overline{B_R}, \end{cases}$$

and then they are in $\mathcal{L}^2(\mathbb{R}^3)^3$.

7. Approximation of the Maxwell system by the eddy currents model.

In this section, we shall compare the asymptotic expansions of the solutions of (2.1)–(2.3) and (2.4)–(2.8). The main idea is that $(\mathbf{E}^m, \mathbf{H}^m)$ can be approximated by $(\mathbf{E}, \mathbf{H})$ as the frequency goes to zero if and only if the first few terms in their asymptotic expansions are equal.

We start then comparing the terms $\{\mathbf{E}_j^m, \mathbf{H}_j^m\}_{j \geq 0}$ and $\{\mathbf{E}_j, \mathbf{H}_j\}_{j \geq 0}$.

• **j = 0.** Comparing the system (5.31)–(5.34) and (6.52) and recalling Lemma 6.3 we have

$$(7.57) \quad \mathbf{E}_0^m = \mathbf{E}_0 \quad \text{in } \mathbb{R}^3.$$

Moreover, if we assume that

$$(7.58) \quad \operatorname{div}(\mathbf{J}_0)|_{\Omega_C} = 0 \quad \text{and} \quad \alpha_i(\mathbf{J}_0) = 0 \quad \forall i = 1, \dots, q,$$

we arrive at $\mathbf{E}_0^m = \mathbf{E}_0 \equiv 0$. In the following, it will be clear that (7.58) is a key assumption in order to show the validity of (2.4)–(2.8) as a low-frequency approximation of the full Maxwell system (2.1)–(2.3).

• **j = 1.** Comparing (5.35)–(5.38), evaluated for $j = 1$, and (6.53), we find that $\mathbf{E}_1^m - \mathbf{E}_1$ satisfies

$$(7.59) \quad \begin{cases} \varepsilon(\mathbf{E}_1^m - \mathbf{E}_1)|_{\Omega_e} \perp \operatorname{grad} \mathcal{H}_0^1(\Omega_e) \oplus D_{\text{ext}}^\varepsilon, \\ \operatorname{curl}(\mathbf{E}_1^m - \mathbf{E}_1) = 0 & \text{in } \mathbb{R}^3, \\ \operatorname{div}(\sigma(\mathbf{E}_1^m - \mathbf{E}_1)) = -i \operatorname{div}(\varepsilon \mathbf{E}_0^m) & \text{in } \Omega_C, \\ \int_{\Omega_C} \sigma(\mathbf{E}_1^m - \mathbf{E}_1) \cdot \operatorname{grad} q_i = -i \alpha_i(\varepsilon \mathbf{E}_0^m) & \text{for } i = 1, \dots, q. \end{cases}$$

Therefore, there exists a function $\psi \in \mathcal{H}^1(\mathbb{R}^3)$ and q real numbers $\{\beta_i\}_{i=1, \dots, q}$ such that

$$(7.60) \quad \mathbf{E}_1^m - \mathbf{E}_1 = \operatorname{grad} \psi + \sum_{i=1}^q \beta_i \operatorname{grad} q_i.$$

In particular, if $\mathbf{E}_0^m = 0$ then the right-hand side of (7.60) is equal to zero and we have

$$(7.61) \quad \text{Assumption (7.58)} \Rightarrow \mathbf{E}_0^m = \mathbf{E}_0 \equiv 0 \Rightarrow \mathbf{E}_1^m = \mathbf{E}_1.$$

Moreover, from (7.59) the following holds:

$$(7.62) \quad \mathbf{H}_0^m = i \mu^{-1} \operatorname{curl} \mathbf{E}_1^m = i \mu^{-1} \operatorname{curl} \mathbf{E}_1 = \mathbf{H}_0.$$

• **j = 2.** If the assumption (7.58) does not hold, nothing can be said about the difference $\mathbf{E}_2^m - \mathbf{E}_2$ since the function $\mathbf{E}_2^m$ is not completely determined by (6.54) evaluated for $j = 1$ (see Lemma 6.3).

Otherwise, if (7.58) is fulfilled, the same argument applied to $\mathbf{E}_0^m$ and $\mathbf{E}_1^m$ in Lemma 6.3 works for $\mathbf{E}_2^m$, which is completely determined by the system (6.54) with $j = 1$. As in the case $j = 1$, the difference $\mathbf{E}_2^m - \mathbf{E}_2$ solves the system

$$(7.63) \quad \begin{cases} \varepsilon(\mathbf{E}_2^m - \mathbf{E}_2)|_{\Omega_e} \perp \operatorname{grad} \mathcal{H}_0^1(\Omega_e) \oplus D_{\text{ext}}^\varepsilon, \\ \operatorname{curl}(\mathbf{E}_2^m - \mathbf{E}_2) = 0 & \text{in } \mathbb{R}^3, \\ \operatorname{div}(\sigma(\mathbf{E}_2^m - \mathbf{E}_2)) = -i \operatorname{div}(\varepsilon \mathbf{E}_1^m) & \text{in } \Omega_C, \\ \int_{\Omega_C} \sigma(\mathbf{E}_2^m - \mathbf{E}_2) \cdot \operatorname{grad} q_i = -i \alpha_i(\varepsilon \mathbf{E}_1^m) & i = 1, \dots, q. \end{cases}$$

In particular, $\mathbf{E}_2^m \neq \mathbf{E}_2$, but we have

$$(7.64) \quad \mathbf{H}_1^m = i\mu^{-1} \operatorname{curl} \mathbf{E}_2^m = i\mu^{-1} \operatorname{curl} \mathbf{E}_2 = \mathbf{H}_1.$$

Now summarizing all these results, our concluding remarks are the following:

- If the assumption (7.58) is fulfilled, (2.4)–(2.8) is a good approximation of (2.1)–(2.3) at low frequencies; we use (7.57), (7.61), (7.62), (7.64), (5.44)–(5.45), and (6.50)–(6.51) in order to obtain the following inequalities:

$$(7.65) \quad \begin{aligned} \|\mathbf{E}^m - \mathbf{E}\|_{\mathcal{L}^2(\Omega_R)^3} &\leq (C + C_m)\omega^2, \\ \|\mathbf{H}^m - \mathbf{H}\|_{\mathcal{L}^2(\Omega_R)^3} &\leq (C + C_m)\omega^2. \end{aligned}$$

Furthermore, roughly speaking, one can also see how the equations (2.4)–(2.8) are deduced from (2.1)–(2.3). Let $\mathbf{E}^m = \omega \mathbf{E}_1 + O(\omega^2)$ and $\mathbf{H}^m = \mathbf{H}_0 + \omega \mathbf{H}_1 + O(\omega^2)$; using these expressions in (2.1)–(2.3), we obtain

$$(7.66) \quad \operatorname{curl} \mathbf{E}^m = i\omega\mu(\mathbf{H}_0 + \omega \mathbf{H}_1) + O(\omega^2),$$

$$(7.67) \quad \begin{aligned} \operatorname{curl} \mathbf{H}^m &= -i\omega\varepsilon(\omega \mathbf{E}_1) + \sigma(\omega \mathbf{E}_1) + \mathbf{J} + O(\omega^2) \\ &= \sigma(\omega \mathbf{E}_1) + \mathbf{J} + O(\omega^2). \end{aligned}$$

From (7.67) one sees that the term coming from the time derivative of the electric field vanishes since it is of higher order.

- If the assumption (7.58) is not fulfilled only the two asymptotic expansions are equal only at order 0. This means that the inequalities (7.65) are replaced by

$$(7.68) \quad \begin{aligned} \|\mathbf{E}^m - \mathbf{E}\|_{\mathcal{L}^2(\Omega_R)^3} &\leq (C + C_m)\omega, \\ \|\mathbf{H}^m - \mathbf{H}\|_{\mathcal{L}^2(\Omega_R)^3} &\leq (C + C_m)\omega. \end{aligned}$$

Moreover, it is important to underline that in this case it turns out to be useless to solve the problem (2.4)–(2.8) in place of the complete static system

$$(7.69) \quad \begin{cases} \mathbf{E}_0 \perp \operatorname{grad} \mathcal{H}_0^1(\Omega_e) \oplus D_{\text{ext}}^\varepsilon, \\ \operatorname{curl} \mathbf{E}_0 = 0 & \text{in } \mathbb{R}^3, \\ \operatorname{curl} \mathbf{H}_0 = \sigma \mathbf{E}_0 + \mathbf{J}_0 & \text{in } \mathbb{R}^3, \\ \operatorname{div}(\mu \mathbf{H}) = 0 & \text{in } \mathbb{R}^3. \end{cases}$$

The well-posedness of this problem is straightforward from section 3 and has already been studied by Ramm et al. in [22].

8. The time-dependent case. This section is concerned with the extension of the same approximation results obtained in the previous sections to the time-dependent version of the equations (2.1)–(2.3) and (2.4)–(2.7). Namely, the system of time dependent Maxwell equations reads as follows:

$$(8.70) \quad \operatorname{curl} \mathbf{H}^m = \varepsilon \frac{\partial \mathbf{E}^m}{\partial t} + \sigma \mathbf{E}^m + \mathbf{J} \quad \text{in } \mathbb{R}^3,$$

$$(8.71) \quad \operatorname{curl} \mathbf{E}^m = \mu \frac{\partial \mathbf{H}^m}{\partial t} \quad \text{in } \mathbb{R}^3,$$

with the initial conditions $\mathbf{E}^m|_{t=0} = 0$, $\mathbf{H}^m|_{t=0} = 0$, and $\mathbf{J}|_{t=0} = 0$.

The eddy currents approximation is obtained by neglecting the term $\varepsilon \frac{\partial \mathbf{E}^m}{\partial t}$ in (8.70).

We first introduce some notation. Let $\mathcal{F}$ be the Fourier transform operator with respect to the time variable t and $\hat{\mathbf{J}}(\cdot, \mathbf{x}) = \mathcal{F}(\mathbf{J}(\cdot, \mathbf{x}))$. In order to shorten our notation, from now on, we shall write $\mathbf{J}(\cdot)$, $\hat{\mathbf{J}}(\cdot)$ instead of $\mathbf{J}(\cdot, \mathbf{x})$ and $\hat{\mathbf{J}}(\cdot, \mathbf{x})$, respectively; the norms and integrals are referred to the time variable or the frequency variable ω , which is the dual variable.

Next, we make the following general assumptions:

- The source current $\mathbf{J}$ satisfies (2.10) and (5.25) $\forall t \geq 0$.
- $\mathbf{J} \in \mathcal{H}^1(\mathbb{R}, \mathcal{L}^2(\mathbb{R}^3))^3$.
- The Fourier transform $\hat{\mathbf{J}}$ of $\mathbf{J}$ is compactly supported. There exists $\bar{\omega}$ such that $\text{supp}\{\hat{\mathbf{J}}\} \subset]-\bar{\omega}, \bar{\omega}[$.
- $\bar{\omega} < \omega_0$, where ω_0 is such that the low-frequency estimates (5.44)–(5.45) and (6.50)–(6.51) hold for any $0 < \omega < \omega_0$.
- The time-dependent Maxwell system (8.70)–(8.71) and the corresponding time-dependent eddy currents model, with the initial conditions $\mathbf{E}^m|_{t=0} = 0$, $\mathbf{H}^m|_{t=0} = 0$, and $\mathbf{J}|_{t=0} = 0$, have a unique solution, denoted by $(\mathbf{E}^m(t), \mathbf{H}^m(t))$ and $(\mathbf{E}(t), \mathbf{H}(t))$, respectively.

The main result of this section is the following theorem.

THEOREM 8.1. *There exists a constant C such that the following estimate holds:*

$$(8.72) \quad \|\mathbf{E}^m(t) - \mathbf{E}(t)\|_{\mathcal{L}^2(0,T,\mathcal{H}(\text{curl},B_R))} \leq C \bar{\omega}^2,$$

$$(8.73) \quad \|\mathbf{H}^m(t) - \mathbf{H}(t)\|_{\mathcal{L}^2(0,T,\mathcal{H}(\text{curl},B_R))} \leq C \bar{\omega}^2.$$

Proof. We only focus our attention on (8.72). The proof of (8.73) follows from the same arguments.

We first state a useful estimate. For $\mathbf{u} = \mathbf{E}$ and $\mathbf{u} = \mathbf{E}^m$ the following stability estimate holds:

$$(8.74) \quad \|\mathbf{u}\|_{\mathcal{L}^2(\Omega_C)}^2 + \int_0^T \|\mathbf{u}\|_{\mathcal{L}^2(\Omega_R)}^2 + \int_0^T \|\text{curl} \mathbf{u}\|_{\mathcal{L}^2(B_R)}^2 \leq C \int_0^T \left\| \frac{\partial \mathbf{J}}{\partial t} \right\|_{\mathcal{L}^2(B_R)}^2,$$

where the constant C is independent of T . For the Maxwell system this estimate is standard, while for the eddy currents model it can be obtained from (5.41).

Now the main idea is to approximate the nonperiodic (with respect to the variable t) density current $\mathbf{J}$ by a sequence $\mathbf{J}_n$ of source density currents where each $\mathbf{J}_n$ is a superposition of periodic modes ($e^{i\omega_1 t}$). Therefore, thanks to the linearity of the equations, we know that the solution of the Maxwell and the eddy currents system are also superpositions of periodic modes and, for these fields, the theory developed in the previous sections can be applied.

Our aim is now to approximate $\hat{\mathbf{J}}$ by a sequence $\{\hat{\mathbf{J}}_n\}_{n \in \mathbb{N}}$ in which every element $\hat{\mathbf{J}}_n$ is a sum of Dirac measures. The inverse Fourier transforms $\mathbf{J}_n$ of $\hat{\mathbf{J}}_n$ are then superpositions of periodic modes.

Let the function $\hat{\mathbf{J}}^p$ be defined by

$$(8.75) \quad \hat{\mathbf{J}}^p(\omega) := \int_{-\bar{\omega}}^{\omega} \hat{\mathbf{J}}(s) ds.$$

We decompose the interval $[-\bar{\omega}, \bar{\omega}]$ into n subintervals of the same size $2\bar{\omega}/n$ and we denote by ω_i , $i = 0, \dots, n$, the nodes of this grid and by χ_i the characteristic

function of the interval $[\omega_i, \omega_{i+1}]$, $i = 0, \dots, n-1$. Then we define a piecewise constant approximation of $\hat{\mathbf{J}}^p$ by

$$(8.76) \quad \hat{\mathbf{J}}_n^p(\omega) := \sum_{i=0}^{n-1} \hat{\mathbf{J}}^p(\omega_i) \chi_i.$$

It is standard to prove that

$$(8.77) \quad \|\hat{\mathbf{J}}^p - \hat{\mathbf{J}}_n^p\|_{\mathcal{L}^2(\mathbb{R})} \leq \frac{2C\bar{\omega}}{n} \|\hat{\mathbf{J}}^p\|_{\mathcal{H}^1(\bar{\omega}, \bar{\omega})} \leq \frac{2C'\bar{\omega}}{n} \|\hat{\mathbf{J}}\|_{\mathcal{L}^2(\mathbb{R})},$$

where the constants C and C' depend neither on n nor on $\bar{\omega}$.

By taking the derivative of (8.76) in the sense of distributions, we construct a sequence $\hat{\mathbf{J}}_n$ approximating $\hat{\mathbf{J}}$,

$$(8.78) \quad \hat{\mathbf{J}}_n(\omega) := \sum_{i=1}^n (\hat{\mathbf{J}}^p(\omega_i) - \hat{\mathbf{J}}^p(\omega_{i-1})) \delta(\omega - \omega_i) = \sum_{i=1}^n \left[\int_{\omega_{i-1}}^{\omega_i} \hat{\mathbf{J}}(s) ds \right] \delta(\omega - \omega_i),$$

where $\delta(\omega - \omega_i)$ denotes the Dirac measure centered in ω_i .

Applying $\mathcal{F}^{-1}$ to both sides of (8.78), we obtain

$$(8.79) \quad \mathbf{J}_n(t) = \sum_{l=1}^n \mathbf{J}_n(l) e^{i\omega_l t} \quad \text{with} \quad \mathbf{J}_n(l) := \int_{\omega_{l-1}}^{\omega_l} \hat{\mathbf{J}}(s) ds.$$

Using (8.79), we easily deduce that

$$(8.80) \quad \frac{\partial^s \mathbf{J}}{\partial t^s} - \frac{\partial^s \mathbf{J}_n}{\partial t^s} = \int_{-\bar{\omega}}^{\bar{\omega}} \hat{\mathbf{J}}(\xi) \left[(i\xi)^s e^{i\xi t} - (i\chi(\xi))^s e^{i\chi(\xi)t} \right] d\xi \quad \text{for } s = 0, 1,$$

where $\chi(\xi) = \omega_i$ for $\xi \in]\omega_{i-1}, \omega_i[$, $i = 1, \dots, n$.

From (8.80), we obtain

$$(8.81) \quad \|\mathbf{J} - \mathbf{J}_n\|_{\mathcal{H}^1([0, T])} \leq \frac{C(T, \|\mathbf{J}\|_{\mathcal{L}^2(\mathbb{R})}, \bar{\omega})}{n},$$

where the quantity $C(T, \|\mathbf{J}\|_{\mathcal{L}^2(\mathbb{R})})$ depends linearly on $\|\mathbf{J}\|_{\mathcal{L}^2(\mathbb{R})}$.

Now we solve the system of Maxwell's equations (8.70)–(8.71) and the corresponding eddy currents model when the data $\mathbf{J}$ is replaced by $\mathbf{J}_n$ defined by (8.79). We denote by $(\mathbf{E}^m(n), \mathbf{H}^m(n))$ and $(\mathbf{E}(n), \mathbf{H}(n))$ the corresponding solutions. We focus our attention on the electric fields, since the estimates on the magnetic fields can be deduced by using the same arguments. Using the stability estimate (8.74) together with the inequality (8.81), we obtain

$$(8.82) \quad \begin{aligned} & \| \mathbf{E}^m - \mathbf{E}^m(n) \|_{\mathcal{L}^2(0, T, \mathcal{H}(\text{curl}, B_R))} \\ & + \| \mathbf{E} - \mathbf{E}(n) \|_{\mathcal{L}^2(0, T, \mathcal{H}(\text{curl}, B_R))} \leq \frac{C'(T, \|\mathbf{J}\|_{\mathcal{L}^2(\mathbb{R}, \mathcal{L}^2(B_R))}, \bar{\omega})}{n}, \end{aligned}$$

where the constant C' depends linearly on $\|\mathbf{J}\|_{\mathcal{L}^2(\mathbb{R}, \mathcal{L}^2(B_R))}$.

In order to apply the results established in the previous sections we first observe that

$$(8.83) \quad \mathbf{E}^m(n) = \sum_{l=1}^n \mathbf{E}_l^m(n) e^{i\omega_l t} \quad \text{and} \quad \mathbf{E}(n) = \sum_{l=1}^n \mathbf{E}_l(n) e^{i\omega_l t},$$

where $\mathbf{E}_l^m(n)$ and $\mathbf{E}_l(n)$ are the solution of the time harmonic systems (2.1)–(2.3) and (2.4)–(2.8), respectively, with $\omega = \omega_l$.

Using the theory developed in sections 5 and 6, we know then that these quantities can be expanded in power series with respect to the frequency ω . Using the results established in section 7, under the suitable hypothesis on the load $\mathbf{J}$ (detailed in section 7) we know that for any $\bar{\omega} \in (0, \omega_0)$

$$(8.84) \quad \mathbf{E}_l^m(n) - \mathbf{E}_l(n) = O(\bar{\omega}^2) \quad \forall l = 1, \dots, n.$$

Using the estimates (8.82) and (8.84), we easily obtain the following approximation result:

$$(8.85) \quad \|\mathbf{E}(t) - \mathbf{E}^m(t)\|_{\mathcal{L}^2(0,T,\mathcal{H}(\text{curl},B_R))} \leq \frac{C'(T, \|\mathbf{J}\|_{\mathcal{L}^2(\mathbb{R},\mathcal{L}^2(B_R))}, \bar{\omega})}{n} + C\bar{\omega}^2.$$

Since the left-hand side does not depend on n , we let $n \rightarrow +\infty$ and we arrive then at the desired estimate (8.72). The proof of the theorem is now complete. $\square$

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