Eddy Current Problem: Attempt #1 PhL 4.2.14

After fumbling around here, I am ready to find someone on the web who tries to solve a problem like this one. It does seem that there will be induced currents, but I cannot really find what they look like. I gave it pretty much a 6 hour shot, and I could not solve the problem. Perhaps I am (as usual) overlooking some important facts about the situation.

You can see that I don't have any kind of "framework" here, as I now have in Appendix P. I am just flailing around here, jumping from one idea to another. I never found a solution to this problem on the web. I think I was lucky to find the thin round plate symmetric solution! I now think this kind of problem is extremely difficult, as suggested by the formal sources talking about combined interior/exterior problems. I did a lot of web scanning, see different doc.

I don't think there is anything useful below in this doc! Last reviewed may 5, 2014.

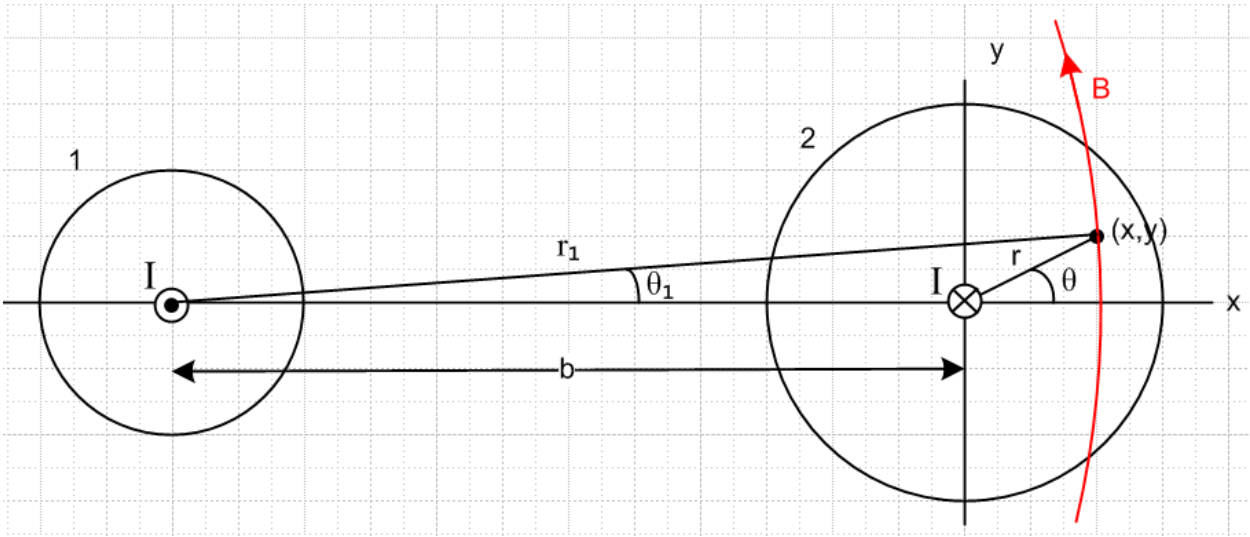
Plan A. This is a screw-up, but keep it for the record and move to Plan B below. 2
Plan B. All I do here is end up with a Helmholtz equation as in Appendix D 3
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Motivation: This is my first attempt at doing such a problem. The web is filled with references to eddy current effects which alter the current distribution in a wire causing it to be stronger in the part closest to "the other wire". This type of distribution is a natural result of my two-cylinder exact solution. I want to know whether the eddy current "method" will give the exact same result. I have this little feeling that the two methods will not agree, call it a "concern". This is the kind of thing that always happens to me.

[But I was never able to solve this problem by the "eddy current method" so I won't get any disagreement. But I do have some graphs from an eddy paper that I think are measurements.]

Scenario. Two widely spaced conductors. This allows me to approximate both conductors as having a uniform J_z , and then I will see what effect the time-changing magnetic field of the left conductor has on the J_z of the right conductor. The uniform J_z approx on the left conductor allows me to know its magnetic field, and then that will be an "externally applied field" for the right conductor.

Solution: My first step always is to "draw a picture (the picture starts simple then accumulates features as they are required)



Plan A. This is a screw-up, but keep it for the record and move to Plan B below.

The current "induced" inside the right conductor will be this:

$\mathbf{J_2}$ = curl $\mathbf{H_1}$

By the usual method, we compute H_1 due to conductor 1

$2\pi r_1 H_1 = I$ $\qquad$ $H_1 = I/(2\pi r_1)$

What is the direction of H_1 at the point shown on the right? It is mostly "up" with a small component to the "left".

$H_{1y} = H_1 \cos\theta_1$
$H_{1x} = -H_1 \sin\theta_1$

I don't yet know now precise this has to be. In order to take the curl, we need H_1 in our cylindrical coordinates on the right. How about this

$\sin\theta_1 = y/r_1$ $\qquad$ $= r\sin\theta/r_1$
$\cos\theta_1 = (x+b)/r_1$ $\qquad$ $= (r\cos\theta - b)/r_1$
$r_1^2 = y^2 + (x+b)^2$ $\qquad$ $= (r\sin\theta)^2 + (b + r\cos\theta)^2 = r^2 + b^2 + 2rb\cos\theta$ $\qquad$ // agrees w LOC

Should we compute curl in Cartesians or cylindricals? Write them both out

$H_{1y} = I/(2\pi r_1) * (x+b)/r_1 = (I/2\pi)\,(x+b)/r_1^2$ $\qquad$ (**)
$H_{1x} = -I/(2\pi r_1) * y/r_1 = -(I/2\pi)\, y/r_1^2$ $\qquad$ $r_1^2 = y^2 + (x+b)^2$

Cartesians seems simple enough. I will let Maple do the work. I have failed to put "curl" into Maple doc, so I will fix that as part of this little effort. **DONE**. So here we go:

```
> restart: with(linalg):alias(I=I,j=sqrt(-1));
Warning, new definition for norm
Warning, new definition for trace
```

$$j$$

```
> H1y := I/(2*Pi)*(x+b)/r1^2;
```

$$H1y := \frac{1}{2}\frac{I(x+b)}{\pi\, r1^2}$$

```
> H1x := -I/(2*Pi)*y/r1^2;
```

$$H1x := -\frac{1}{2}\frac{Iy}{\pi\, r1^2}$$

```
> H1 := [H1x,H1y,0];
```

$$H1 := \left[-\frac{1}{2}\frac{Iy}{\pi\, r1^2}, \frac{1}{2}\frac{I(x+b)}{\pi\, r1^2}, 0\right]$$

```
> r1 := sqrt(y^2+(x+b)^2);
```

$$r1 := \sqrt{y^2+x^2+2xb+b^2}$$

```
> c := [x,y,z];
```

$$c := [x, y, z]$$

```
> C := curl(H1,c);
```

$$C := \left[0, 0, \frac{I}{\pi\,(y^2+x^2+2xb+b^2)} - \frac{1}{2}\frac{I(x+b)(2x+2b)}{\pi\,(y^2+x^2+2xb+b^2)^2} - \frac{Iy^2}{\pi\,(y^2+x^2+2xb+b^2)^2}\right]$$

```
> C[3]; simplify(%);
```

$$\frac{I}{\pi\,(y^2+x^2+2xb+b^2)} - \frac{1}{2}\frac{I(x+b)(2x+2b)}{\pi\,(y^2+x^2+2xb+b^2)^2} - \frac{Iy^2}{\pi\,(y^2+x^2+2xb+b^2)^2}$$

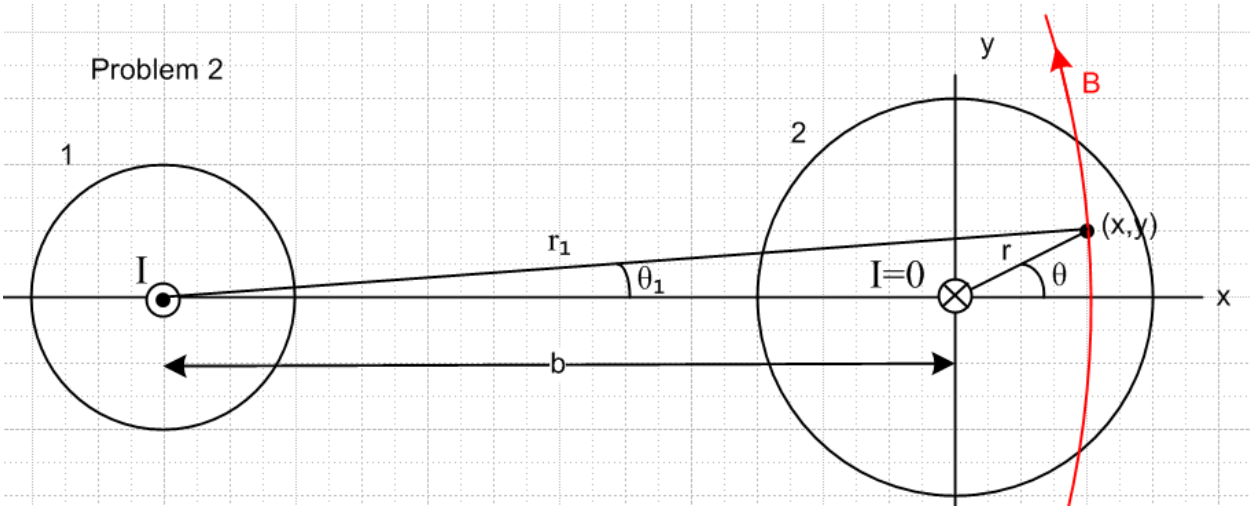
$$0$$

Oops! We get curl $\mathbf{H_1} = 0$. But that is just what one should expect if the wire on the right is missing, because in that case, there certainly is no current there. Adding the wire does not change curl $\mathbf{H_1}$

[Had this come out curl H ≠0 , I would have demonstrated a DC eddy current effect, but we know there is no such thing.]

Plan B. All I do here is end up with a Helmholtz equation as in Appendix D

Let's try a superposition concept. New picture where right has I = 0 and just sits there. So this is what I called Problem 2 in another doc:



I think the time-changing B field from the left conductor creates an E field on the right. We know that

$$\text{curl } \mathbf{H}_1 = \partial_t \mathbf{D}_2 + \mathbf{J}_2 = j\omega\varepsilon_0 \mathbf{E}_2 + \mathbf{J}_2$$

Here $\mathbf{E}_2$ and $\mathbf{J}_2$ are inside the conductor on the right, created by the time changing $\mathbf{H}_1$ field on the left. We have two unknowns, but let's use regular Ohm's Law to get

$$\text{curl } \mathbf{H}_1 = \partial_t \mathbf{D}_2 + \mathbf{J}_2 = j\omega\varepsilon_0 \mathbf{E}_2 + \sigma \mathbf{E}_2 = (j\omega\varepsilon_0 + \sigma)\mathbf{E}_2$$

But I know from Plan A that curl $\mathbf{H}_1 = 0$, so we end up with $E_2 = 0$ and then $J_2 = 0$ and then there is nothing happening on the right. This is not what I expect. I expect "eddy currents" on the right. Where did they go?

OK, here is where Eddy Currents are supposed to come from: Put a horizontal rectangular math loop in the right conductor in the x = 0 plane. This loop sees a changing flux, so we get

$$\text{curl } \mathbf{E} = -\partial_t \mathbf{B} \quad \Leftrightarrow \quad \oint_C \mathbf{E} \bullet d\mathbf{s} = -\partial_t \left[\int_S \mathbf{B} \bullet d\mathbf{S} \right] . \qquad (1.1.36)$$

and then

$$\mathcal{E}_{emf} = \oint_C \mathbf{E} \bullet d\mathbf{s} = -j\omega \int_S \mathbf{B} \bullet d\mathbf{S} \neq 0$$

Doesn't this imply there exists some $\mathbf{E} \neq 0$ in the conductor??? Perhaps it is the Lenz Law fact that this current creates its on H field $\mathbf{H}_2$ which I am ignoring? Then we would have

$$\text{curl } (\mathbf{H}_1 + \mathbf{H}_2) = \text{curl } \mathbf{H}_2 = (j\omega\varepsilon_0 + \sigma)\mathbf{E}_2 \qquad \text{[compare to P.1.9 second line]}$$

Well then I am stuck with this equation with two unknowns,

curl $\mathbf{H_2} = (j\omega\varepsilon_0+\sigma)\mathbf{E_2}$

We need more equations! Such as

curl $\mathbf{E_2} = -j\omega\mathbf{B} = -j\omega(\mathbf{B_1} + \mathbf{B_2}) = -(j\omega\mu_0)(\mathbf{H_1} + \mathbf{H_2})$ [compare to P.1.9 first line]

where $\mathbf{H_1}$ is from the left conductor, and $\mathbf{H_2}$ is that due to the induced current in the right conductor.

Take the curl of this to get

curl curl $\mathbf{E_2} = -(j\omega\mu_0)$ curl $\mathbf{H_2}$ // since curl $\mathbf{H_1} = 0$

$= -(j\omega\mu_0)(j\omega\varepsilon_0+\sigma)\mathbf{E_2}$

But this is just taking us to the Helm equation for E_2 . But let's do it

$\nabla \times (\nabla \times \mathbf{E_2}) = \nabla(\nabla \bullet \mathbf{E_2}) - \nabla^2 \mathbf{E_2} = -\nabla^2 \mathbf{E_2}$ since div $\mathbf{E_2} = 0$ we presume inside conductor

Then we have

$-\nabla^2 \mathbf{E_2} = -(j\omega\mu_0)(j\omega\varepsilon_0+\sigma)\mathbf{E_2} \approx -(j\omega\sigma\mu_0)\mathbf{E_2}$

$[\nabla^2 - (j\omega\sigma\mu_0)]\mathbf{E_2} = 0$

According to my lines (1.5.27) I should have

$(\nabla^2+\beta_2^2)\mathbf{E} = 0$ $(\nabla^2+\beta_2^2)\mathbf{B} = 0$ // region 2

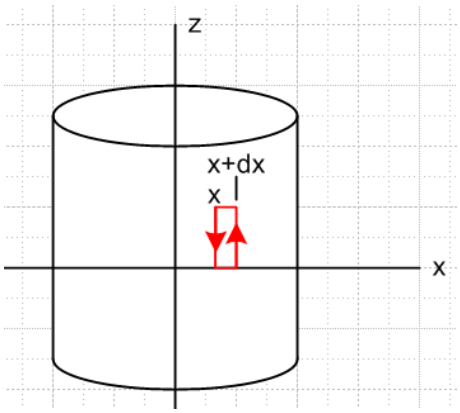
$\beta_2^2 = \omega^2\mu_0\xi_2 = \omega^2\mu_0[\varepsilon_0 - j\sigma/\omega] \approx \omega^2\mu_0[-j\sigma/\omega] = -j\mu_0\sigma\omega$

which agrees with what I just got. So now I am stuck with a **vector Helmholtz equation** to solve for E_2 in the right wire. There must be some simple approximate way to solve this problem. Otherwise I have to use all of Appendix D to write down the solution of this Helm equation.

Plan C. Try it with loops.

[I think this calculation is similar to Section P.5 Non-uniform B_{ext} , but instead of using a linear gradient as in Section P.5, I am using the actual gradient of the other wire. The *problem* is that the proximity effect and skin effect must include the self-induced eddy current in the right wire which here I ignore. So the log function I get here is then probably analogous to the $Ax + Bx^2$ result of (P.5.6)]

Consider only a horizontal loop inside the right side wire. Here is a top view of that loop:



If the induced $\mathbf{E_2}$ is independent of z, then the top and bottom segments should cancel and then

$$\mathcal{E}_{emf} = \oint_C \mathbf{E} \bullet d\mathbf{s} = -j\omega \int_S \mathbf{B} \bullet d\mathbf{S} \qquad \text{from curl } \mathbf{E} = -j\omega\mathbf{B}$$

$$\mathcal{E}_{emf} = \oint_C \mathbf{E} \bullet d\mathbf{s} = E_z(x+dx)\, h - E_z(x)h$$

The flux through the loop is [I am assuming the left wire makes H_{1y} which makes eddies in the right wire #2 where I have drawn my loop.]

$$\int_S \mathbf{B} \bullet d\mathbf{S} = H_{1y}(x,y)h\, dx \qquad \text{// I ignore induced } \mathbf{B_2} \text{ which maybe is small}$$

so then

$$E_z(x+dx)\, h - E_z(x)h = -j\omega\, H_{1y}(x,y)h\, dx$$

$$\partial_x E_z(x,y) = -j\omega\, H_{1y}(x) = -j\omega\, (I/2\pi)\, (x+b)/r_1^2 \qquad [\text{ from (**) above }]$$

Let's integrate this from x_1 to x_2 so that

$$E_z(x_2,y) - E_z(x_1,y) = \int_{x_1}^{x_2} dx\, [-j\omega\, (I/2\pi)\, (x+b)/r_1^2]$$

$$= -j\omega\, (I/2\pi) \int_{x_1}^{x_2} dx\, (x+b)/r_1^2 \qquad r_1^2 = y^2 + (x+b)^2$$

$$= -j\omega\, (I/2\pi)\, (1/2) \ln \left[\frac{y^2 + (x_2+b)^2}{y^2 + (x_1+b)^2} \right]$$

where we use:

```
Int((x+b)/(y^2+(x+b)^2),x);
```

$$\int \frac{x+b}{y^2+(x+b)^2}\,dx$$

```
value(%);
```

$$\frac{1}{2}\ln(y^2+x^2+2\,x\,b+b^2)$$

Our math loop could be inside or outside the wire, the result is the same, but a current will only flow one would think if the loop were inside the wire.

Is there some way to extract $E_z(x,y)$ inside the wire from this result?? Here is what I know

$$E_z(x,y) = E_z(x_1,y) - j\omega\,(I/2\pi)\,(1/2)\ln\left[\frac{y^2+(x+b)^2}{y^2+(x_1+b)^2}\right]$$

This also says

$$J_z(x,y) = J_z(x_1,y) - j\omega\,(I/2\pi)\,(1/2)\ln\left[\frac{y^2+(x+b)^2}{y^2+(x_1+b)^2}\right]$$

Idea: Maybe I can require this to be true

$$\int_0^a r\,dr \int_{-\pi}^{\pi} d\theta\; J_z(x,y) = 0$$

No go! There is no reason why the induced I would be 0, and probably it is not 0 !

It is 4.2.14 at 1:45PM and I am stumped, see no way forward.

Idea: Suppose ω is very low. Then maybe argue that $E_z = 0$ on the left and right edges of the right conductor. Let's just try $x_1 = 0$ and then have

$$E_z(x,y) = E_z(0,y) - j\omega\,(I/2\pi)\,(1/2)\ln\left[\frac{y^2+(x+b)^2}{y^2+b^2}\right]$$

What does this function actually look like? Let's try if for y = 0 first:

$$E_z(x,0) = E_z(0,0) - j\omega\,(I/2\pi)\,(1/2)\ln\left[\frac{(x+b)^2}{b^2}\right] \qquad \text{where } -a < x < a \text{ for conductor}$$

But this is a monotonic function so you could not make it be 0 on both edges of the right conductor:

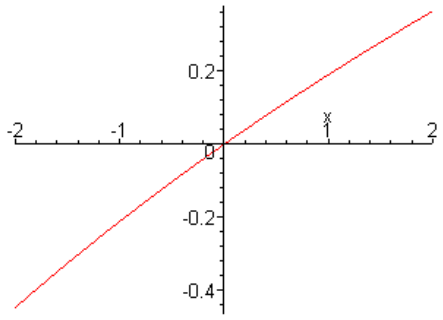
```
f := (x+b)^2/b^2;
```

$$f := \frac{(x+b)^2}{b^2}$$

```
b := 10;
```

$$b := 10$$

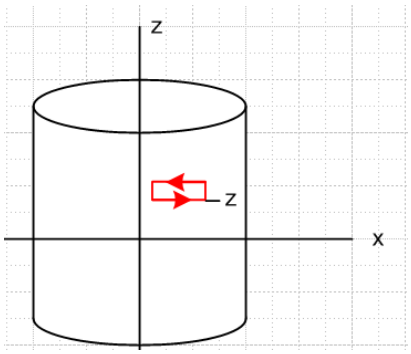
```
plot(ln(f), x=-2..2);
```



So that plan won't work.

What about div **E** = 0 inside the conductor? I already have $\partial_z E_z = 0$ and no other components, so this seems to add nothing.

What happens if you put the little loop like this:



Then

$\mathcal{E}_{emf} = \oint_C \mathbf{E} \bullet d\mathbf{s} = -j\omega \int_S \mathbf{B} \bullet d\mathbf{S}$ from curl **E** = -jω**B**

$\mathcal{E}_{emf} = \oint_C \mathbf{E} \bullet d\mathbf{s} = E_x(z)\, w - E_x(z+dz)\, w$

The flux through the loop is

$\int_S \mathbf{B} \bullet d\mathbf{S} = H_{1y}(x,y)\, w\, dz$ // I ignore induced $\mathbf{B_2}$ which maybe is small

so then

$$E_x(z)\, w - E_x(z+dz)\, w = -j\omega H_{1y}(x,y) w\, dz$$
$$\partial_z E_x(x,y) = +j\omega H_{1y}(x) = +j\omega (I/2\pi)(x+b)/r_1^2$$

This says E_x exists, and also that it depends on z, and in fact

$$E_x(x,y,z) = [j\omega (I/2\pi)(x+b)/r_1^2]\, z$$

which certainly seems strange. Maybe the problem is that I know there is a wave going down this transmission line even though the current is imbalanced, but I am here ignoring this wave

Suppose instead of our fat cylinder we had an actual wire that was the red math loop. We would then know EMF around the loop. But if it is real wire, then I is the same everywhere in the wire, and therefore J is the same, and therefore E is the same, and you could conclude that $|\mathbf{E}|$ = constant everywhere in the math loop. Is it possible to "model" the solid cylinder as a sum of math loops? I don't think so. Current would not "stay in its loop", etc. You are then again looking at Mr. Helmholtz.

****

Basically any kind of EMF-based analysis is not capable of setting a DC level anywhere for the E field. Suppose *for the moment* that I arbitrarily set $E_z = 0$ in the plane (x,y) = (0,0). So I set $x_1 = 0$ to get

$$E_z(x,y) = -j\omega (I/4\pi) \ln\left[\frac{y^2 + (x+b)^2}{y^2 + b^2}\right]$$

But this will never tell me the true current in the wire. I need something else.

But I do have something else, namely,

$$\text{curl}\ \mathbf{H_2} = (j\omega\varepsilon_0 + \sigma)\mathbf{E_1}$$

But

$$\text{curl}\ \mathbf{H} = \hat{\mathbf{x}}(\partial_y H_z - \partial_z H_y) + \hat{\mathbf{y}}(\partial_z H_x - \partial_x H_z) + \hat{\mathbf{z}}(\partial_x H_y - \partial_y H_x)$$

But nothing varies in z, so we simplify

$$\text{curl}\ \mathbf{H} = \hat{\mathbf{x}}(\partial_y H_z) + \hat{\mathbf{y}}(-\partial_x H_z) + \hat{\mathbf{z}}(\partial_x H_y - \partial_y H_x)$$

But I think $H_{2z} = 0$ almost exactly so then

$$\text{curl}\ \mathbf{H} = \hat{\mathbf{z}}(\partial_x H_y - \partial_y H_x)$$

and then

$$(j\omega\varepsilon_0+\sigma)E_{1z}(x,y) = \partial_x H_{2y}(x,y) - \partial_y H_{2x}(x,y)$$

But I know nothing about H_2 so what good does this do me?

****

BUG: If y = 0, this seems to say $E_z(x_2,y) - E_z(x_1,y) = 0$ which seems wrong. Let's back up and start off with y = 0 and see what happens:

$$E_z(x_2,0) - E_z(x_1,0) = \int_{x_1}^{x_2} dx\, [-j\omega\, (I/2\pi)\, (x+b)/(x+b^2)] = -j\omega\, (I/2\pi) \int_{x_1}^{x_2} dx\, 1/(x+b)$$

$$= -j\omega\, (I/2\pi)\, [\ln(x_2+b) - \ln(x_1+b)] = -j\omega\, (I/2\pi) \ln\left(\frac{x_2+b}{x_1+b}\right) \neq 0 !$$

where we used

```
Int(1/(x+b),x);
```

$$\int \frac{1}{x+b}\, dx$$

```
value(%);
```

$$\ln(x+b)$$

****

$$= -j\omega\, (I/2\pi)\, [\theta_2 - \theta_1]$$

I show on scratch that

$$\tan(\theta_2-\theta_1) = y\, \frac{x_1-x_2}{(x_1+b)(x_2+b)+y^2}$$

$$\theta_2-\theta_1 = \tan^{-1}\left[y\, \frac{x_1-x_2}{(x_1+b)(x_2+b)+y^2}\right]$$

and then

$$E_z(x_2,y) - E_z(x_1,y) = -j\omega (I/2\pi) \tan^{-1}\left[y \frac{x_1-x_2}{(x_1+b)(x_2+b)+y^2}\right]$$

So at least this is something. But if y = 0, why does it give 0 ?

where we use

```
Int((x+b)/(y^2+(x+b)^2),y);
```

$$\int \frac{x+b}{y^2+(x+b)^2}\,dy$$

```
value(%): simplify(%);
```

$$\arctan\left(\frac{y}{x+b}\right)$$

From Appendix A below we integrate to get

$$E_z(x,y) = j\omega k \tan^{-1}[y/(x+b)] + C_2(y)$$

**********Appendix A**********

curl **E** = - jω**B**

How do you solve this for the E field? We know B from above,

$B_{1y} = (\mu_0 I/2\pi)\,(x+b)/r_1^2$

$B_{1x} = -(\mu_0 I/2\pi)\,y/r_1^2$ $\qquad\qquad$ $r_1^2 = y^2 + (x+b)^2$

$B_{1z} = = 0$

curl $\mathbf{E} = \hat{\mathbf{x}}\,(\partial_y E_z - \partial_z E_y) + \hat{\mathbf{y}}\,(\partial_z E_x - \partial_x E_z) + \hat{\mathbf{z}}\,(\partial_x E_y - \partial_y E_x)$

But nothing varies in z, so we simplify

curl $\mathbf{E} = \hat{\mathbf{x}}\,(\partial_y E_z) + \hat{\mathbf{y}}\,(-\partial_x E_z) + \hat{\mathbf{z}}\,(\partial_x E_y - \partial_y E_x)$

We then have to solve these three equations:

$\partial_y E_z = j\omega k y / r_1^2$	x	$k \equiv (\mu_0 I/2\pi)$
$\partial_x E_z = j\omega k\,(x+b)/r_1^2$	y	
$\partial_x E_y - \partial_y E_x = 0$	z	

Let's just see if Maple can hit a homer on this problem. But no, this is a PDE and I don't want Maple to try that. I will do it first myself. Integrate the first equation

$$E_z(x,y) = j\omega k \int dy\; y / (y^2 + (x+b)^2) + C_1(x) = (j\omega k/2)\ln(y^2 + (x+b)^2) + C_1(x)$$

Now integrate the second,

$$E_z(x,y) = j\omega k \int dx\; (x+b) / (y^2 + (x+b)^2) + C_2(y) = j\omega k \tan^{-1}(y/(x+b)) + C_2(y)$$

Here are these two integrals from Maple:

```
Int(y/(y^2+(x+b)^2),y);
```

$$\int \frac{y}{y^2 + (x+b)^2}\,dy$$

```
value(%);
```

$$\frac{1}{2}\ln(y^2 + x^2 + 2xb + b^2)$$

```
Int((x+b)/(y^2+(x+b)^2),y);
```

$$\int \frac{x+b}{y^2 + (x+b)^2}\,dy$$

```
value(%): simplify(%);
```

$$\arctan\left(\frac{y}{x+b}\right)$$

So far then we know that

$$(j\omega k/2)\ln(y^2 + (x+b)^2) + C_1(x) = j\omega k \tan^{-1}(y/(x+b)) + C_2(y)$$

or let $D_i = j\omega k C_i$ so then we know [this certainly looks fishy!]

$$(1/2)\ln(y^2 + (x+b)^2) + D_1(x) = \tan^{-1}(y/(x+b)) + D_2(y)$$

Now what does our third equation say?

$$\partial_{\mathbf{x}} E_{\mathbf{y}} - \partial_{\mathbf{y}} E_{\mathbf{x}} = 0$$