Perturbation Attempt PhL 5.2.14

This was an early precursor to Section P.1's perturbation theory approach. Here I have a formal dimensionless α, but there I just fudge the smallness parameter and say it is ω because I don't want to have to worry about some ω_1 though I could no doubt do it using skin effect δ to set the ω scale.

We now imagine a scenario where some "external" B field $\mathbf{B_{ext}}$ is created by some external current source $\mathbf{J_{ext}}$. A system of interest (call it **DUT** for Device Under Test), is immersed in this external field $\mathbf{B_{ext}}$ and this time-changing field "induces" currents $\mathbf{J_{eddy}}$ in the DUT. With this partitioning, we write the previous two equations as

$$\text{curl}\,(\mathbf{B_{ext}} + \mathbf{B_{eddy}}) = \mu(\mathbf{J_{ext}} + \mathbf{J_{eddy}}) \qquad \text{(P.1.3)}$$
$$\text{curl}\,(\mathbf{J_{ext}} + \mathbf{J_{eddy}}) = -\,j\omega\sigma(\mathbf{B_{ext}} + \mathbf{B_{eddy}}) \quad . \qquad \text{(P.1.4)}$$

We assume next that $\mathbf{J_{eddy}}$ in the DUT exists in a region of space which is disjoint from the region where $\mathbf{J_{ext}}$ flows in the external generating apparatus. In the DUT region, $\mathbf{J_{ext}} = 0$, and in the generating region we have $\mathbf{J_{eddy}} = 0$, following our definitions of $\mathbf{J_{ext}}$ and $\mathbf{J_{eddy}}$. Any eddy currents which the generating system induces into itself we assume are included in $\mathbf{J_{ext}}$. Then we can write (P.1.3) and (P.1.4) as

$$\text{curl}\,(\mathbf{B_{ext}} + \mathbf{B_{eddy}}) = \mu\mathbf{J_{eddy}} \qquad \text{// within the DUT} \qquad \text{(P.1.3)'}$$
$$\text{curl}\,\mathbf{J_{eddy}} = -\,j\omega\sigma(\mathbf{B_{ext}} + \mathbf{B_{eddy}}) \quad . \qquad \text{// within the DUT} \qquad \text{(P.1.4)'}$$

In principle, $\mathbf{B_{eddy}}$ could alter $\mathbf{J_{ext}}$ in the external apparatus, but we shall assume that $\mathbf{J_{ext}}$ is regulated by some kind of current source and so $\mathbf{J_{ext}}$ does not change when the DUT is put in place. Correspondingly, we assume that $\mathbf{B_{ext}}$ does not change either. Assuming also that the DUT and the region of space into which it is inserted both have permeability μ, we have the following equation applying both before and after the DUT is put in place:

$$\text{curl}\,\mathbf{B_{ext}} = \mu\mathbf{J_{ext}} = 0 \qquad \text{// within the DUT}$$

Thus we once more rewrite our pair of equations using this fact,

$$\text{curl}\,\mathbf{B_{eddy}} = \mu\mathbf{J_{eddy}} \qquad \text{// within the DUT} \qquad \text{(P.1.3)''}$$
$$\text{curl}\,\mathbf{J_{eddy}} = -\,j\omega\sigma(\mathbf{B_{ext}} + \mathbf{B_{eddy}}) \quad . \qquad \text{// within the DUT} \qquad \text{(P.1.4)''}$$

We now define ω_0 to be some frequency, and $\alpha \equiv \omega/\omega_0$ is a dimensionless parameter which is small at low frequency. The above equations are then

$$\text{curl}\,\mathbf{B_{eddy}} = \mu\mathbf{J_{eddy}} \qquad \text{// within the DUT}$$
$$\text{curl}\,\mathbf{J_{eddy}} = -\,j\omega_0\sigma\,\alpha\,(\mathbf{B_{ext}} + \mathbf{B_{eddy}}). \qquad \text{// within the DUT}$$

We next expand $\mathbf{B_{eddy}}$ and $\mathbf{J_{eddy}}$ in this smallness parameter α,

$$\mathbf{B_{eddy}} = \alpha\,\mathbf{B_{eddy}}^{(1)} + \alpha^2\,\mathbf{B_{eddy}}^{(2)} + \ldots\ldots = \Sigma_{n=1}^{\infty}\,\alpha^n\,\mathbf{B_{eddy}}^{(n)}$$

$$\mathbf{J_{eddy}} = \alpha\, \mathbf{J_{eddy}}^{(1)} + \alpha^2\, \mathbf{J_{eddy}}^{(2)} + \ldots\ldots = \Sigma_{n=1}^{\infty}\, \alpha^n\, \mathbf{J_{eddy}}^{(n)}$$

Inserting these expansions into (P.1.3)" and (P.1.4)" then gives

$$\text{curl}\ (\alpha\, \mathbf{B_{eddy}}^{(1)} + \alpha^2\, \mathbf{B_{eddy}}^{(2)} + \ldots\ldots) = \mu(\alpha\, \mathbf{J_{eddy}}^{(1)} + \alpha^2\, \mathbf{J_{eddy}}^{(2)} + \ldots\ldots)$$

$$\text{curl}\ (\alpha\, \mathbf{J_{eddy}}^{(1)} + \alpha^2\, \mathbf{J_{eddy}}^{(2)} + \ldots\ldots) = -j\omega_0\sigma\, \alpha\, (\mathbf{B_{ext}} + \alpha\, \mathbf{B_{eddy}}^{(1)} + \alpha^2\, \mathbf{B_{eddy}}^{(2)} + \ldots\ldots)$$

If we now equate the coefficients of α in the two equations, and alternate back and forth between the equations, we obtain this sequence:

$$\text{curl}\ \mathbf{J_{eddy}}^{(1)} = -j\omega_0\sigma\, \mathbf{B_{ext}}$$

$$\text{curl}\ \mathbf{B_{eddy}}^{(1)} = \mu\, \mathbf{J_{eddy}}^{(1)}$$

$$\text{curl}\ \mathbf{J_{eddy}}^{(2)} = -j\omega_0\sigma\, \mathbf{B_{eddy}}^{(1)}$$

$$\text{curl}\ \mathbf{B_{eddy}}^{(2)} = \mu\, \mathbf{J_{eddy}}^{(2)}$$

etc.

$$\text{curl}\ (\mathbf{B_{eddy}}^{(0)} + \alpha\, \mathbf{B_{eddy}}^{(1)} + \alpha^2\, \mathbf{B_{eddy}}^{(2)} + \ldots.) = \mu(\mathbf{J_{eddy}}^{(0)} + \alpha\, \mathbf{J_{eddy}}^{(1)} + \alpha^2\, \mathbf{J_{eddy}}^{(2)} + \ldots\ldots)$$

$$\text{curl}\ (\mathbf{J_{eddy}}^{(0)} + \alpha\, \mathbf{J_{eddy}}^{(1)} + \alpha^2\, \mathbf{J_{eddy}}^{(2)} + \ldots\ldots) = -j\omega\sigma(\mathbf{B_{ext}} + \mathbf{B_{eddy}}^{(0)} + \alpha\, \mathbf{B_{eddy}}^{(1)} + \alpha^2\, \mathbf{B_{eddy}}^{(2)} + \ldots\ldots)$$

We can then regard each of these equations as a set of equations in each order of α :

$0 = 0$ order α^0

$\text{curl}\ \mathbf{B_{eddy}}^{(1)} = \mu\, \mathbf{J_{eddy}}^{(1)}$ order α^1

$\text{curl}\ \mathbf{B_{eddy}}^{(2)} = \mu\, \mathbf{J_{eddy}}^{(2)}$ order α^2

....

$0 = -j\omega\sigma\mathbf{B_{ext}}$ order α^0

$\text{curl}\ \mathbf{J_{eddy}}^{(1)} = -j\omega\sigma\, \mathbf{B_{eddy}}^{(1)}$ order α^1

$\text{curl}\ \mathbf{J_{eddy}}^{(2)} = -j\omega\sigma\, \mathbf{B_{eddy}}^{(2)}$ order α^2

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