Simple eddy current problems PhL 4.3.14

This is the basis for the examples of Sections P.2 and P.3 of Appendix P. These were the first eddy current problems I ever did (and probably they will be the last). There is more detail here than in Appendix P, and I explore extra subjects here like magnetic Ohm's and electric version of Biot Savart. For the harder problem I obtained here the coupled PDE's but I never got a good solution to them directly. I eventually got a solution using the stream function method in item 3 below. I think that everything useful in this doc is not included in Appendix P. [May 5, 2014]

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1. Eddy Problem 1. Circular Plate with Uniform B field

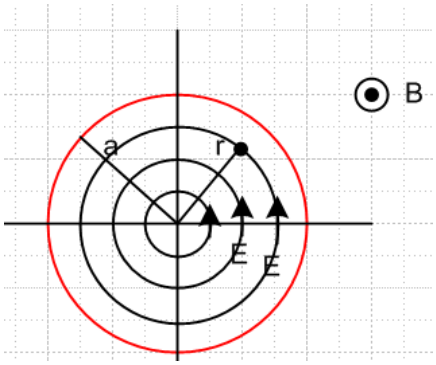
A thin round plate lies in a uniform ω external B field. Find the induced E field in the plate.

Faraday and symmetry tells us that we get circular closed electric field lines in the plate! Assume low ω so δ >> plate thickness. Then those E lines are uniform in the plate thickness.

I am not used to closed E field lines! I am used to them starting and ending on charges, but if there is no ρ inside the plate, then div E = 0 and that means the E lines are closed. We know they are there because that is what curl E = -jωB tells.

The useful new thing I learn from pondering this problem is that you have these closed lines. Remember that a field line is a direction thing, and field strength can vary along a line.

But in this simple problem, we know that field strength will be constant on each circle.



In my infinite cylinder problem, I think things were complicated by the infinite aspect since the lines have to stretch way down the line and somehow eventually close back. Here with a finite plate, we don't have such a problem.

So OK, here is Faraday:

$$\text{curl } \mathbf{E} = -\partial_t \mathbf{B} \quad \Leftrightarrow \quad \oint_C \mathbf{E} \bullet d\mathbf{s} = -\partial_t \left[\int_S \mathbf{B} \bullet d\mathbf{S} \right] \quad . \qquad (1.1.36)$$

So consider a circle at radius r. We then have

$$E_\theta * 2\pi r = -\dot{B}_z \pi r^2 \quad \Rightarrow \quad E_\theta(r) = -\dot{B}_z \pi r^2 / 2\pi r = (-\dot{B}_z / 2)\, r$$

This has a lot of similarity to our famous magnetic problem

$$H * 2\pi r = I\, \pi r^2 / \pi a^2 \quad =< \quad H(r) = I\, r^2/a^2 \,/\, 2\pi r = (I/2\pi a^2) r$$

The big similarity is a linear field growing from the center.

Notice that these E circles continue outside the plate and under and over it! It's just that there will only be current in the plate. So here are our findings:

$$\mathbf{E(r)} = E_\theta(r)\hat{\boldsymbol{\theta}} \qquad E_\theta(r) = (-\dot{B}_z / 2)\, r \qquad J_\theta(r) = \sigma\, (-\dot{B}_z / 2)\, r \quad \text{OK } \omega\tau << 1$$

So OK, how much heat is generated in this plate? Think of a thin cylinder shell of height dz and radius r and thickness dr. Think of this as a wire of rectangular cross section dA = dzdr. The current in this wire is given by,

$$dI = J_\theta \, dA = J_\theta \, dzdr$$

Just for fun, the total current going around the circle as observed through any azimuthal slice:

$$I = h \int J_\theta \, dr = h \sigma (-\dot{B}_z /2) \int r dr = h \sigma (-\dot{B}_z /4) a^2 = - (1/4) \, h \sigma a^2 \dot{B}_z$$

The resistance of the wire is $dR = \rho L/dA = \rho \, 2\pi r/(dzdr)$. The power burned from $P = I^2R$ is

$$dP = (dI)^2 dR = [J_\theta \, dzdr]^2 * \rho \, 2\pi r/(dzdr) = J_\theta^2 \, [dzdr] * \rho \, 2\pi r$$

$$= J_\theta^2 \, 2\pi r \rho \, dr \, dz$$

$$= [\sigma (-\dot{B}_z /2) \, r]^2 \; 2\pi r \rho \, dr \, dz = (\pi/2) \sigma \dot{B}_z^2 \, r^3 \, drdz$$

Now integrate over the plate thickness h to replace dz by h. Then integrate r from 0 to a to get

$$P = (\pi/2) \sigma \dot{B}_z^2 \, h(a^4/4) = (\pi/8) \sigma \dot{B}_z^2 a^4 h$$

Dimensions:

RHS = $ohm^{-1} m^{-1} \; sec^{-2} \, [volt\text{-}sec/m^2]^2 \, m^5 = ohm^{-1} \; sec^{-2} \, [volt\text{-}sec]^2 = volt^2/ohm = watts$

So THIS is the kind of "simple eddy problem" I have been looking for that no one bellies up with on the web. Notice the ω^2 factor here, as well as B^2. Metal with higher σ (more conductive) has higher power loss.

<u>Verification</u> from the two methods paper !! I had

$$J_\theta(r) = \sigma (-\dot{B}_z /2) \, r$$

$$I = - (1/4) \, h \sigma a^2 \dot{B}_z$$

$$P = (\pi/8) \sigma \dot{B}_z^2 a^4 h$$

while he has (S = area = πa^2) . All three of my results agree with his circle results!

$$j(r,t) = -\sigma\dot{B}(t)\frac{S}{2\pi}\frac{r}{r_0^2}$$
$$= -\tfrac{1}{2}\sigma\dot{B}(t)r \qquad (36)$$
$$I(t) = -\tfrac{1}{2}\sigma h\frac{S}{2\pi}\dot{B}(t)$$
$$= -\frac{1}{4}\sigma h r_0^2\dot{B}(t) \qquad (37)$$
$$P(t) = \frac{1}{4}\sigma h\frac{S^2}{2\pi}\dot{B}^2(t)$$
$$= \frac{\pi}{8}\sigma h r_0^4\dot{B}^2(t). \qquad (38)$$

More on Power in the ω-domain (but I changed the above to the time domain!) . You have to be careful with a power calculation since non-linear, just dim memories that should get cleared up right here. In the world of phasor time dependence, how does this all work? For example one might have

$I = I_0 e^{j\omega t} e^{j\alpha}$ $V = V_0 e^{j\omega t} e^{j\beta}$ $I_{phys} = I_0\cos(\omega t+\alpha)$ $V_{phys} = V_0\cos(\omega t+\beta)$

Then

$P_{phys} = I_{phys} V_{phys} = I_0 V_0 \cos(\omega t+\alpha)\cos(\omega t+\beta)$

$= I_0 V_0 \ 1/2 \ \{ \cos(\alpha-\beta) + \cos(2\omega t) \}$

$\langle P_{phys}\rangle = (1/2) I_0 V_0 \cos(\alpha-\beta)$

So there is an overall (1/2) and there is a phase factor as well.

But in my case I have $P = I^2R$ and so we are talking I*I and so $\alpha = \beta$ and there is no extra phase factor. So I think we just add the 1/2 to get $P = (1/2) I_0^2 R$ where I_0 is magnitude. Then

$\langle P\rangle = (\pi/4)\sigma\omega^2 B^2 a^4 h$

But my two theories PDF guy has a (π/8), so why is that?

Resume:

Interpretation? Think of the conduction electrons trying to do their circular motion using the Lorentz Force equation from the Hall effect.

$$mj\omega\mathbf{v} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) - (m/\tau)\mathbf{v}$$

Write this as

$\kappa\,\mathbf{v} = \mathbf{E} + \mathbf{v}\times\mathbf{B}$ $\kappa = (1+j\omega\tau)(1/\mu)$

and then

$$\kappa v_1 = E_1 + v_2B_3 - v_3B_2$$
$$\kappa v_2 = E_2 + v_3B_1 - v_1B_3$$
$$\kappa v_3 = E_3 + v_1B_2 - v_2B_1$$

In our little problem here we have B_3 only

$$\kappa v_1 = E_1 + v_2B_3$$
$$\kappa v_2 = E_2 - v_1B_3$$
$$\kappa v_3 = E_3$$

and we assume $v_3 = E_3 = 0$ so these are the only equations

$$\kappa v_1 = E_1 + v_2B_3$$
$$\kappa v_2 = E_2 - v_1B_3$$

which in cylindricals I write as

$$\kappa v_r = E_r + v_\theta B_3 \qquad 1$$
$$\kappa v_\theta = E_\theta - v_r B_3 \qquad 2$$

or

$$\kappa n q v_r = nqE_r + nqv_\theta B_3$$
$$\kappa n q v_\theta = nqE_\theta - nqv_r B_3$$

or

$$\kappa J_r = nqE_r + J_\theta B_3$$
$$\kappa J_\theta = nqE_\theta - J_r B_3$$

But we certainly need $J_r = 0$ at r = a, (also $v_r = 0$) and seems true everywhere if we get circular lines, so then our two equations become

$$0 = nqE_r + J_\theta B_3 \qquad 1 \qquad J_\theta = nqv_\theta \qquad J_r = nqv_r = 0$$
$$\kappa J_\theta = nqE_\theta \qquad 2$$

The *first* equation tells us that

$$E_r = - v_\theta B_3$$

Digression. This is forcing the current to travel in a circle I think. The Lorentz **v x B** is radial in direction and it is neutralized by this Hall-like field, so v_r then does not change, the particle has $\dot{v}_r = 0$. But that seems wrong for circular motion? Let's check on this:

$$\mathbf{v} = v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}} \qquad \dot{\hat{\mathbf{r}}} = d\hat{\mathbf{r}}/dt = d\hat{\mathbf{r}}/d\theta\, d\theta/dt = \hat{\boldsymbol{\theta}}\,\Omega \qquad \Omega \equiv d\theta/dt$$

$$\dot{\hat{\boldsymbol{\theta}}} = d\hat{\boldsymbol{\theta}}/dt = d\hat{\boldsymbol{\theta}}/d\theta\ d\theta/dt = -\hat{\mathbf{r}}\,\Omega$$

$$\dot{\mathbf{v}} = \mathbf{a} = \dot{v}_r\hat{\mathbf{r}} + v_r\,\hat{\boldsymbol{\theta}}\,\Omega + \dot{v}_\theta\hat{\boldsymbol{\theta}} - v_\theta\,\hat{\mathbf{r}}\,\Omega$$

$$= (\dot{v}_r - \Omega\, v_\theta)\,\hat{\mathbf{r}} + (\dot{v}_\theta + \Omega\, v_r)\,\hat{\boldsymbol{\theta}}$$

What is Ω for our problem? If everything is doing $e^{j\omega t}$ including angle θ, then I guess

$$\Omega = d\theta/dt = j\omega\theta$$

So then we have

$$\dot{\mathbf{v}} = \mathbf{a} = (\dot{v}_r - j\omega\theta\, v_\theta)\,\hat{\mathbf{r}} + (\dot{v}_\theta + j\omega\theta\, v_r)\,\hat{\boldsymbol{\theta}}$$

For circular motion, we have only v_θ = constant and $v_r = 0$ so this says

$$\dot{\mathbf{v}} = \mathbf{a} = (-j\omega\theta\, v_\theta)\,\hat{\mathbf{r}} + (\dot{v}_\theta)\,\hat{\boldsymbol{\theta}} = -j\omega\theta\, v_\theta\,\hat{\mathbf{r}}$$

and there is your expected acceleration toward the center! End of digression.

The *second* equation tells us that

$$\kappa J_\theta = nqE_\theta$$

or

$$J_\theta = nq\mu\, E_\theta / (1+j\omega\tau) = \sigma\, E_\theta/(1+j\omega\tau) \approx \sigma\, E_\theta$$

So this tells us that we have the regular Hall effect in the θ direction if $\omega\tau << 1$. Aside:

$$v_\theta = J_\theta/nq = \mu\, E_\theta / (1+j\omega\tau) =$$

The current is therefore

$$J_\theta(r) = \sigma\, E_\theta/(1+j\omega\tau) = \sigma\,(1/2)\,(-j\omega B)\, r\,/(1+j\omega\tau)$$

So here is my complete solution to this problem:

$v_r = 0$		
$v_\theta = \mu\, E_\theta / (1+j\omega\tau)$	$= -(1/2)\, j\omega B\, \mu\, r / (1+j\omega\tau)$	linear in r
$v_z = 0$		
$E_r = -v_\theta B$	$= (1/2)\, j\omega B^2\, \mu\, r / (1+j\omega\tau)$	linear in r
$E_\theta = (1/2)\,(-j\omega B)\, r$		
$E_z = 0$		

$J_r = 0$

$J_\theta = \sigma E_\theta/(1+j\omega\tau)$ $= -(1/2)\, j\omega B\sigma\, r/(1+j\omega\tau)$ linear in r

$J_z = 0$

You could calculate the total current going around the plate by integrating J_θ but I don't think that really means anything.

Is there a charge density inside as in my radial Hall effect?

$$\rho/\varepsilon_0 = \text{div}\,\mathbf{E} = r^{-1}\partial_r(rE_r) + r^{-1}\partial_\theta E_\theta + \partial_z E_z$$

$$= r^{-1}\partial_r(rE_r) + r^{-1}\partial_\theta E_\theta$$

$$= r^{-1}\partial_r(r[\ (1/2)j\omega B^2\ \mu\, r\,/\,(1+j\omega\tau)]) + r^{-1}(1/2)\partial_\theta[(-j\omega B)\, r]$$

$$= (1/2)j\omega B^2\ \mu/\,(1+j\omega\tau) * r^{-1}\partial_r(r^2) - (1/2)j\omega B\, r^{-1}\partial_\theta(r)$$

$$= (1/2)\, j\omega B^2\ \mu/\,(1+j\omega\tau)\, r^{-1}\, 2r - 0$$

$$= j\omega B^2\ \mu/\,(1+j\omega\tau)$$

So yes, there is a uniform inside charge density

$$\rho = \varepsilon_0\, j\omega B^2\ \mu/\,(1+j\omega\tau) \approx \varepsilon_0\, \mu\, j\omega B^2$$

and then there will be a corresponding surface charge on the surface of the plate rim.

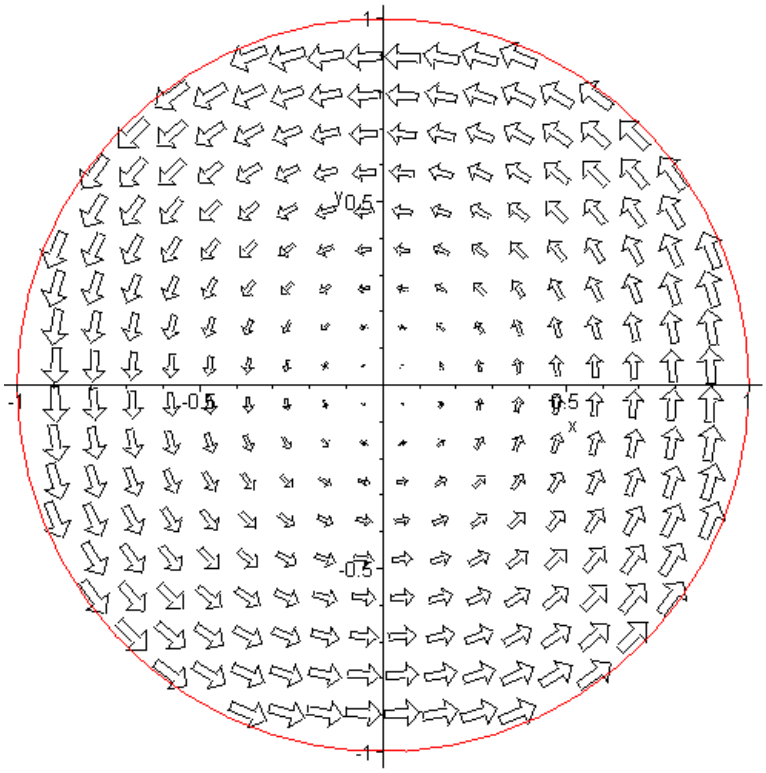
Now let's plot the current vector. We can say

$$J_x = J_\theta\hat{\boldsymbol{\theta}}\bullet\hat{\mathbf{x}} = -J_\theta\sin\theta = -J_\theta\,(y/r) = +(1/2)j\omega B\sigma\, y/(1+j\omega\tau) = ky\ \theta(r<a)$$

$$J_y = J_\theta\hat{\boldsymbol{\theta}}\bullet\hat{\mathbf{y}} = J_\theta\cos\theta = J_\theta\,(x/r) = -(1/2)j\omega B\sigma\, x/(1+j\omega\tau) = -kx\ \theta(r<a)$$

Then here is Maple:

```
restart;with(plots):with(plottools):
k := 1:
a := 1:
Jx := -k*y*Heaviside(a-r):
Jy := k*x*Heaviside(a-r):
r := sqrt(x^2+y^2):
p1 := circle([0,0],a,color=red):
p2 := fieldplot([Jx,Jy], x=-a..a, y = -a..a, scaling=constrained, arrows=thick):
display(p1,p2);
```



So there are your eddy currents in the plate! The do of course produce a Lenz's Law B field that is opposite the applied and, and we do neglect this correction to B.

2. Electric Biot-Savart ? Maybe $\nabla^2\mathbf{E} = \mathbf{0}$?

Suppose B is non uniform in Problem 1. We now lose our symmetry benefit, but I have a dim idea. Look at lines below (B.2.12) where I derive the Biot Savart Law. I compute H in terms of J. Could there be some analogous development starting with a Helm equation for E? Start back at B.2.1 and copy paste and edit:

Start with Maxwell's equation (1.1.1), and we are now working inside a conductor so **E** = 0 and then

$$\text{curl } \mathbf{E} = -j\omega\mathbf{B} \qquad \text{(B.2.1)}$$

where **B** is the magnetic field. Apply curl to both sides and use curl curl = grad div $-\nabla^2$ to get

$$\text{grad div } \mathbf{E} - \nabla^2\mathbf{E} = -j\omega \text{ curl } \mathbf{B} \,. \qquad \text{(B.2.2)}$$

But in a medium with no charge density we have div **E** = 0 so we get

$$\nabla^2\mathbf{E} = j\omega \text{ curl } \mathbf{B} \qquad \text{(B.2.3)}$$

Now jump down below B.2.12 and continue this process. We basically makes these replacements:

$H \rightarrow E$ and $-\text{curl } \mathbf{J} \rightarrow +j\omega \text{ curl } \mathbf{B}$

The very first step is then

$$\mathbf{E(x)} = j\omega \int d^3x' \left[\frac{1}{4\pi R} \right] \text{curl}' \mathbf{B}(\mathbf{x}') \qquad R \equiv |\mathbf{x}-\mathbf{x}'| \tag{B.2.13}$$

But

$$\text{curl } \mathbf{B} = \hat{\mathbf{x}} (\partial_y B_z - \partial_z B_y) + \hat{\mathbf{y}} (\partial_z B_x - \partial_x B_z) + \hat{\mathbf{z}} (\partial_x B_y - \partial_y B_x)$$

$= 0$ because the B field is uniform.

Thus right here you get $\mathbf{E(x)} = 0$ so you really have

We then sort of have

$$\nabla^2 \mathbf{E} = \mathbf{0} \tag{B.2.3}$$

as your equation of interest. For our circular plate in Cartesian this seems to say

$$\nabla^2 E_x = 0$$
$$\nabla^2 E_y = 0$$

In cylindricals this vector Helm becomes (as in Appendix D.1.12)

$$(\nabla^2 \mathbf{E})_r = \nabla^2 E_r - (2/r^2) \partial_\theta E_\theta - (1/r^2) E_r$$
$$(\nabla^2 \mathbf{E})_\theta = \nabla^2 E_\theta + (2/r^2) \partial_\theta E_r - (1/r^2) E_\theta$$

For our Problem 1 case we have

$$E_r(r) = (1/2) j\omega B^2 \mu r / (1+j\omega\tau)$$
$$E_\theta(r) = (1/2) (-j\omega B) r$$

so then the vector Helm is

$$(\nabla^2 \mathbf{E})_r = \nabla^2 E_r - (1/r^2) E_r = r^{-1} \partial_r (r \partial_r E_r) - (1/r^2) E_r$$
$$(\nabla^2 \mathbf{E})_\theta = r^{-1} \partial_r (r \partial_r E_\theta) - (1/r^2) E_\theta$$

Well, either equation has the form $E = k r$ so we get

$$(\nabla^2 \mathbf{E})_r = r^{-1} \partial_r (r \partial_r E_r) - (1/r^2) E_r = k\, r^{-1} \partial_r (r \partial_r r) - k\, (1/r^2) r$$

$$= k\, r^{-1}\partial_r(r) - k/r = k\, r^{-1} - k/r = 0$$

so YES, my circular plate solution does explicitly satisfy $\nabla^2\mathbf{E} = \mathbf{0}$. I just derived this equation above with the uniform B field only which made curl $\mathbf{B} = 0$, so this was a special case. So how does this gibe with

$$\begin{aligned}
&(\nabla^2+\beta_1^2)\mathbf{E} = (1/\varepsilon_1)\,\Sigma_{i=2}^{N+1}\text{grad}\,\rho_i && (\nabla^2+\beta_1^2)\mathbf{B} = 0 && \text{// region 1}\\
&(\nabla^2+\beta_2^2)\mathbf{E} = 0 && (\nabla^2+\beta_2^2)\mathbf{B} = 0 && \text{// region 2}\\
&(\nabla^2+\beta_3^2)\mathbf{E} = 0 && (\nabla^2+\beta_3^2)\mathbf{B} = 0 && \text{// region 3}
\end{aligned} \qquad (1.5.27)$$

where in a conductor region 2 it seems you really should have

$$(\nabla^2+\beta_2^2)\mathbf{E} = 0 \qquad \beta_2^2 = \omega^2\mu_2\xi_2 \approx -j\omega\mu\sigma$$

I guess since I ignored the Lenz induced B field, that is the same as thinking "low ω" and then $\beta_2 = 0$.

But let's continue anyway from (B.2.13) above. We then just go through all the same steps to end up with

$$\mathbf{E}(\mathbf{x}) = -j\omega \int d^3x' \, \frac{1}{4\pi R^3}\, \mathbf{B}(\mathbf{x}') \times \mathbf{R} \; . \qquad \mathbf{R} \equiv \mathbf{x} - \mathbf{x}' \qquad (B.2.18)$$

and this is the 3D Biot-Savart Analog. If E does not depend on z (only on x,y) I think you can also write down the 2D Biot-Savart Analog,

$$\mathbf{E}(x,y) = -j\omega \int d^2x' \, \frac{1}{2\pi R^2}\, \mathbf{B}(\mathbf{x}') \times \mathbf{R} \qquad \mathbf{R} \equiv \mathbf{x} - \mathbf{x}' \qquad (B.2.24)$$

Wow, I wonder if I can do the circular plate with one of these guys? There is one painful problem, however. In regular Biot-Savart, $\mathbf{J}(\mathbf{x}')$ is usually a well-defined limited "source" such as a wire or maybe a cylinder. Here we have to integrate over all space where $\mathbf{B}(\mathbf{x}')$ exists and that sounds like trouble.

For example, suppose

$\mathbf{B}(\mathbf{x}') = B\,\hat{\mathbf{z}}$. We then get

$$\mathbf{E}(x,y) = -j\omega B\,\hat{\mathbf{z}} \times \{ \int d^2x' \, \frac{1}{2\pi R^2}\, \mathbf{R} \} \qquad \mathbf{R} \equiv \mathbf{x} - \mathbf{x}' \qquad (B.2.24)$$

and this looks log divergent. Well, write it out

$$R^2 = (x-x')^2 + (y-y')^2 \qquad \mathbf{R} = (x-x')\hat{\mathbf{x}} + (y-y')\hat{\mathbf{y}}$$

$$\hat{\mathbf{z}} \times \mathbf{R} = \hat{\mathbf{z}} \times [(x-x')\hat{\mathbf{x}} + (y-y')\hat{\mathbf{y}}] = (x-x')\,\hat{\mathbf{y}} - (y-y')\,\hat{\mathbf{x}}$$

$$\mathbf{E}(x,y) = -j\omega B \frac{1}{2\pi} \int d^2x' \frac{1}{(x-x')^2 + (y-y')^2} [(x-x')\hat{y} - (y-y')\hat{x}]$$

$$= -j\omega B \frac{1}{2\pi} \{ \hat{y} [\int dx'dy' \frac{(x-x')}{(x-x')^2 + (y-y')^2} - \hat{x} \int dx'dy' \frac{(y-y')}{(x-x')^2 + (y-y')^2} \}$$

Look at the first integration: ($\pm\infty$ say)

$$\int dy' \frac{(x-x')}{(x-x')^2 + (y-y')^2} = \text{signum}(x-x_p)\,\pi$$

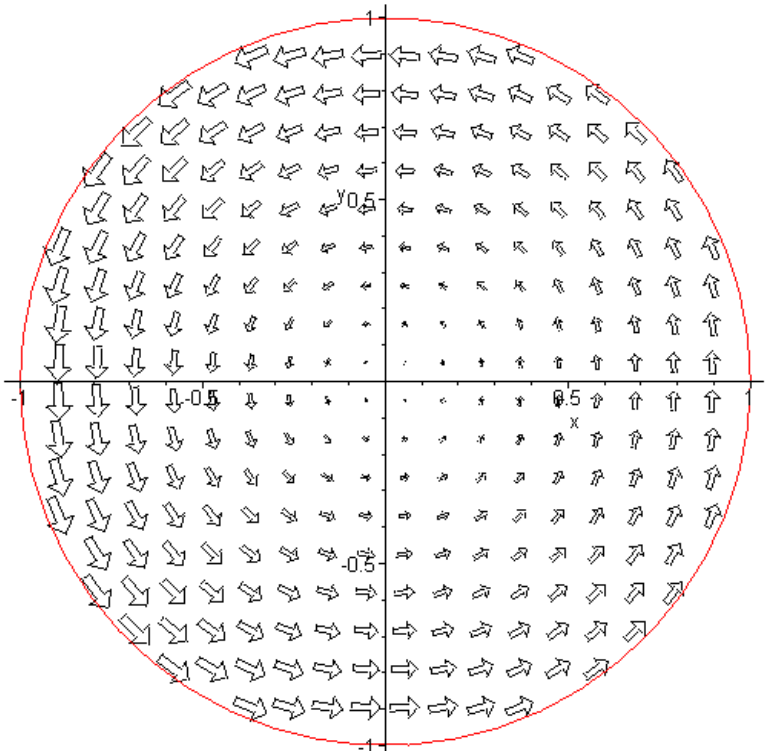
Then we are left with

$$\int dx' \,\text{signum}(x-x_p)\,\pi = \pi \int ds \,\text{signum}(s) = 0 \text{ since signum(s) is odd.}$$

So we get **E** = 0. This assumed that our uniform B field was everywhere! At least it agrees with the initial result noted earlier!

2. Eddy Problem 2: B field with linear gradient, still circular plate

Continue with the circular plate and assume that B is stronger to the left of the plate. If you drew little loops in various places, you would find generally that EMF's were greater to the left, so I might expect a picture to look *something* like this: [created with k := 1 - (1/4)*(x/r) added as a factor] [but this picture is wrong because the correct lines don't have the origin as their center]



Maybe the E circles are the same, but the field mag is just larger on the left side? [wrong]

Suppose we assume this for the moment. Then looking back at the derivation for Problem 1 :

$$\oint_C \mathbf{E} \bullet d\mathbf{s} \;=\; \oint_C E_\theta(r,\theta)\, r d\theta \;=\; -\dot{B}_z\, \pi r^2$$

or

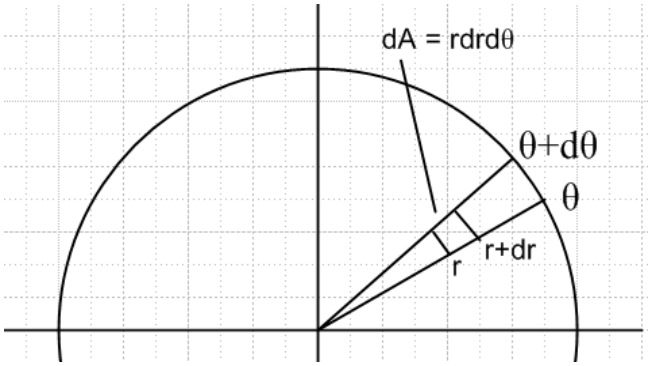
$$r \oint_C E_\theta(r,\theta)\, d\theta \;=\; -\dot{B}_z\, \pi r^2$$

or

$$\int_{-\pi}^{\pi} E_\theta(r,\theta)\, d\theta \;=\; -\dot{B}_z(x)\, \pi r$$

But now with no symmetry, I am stuck. But this is just a result applied to a single selected loop, and I can maybe learn more (extract more) by taking some smaller math loops as I did in my previous doc.

(a) Obtain PDE#1 two different ways and write down PDE #2 as well



Assume that B is only in the z direction and that it varies linearly in the x direction

$B(x) = B_0 - \alpha x$ $\alpha > 0$ B(x) bigger for x < 0 (on left)

Then apply Faraday,

$\mathcal{E}_{emf} = \oint_C \mathbf{E} \bullet d\mathbf{s} = -j\omega \int_S \mathbf{B} \bullet d\mathbf{S}$ from curl $\mathbf{E} = -j\omega\mathbf{B}$

$\oint_C \mathbf{E} \bullet d\mathbf{s}$

$= E_r(r + dr/2, \theta)dr + E_\theta(r+dr, \theta + d\theta/2)([r+dr]d\theta) - E_r(r+dr/2,\theta+d\theta)dr - E_\theta(r, \theta + d\theta/2)(rd\theta)$

$= [E_r(r + dr/2, \theta) - E_r(r+dr/2,\theta+d\theta)]\, dr + [(r+dr)E_\theta(r+dr, \theta + d\theta/2) - rE_\theta(r, \theta + d\theta/2)]\, (d\theta)$

The right side is

$\int_S \mathbf{B} \bullet d\mathbf{S} = [\, B_0 - \alpha x \,]\, dr\, rd\theta = [\, B_0 - \alpha r\cos\theta \,]\, dr\, rd\theta$

So then

$[E_r(r + dr/2, \theta) - E_r(r+dr/2,\theta+d\theta)]\, dr + [(r+dr)E_\theta(r+dr, \theta + d\theta/2) - rE_\theta(r, \theta + d\theta/2)]\, d\theta$

$= -j\omega\, [\, B_0 - \alpha r\cos\theta \,]\, dr\, rd\theta$

Divide through by drdθ to get

$[E_r(r + dr/2, \theta) - E_r(r+dr/2,\theta+d\theta)]/d\theta + [(r+dr)E_\theta(r+dr, \theta + d\theta/2) - rE_\theta(r, \theta + d\theta/2)]\, /dr$

$= -j\omega\, [\, B_0 - \alpha r\cos\theta \,]\, r$

or

$\partial_\theta E_r(r + dr/2, \theta) + \partial_r(r\, E_\theta(r, \theta + d\theta/2)) = -j\omega\, [\, B_0 - \alpha r\cos\theta \,]\, r$

which is at least *something*. Perhaps at this point we can simplify it to say

$$\partial_\theta E_r(r, \theta) + \partial_r(r E_\theta(r, \theta)) = -j\omega [B_0 - \alpha r\cos\theta] r \qquad (*)$$

So what exactly IS this equation? Probably just curl **E** = -jω**B** in the z component.

$$\text{curl } \mathbf{E} = \hat{\mathbf{r}} [r^{-1}\partial_\theta E_z - \partial_z E_\theta] + \hat{\boldsymbol{\theta}} [\partial_z E_r - \partial_r E_z] + \hat{\mathbf{z}} [r^{-1}\partial_r(rE_\theta) - r^{-1}\partial_\theta E_r]$$

$$= \hat{\mathbf{z}} [r^{-1}\partial_r(rE_\theta) - r^{-1}\partial_\theta E_r]$$

so then we get from curl **E** = -jω**B** the result that,

$$r^{-1}\partial_r(rE_\theta) - r^{-1}\partial_\theta E_r = -j\omega [B_0 - \alpha r\cos\theta]$$

or

$$\partial_r(rE_\theta) - \partial_\theta E_r = -j\omega [B_0 - \alpha r\cos\theta] r \qquad (**) = \text{same!}$$

Just checking. So the main point is that we have this PDE to somehow solve

$$\partial_r(rE_\theta) - \partial_\theta E_r = -j\omega [B_0 - \alpha r\cos\theta] r \qquad \text{PDE\#1}$$

We also know that div **E** = 0 inside the circular plate, and that says

$$\text{div } \mathbf{E} = r^{-1}\partial_r(rE_r) + r^{-1}\partial_\theta E_\theta + \partial_z E_z \approx 0$$

I am expecting some small radial Hall ρ but I will neglect it. Then we get

$$\partial_r(rE_r) + \partial_\theta E_\theta = 0 \qquad \text{PDE\#2}$$

I think the above two PDE's involve no extra assumptions like $J_r = 0$. They are just two Maxwell equations! [the above two PDE's are correct]

(b) What does Magnetic Ohm's Law say?

What do I know from the **Magnetic Ohm's Law** type stuff? I just look back over the Problem 1 work and think I can steal some things:

$$\kappa v_r = E_r + v_\theta B_3(x) \qquad 1$$
$$\kappa v_\theta = E_\theta - v_r B_3(x) \qquad 2$$

or

$$\kappa n q v_r = n q E_r + n q v_\theta B_3(x)$$
$$\kappa n q v_\theta = n q E_\theta - n q v_r B_3(x)$$

or

$$\kappa J_r = n q E_r + J_\theta B_3(x)$$
$$\kappa J_\theta = n q E_\theta - J_r B_3(x)$$

There is more to steal, but the above is enough for the next section.

(c) Try making the same assumption made in Problem 1, that $J_r \equiv 0$ everywhere in cylinder

[this ansatz turns out to be no good, but I don't find that out until section (f) below, so reader may skip to that point.]

I can TRY the idea that $J_r = 0$ (also $v_r = 0$) and see what happens:

$0 = nqE_r + J_\theta B_3(x)$ 1 $J_\theta = nqv_\theta$ $J_r = nqv_r = 0$

$\kappa J_\theta = nqE_\theta$ 2

The *first* says

$$E_r = - v_\theta B_3(x)$$

The *second* equation tells us that

$$\kappa J_\theta = nqE_\theta$$

or

$$J_\theta = nq\mu E_\theta / (1+j\omega\tau) = \sigma E_\theta/(1+j\omega\tau) \approx \sigma E_\theta$$

$$v_\theta = J_\theta/(Nq) = \mu E_\theta / (1+j\omega\tau) \approx \mu E_\theta$$

So here is what I know from just doing this stuff

$$v_r = 0$$
$$v_\theta = \mu E_\theta / (1+j\omega\tau) \approx \mu E_\theta$$
$$v_z = 0$$

$$E_r = - v_\theta B_3(x) = - \mu E_\theta / (1+j\omega\tau) * B_3(x) \qquad (*)$$
$$E_\theta = \text{unknown!}$$
$$E_z = 0$$

$$J_r = 0$$
$$J_\theta = \sigma E_\theta/(1+j\omega\tau)$$
$$J_z = 0$$

Everything is in terms of E_θ. But equation is very helpful because it says

$E_r = [-\mu B_3(x)/(1+j\omega\tau)] E_\theta \equiv \beta(x) E_\theta$ β = no dimensions $\beta' = m^{-1}$

and then instead of having two PDE's with two unknown functions, we have only 1 fctn! In other words, we get this very simple relationship between the two field components with our Assumption above.

(d) Process the two PDE's to obtain a single ODE for E_r with $J_r \equiv 0$

In this section, I continue with the $J_r \equiv 0$ ansatz to see if we end up in contradiction.

So here then are those two PDE's again:

$$\partial_r(rE_\theta) - \partial_\theta E_r = -j\omega\,[B_0 - \alpha\, r\cos\theta]\, r \qquad \text{PDE\#1}$$

or

$$\partial_r(rE_\theta) - \partial_\theta[\beta(x)\, E_\theta] = -j\omega\,[B_0 - \alpha\, r\cos\theta]\, r \qquad \text{// Jr≡0 used here}$$

$$\partial_r(rE_r) + \partial_\theta E_\theta = 0 \qquad \text{PDE\#2}$$

or

$$\partial_r(r[\beta(x)\, E_\theta]) + \partial_\theta E_\theta = 0 \qquad \text{// Jr≡0 used here}$$

so here are the two PDE's now:

$$\partial_r(rE_\theta) - \partial_\theta[\beta(r\cos\theta)\, E_\theta] = -j\omega\,[B_0 - \alpha\, r\cos\theta]\, r$$
$$\partial_r(\,[\beta(r\cos\theta)\, rE_\theta]) + \partial_\theta E_\theta = 0$$

Now consider

$$\partial_\theta\, \beta(r\cos\theta) = \beta'(r\cos\theta) * (-r\sin\theta)$$

$$\partial_r\, \beta(r\cos\theta) = \beta'(r\cos\theta)*\cos\theta$$

Then our equations become

$$\partial_r(rE_\theta) - \beta(r\cos\theta)\, \partial_\theta E_\theta + \beta'(r\cos\theta)\, (r\sin\theta)\, E_\theta = -j\omega\,[B_0 - \alpha\, r\cos\theta]\, r \qquad \text{PDE\#1}$$

$$\beta(r\cos\theta)\, \partial_r(rE_\theta) + \beta'(r\cos\theta)*\cos\theta\, (rE_\theta) + \partial_\theta E_\theta \approx 0 \qquad \text{PDE\#2}$$

or more compactly

$$\partial_r(rE_\theta) - \beta\, \partial_\theta E_\theta + \beta'\, (r\sin\theta)\, E_\theta = -j\omega\,[B_0 - \alpha\, r\cos\theta]\, r \qquad \text{PDE\#1}$$

$$\beta\, \partial_r(rE_\theta) + \beta'\cos\theta\, (rE_\theta) + \partial_\theta E_\theta \approx 0 \qquad \text{PDE\#2}$$

and this seems a lot to know about our one function E_θ . Suppose I use #2 to eliminate $\partial_\theta E_\theta$

$$-\partial_\theta E_\theta = \beta\, \partial_r(rE_\theta) + \beta'\cos\theta\, (rE_\theta)$$

so # 1 becomes

$\partial_r(rE_\theta) + \beta\,[\beta\,\partial_r(rE_\theta) + \beta'\cos\theta\,(rE_\theta)] + \beta'\,(r\sin\theta)\,E_\theta = -j\omega\,[B_0 - \alpha\, r\cos\theta]\,r$

or

$[\,1+\beta^2\,]\,\partial_r(rE_\theta) + [\beta\beta'\,r\cos\theta + \beta'\,r\sin\theta]\,E_\theta = -j\omega\,[B_0 - \alpha\, r\cos\theta]\,r$

or

$[\,1+\beta^2\,]\,\partial_r(rE_\theta) + \beta' r\,[\beta\cos\theta + \sin\theta]\,E_\theta = -j\omega\,[B_0 - \alpha\, r\cos\theta]\,r$

and this is then an ODE!

So how big is β ?

$\beta = [\,-\mu B_3(x)/(1+j\omega\tau)] \sim \mu B$ = small if B is << 569 tesla,

I remember this from "Hall Effect for General B". We can then simplify to get

$\partial_r(rE_\theta) + \beta' r\,(\sin\theta)\,E_\theta = -j\omega\,[B_0 - \alpha\, r\cos\theta]\,r$

Now

$\beta(x) = [\,-\mu B_3(x)/(1+j\omega\tau)] \approx -\mu B_3(x) = -\mu\,[B_0 - \alpha\, r\cos\theta] = -\mu[B_0 - \alpha x]$

$\partial_x\beta(x) = -\mu\partial_x B_3(x) = -\mu\partial_x[B_0 - \alpha x] = \mu\alpha = \beta'$

So again

$\partial_r(rE_\theta) + \mu\alpha\, r\sin\theta\, E_\theta = -j\omega\,[B_0 - \alpha\, r\cos\theta]\,r$ **Here is the ODE based on $J_r \equiv 0$**

First, as a test, suppose I set α = 0. This should give the uniform B result:

$\partial_r(rE_\theta) = -j\omega\,[B_0]\,r$

Integrate both sides to get

$rE_\theta(r) - 0E_\theta(0) = -j\omega B_0\,[\,(r^2/2) - 0]$

or

$rE_\theta(r) = -j\omega B_0\,[\,(r^2/2)$

$E_\theta(r) = -j\omega B_0\,[\,(r/2)\,]$ // agrees with my Problem 1 solution !!

So that is very encouraging. Now let's give Maple a shot at the ODE. But once gain we have

$\partial_r(rE_\theta) + \mu\alpha\, r\sin\theta\, E_\theta = -j\omega\,[B_0 - \alpha\, r\cos\theta]\,r$

$r\partial_r E_\theta + E_\theta + \mu\alpha\, r\sin\theta\, E_\theta = -j\omega\,[B_0 - \alpha\, r\cos\theta]\,r$

$r\partial_r E_\theta + (1+ \mu\alpha\, r\sin\theta)\, E_\theta = -j\omega\,[B_0 - \alpha\, r\cos\theta]\,r$

$\partial_r E_\theta + (1/r+ \mu\alpha\, \sin\theta)\, E_\theta = -j\omega\,[B_0 - \alpha\, r\cos\theta]$ dim(α) = B/m dim(μ) = 1/B

(e) Try to solve the single ODE for E_θ : still using $J_r \equiv 0$ ansatz

$$\left(\frac{\partial}{\partial r}u(r)\right) r + r\,u(r)\,\mu\,\alpha\,\sin(\theta) + u(r) = -j\,r\,\omega\,B0 + j\,\omega\,\cot(\theta)\,\alpha\,\sin(\theta)\,r^2$$

$-\partial_\theta E_\theta = \beta\,\partial_r(rE_\theta) + \beta'\cos\theta\,(rE_\theta)$

Maple DOES get a solution for the first equation, but it is a big mess and does not seem to have a limit as $\alpha \rightarrow 0$ as it should. I think Maple has a problem.

OK, let's simplify the first ODE a bit. I have it down to

$\partial_r u + (s+1/r)u = c+bsr$

where

$b = j\omega\cot\theta/\mu$		dim(b) = B/sec
$c = -j\omega B_0$		dim(c) = B/sec
$s = \mu\alpha\sin\theta$	$bs = j\omega\cot\theta/\mu * \mu\alpha\sin\theta = j\omega\alpha\cos\theta$	dim(s) = 1/m
		dim(sr) = 1

Dimension check of the ODE.

Note added. For very small r, the above ODE is roughly

$\partial_r u + (1/r)u = c$

and Maple says the solution to this is [it is the homo solution that blows up]

$$u(r) = \frac{1}{2}C\,r + \frac{_C1}{r}$$

So I can set C1 = 0 to get a non-divergent u = (C/2)r. Fine.

I enter this and Maple now finds a relatively simple solution:

$\partial_r u + (s+1/r)u = c+bsr$

```
eq := diff(u(r),r) + (s + 1/r)*u(r) = c + b*s*r;
```

$$eq := \left(\frac{\partial}{\partial r}u(r)\right) + \left(s + \frac{1}{r}\right)u(r) = c + b\,s\,r$$

```
dsolve(eq);
```

$$u(r) = \frac{c}{s} - \frac{c}{r\,s^2} + b\,r - 2\,\frac{b}{s} + 2\,\frac{b}{r\,s^2} + \frac{e^{(-s\,r)}_C1}{r}$$

which I now write out as

$$u(r) = br - \frac{2b-c}{s} + \frac{2b-c}{s^2 r} + C_1\, e^{-sr}/r$$

Note added. I have verified that this really does solve the ODE. But for very small r, this diverges, which seems to contradict what happened above where I *first* considered small r, *then* solves. But if I just look at the ODE by hand and try to make the solution be u = Ar, because parameter s *appears on both sides* of the equation, I find that I need A = c/2 and A = b *at the same time*, which does not fly. To wit:

$\partial_r u + (s+1/r)u = c+bsr$ Try u = Ar

$A + (s+1/r)Ar = c + bsr$

$A + Asr + A = c + bsr$ => $2A = c$ and $A = b$

This then is why the 1/r divergent term goes away if 2b = c!

I then seek the α = 0 solution this way

```
s :=0: u1 := dsolve(eq); unassign('s');
```

$$u1 := u(r) = \frac{1}{2} c\, r + \frac{_C1}{r}$$

which is correct. Then I enter the constants

```
b := j*omega*cot(theta)/mu;
```

$$b := \frac{j\,\omega\,\cot(\theta)}{\mu}$$

```
c := -j*omega*B0;
```

$$c := -j\,\omega\,B0$$

```
s := mu*alpha*sin(theta);
```

$$s := \mu\,\alpha\,\sin(\theta)$$

I verify that the ODE is correct

```
r*eq;
```

$$r\left(\left(\frac{\partial}{\partial r} u(r)\right) + \left(\mu\,\alpha\,\sin(\theta) + \frac{1}{r}\right) u(r)\right) = r\,(-j\,\omega\,B0 + j\,\omega\,\cot(\theta)\,\alpha\,\sin(\theta)\,r)$$

```
expand(%);
```

$$\left(\frac{\partial}{\partial r} u(r)\right) r + r\,u(r)\,\mu\,\alpha\,\sin(\theta) + u(r) = -j\,r\,\omega\,B0 + j\,\omega\,\cot(\theta)\,\alpha\,\sin(\theta)\,r^2$$

And then I verify that the simple solution is correct

```
u1;
```

$$u(r) = -\frac{1}{2} j\,r\,\omega\,B0 + \frac{_C1}{r}$$

So good, at least we have some kind of solution. Let's now examine it!

April 4, 2014. Well the solution was this

$$r\,b - 2\frac{b}{s} + \frac{c}{s} + \frac{2\frac{b}{s^2} - \frac{c}{s^2} + e^{(-s\,r)}_C1}{r}$$

But it has a piece I cannot remove which blows up at r = 0 unless 2b = c

$b = j\omega\cot\theta/\mu$
$c = -j\omega B_0$

which is not realized. So how can we have $E_\theta(r)$ blow up at r = 0 ???

(f) Conclusion: the assumption that $J_r \equiv 0$ made in part (c) is invalid

I first went through all the math above (single full pass) looking for math errors and I found none, so things are restored to black.

I often say that an ansatz will prove itself either right (self consistent solution) or wrong (contradiction). So here the ansatz leads to a divergent E_θ field at r = 0. So ansatz is no good.

In retrospect, you can see that the ansatz was OK for the symmetric uniform B field case, because the E field contours are exactly circles. In this problem they are NOT going to be circles. But perhaps more to the point, the *current flow* will not be in perfect circles, so we won't have $v_r \equiv 0$ which goes with $J_r \equiv 0$. My quick picture above is a little misleading in this respect. It seems totally reasonable NOW that the solution cannot really have $J_r \equiv 0$. We will still have $J_r(a) = 0$, however.

(g) Try partial waves on general PDF (*)

We still have our two PDF equations which just follow from curl **E** = -jω**B** and div **E** = 0:

$$\partial_r(rE_\theta) - \partial_\theta E_r = -j\omega\,[B_0 - \alpha\,r\cos\theta]\,r \qquad \text{PDE\#1}$$

$$\partial_r(rE_r) + \partial_\theta E_\theta = 0 \qquad \text{PDE\#2}$$

We are ignoring the induced B field here. So I will probably be taking a little boat trip headed toward Helmholtz Island, but just do it anyway. First, mult #1 by r

$$r\partial_r(rE_\theta) - \partial_\theta(rE_r) = -j\omega\,[B_0 - \alpha\,r\cos\theta]\,r^2$$

Then apply ∂_r to this thing:

$$\partial_r[r\partial_r(rE_\theta)] - \partial_\theta\,\partial_r(rE_r) = -j\omega\,\partial_r\,\{[B_0 - \alpha\,r\cos\theta]\,r^2\}$$

Now use #2 to replace the second term here

$$\partial_r[r\partial_r(rE_\theta)] - \partial_\theta[-\partial_\theta E_\theta] = -j\omega\,\partial_r\{[B_0 - \alpha\, r\cos\theta]\, r^2\}$$

or

$$\partial_r[r\partial_r(rE_\theta)] + \partial_\theta^2 E_\theta = -j\omega\,\partial_r\{[B_0 - \alpha\, r\cos\theta]\, r^2\} \qquad (*)$$

So now at least we have only one unknown function. This is a PDE: function is E_θ and variables are r,θ. This does not quite seem to be Helmholtz, but has similarities. It follows from the two Maxwell's and no other assumptions.

Boundary Condition

Meanwhile, magnetic Ohm's still has this stuff

$$\kappa v_r = E_r + v_\theta B_3(x) \qquad 1$$
$$\kappa v_\theta = E_\theta - v_r B_3(x) \qquad 2$$

or

$$\kappa n q v_r = nqE_r + nqv_\theta B_3(x)$$
$$\kappa n q v_\theta = nqE_\theta - nqv_r B_3(x)$$

or

$$\kappa J_r = nqE_r + J_\theta B_3(x)$$
$$\kappa J_\theta = nqE_\theta - J_r B_3(x)$$

So this does not generate any obvious BC on the function E_θ . The BC for J_r is certainly more obvious, and we have

$$\kappa J_r(a,\theta) = nqE_r(a,\theta) + J_\theta(a,\theta)\, B_3(x=a\cos\theta) = 0$$
$$\kappa J_\theta(a,\theta) = nqE_\theta(a,\theta) - J_r(a,\theta)B_3(x = a\cos\theta) = nqE_\theta(a,\theta)$$

So these equations tell us that

$$nqE_r(a,\theta) + J_\theta(a,\theta)\, B_3(x=a\cos\theta) = 0 \qquad // J_r(a,\theta) = 0 \text{ BC}$$
$$J_\theta(a,\theta) = \sigma\, E_\theta(a,\theta) \qquad // \text{regular Ohm's in } \theta$$

Insert the second into the first to get

$$nqE_r(a,\theta) + \sigma\, E_\theta(a,\theta)B_3(x=a\cos\theta) = 0 \qquad \mu = (q\tau/m) \quad \sigma = (nq^2\tau/m) = nq\mu$$

or

$$E_r(a,\theta) + \mu\, E_\theta(a,\theta)B_3(x=a\cos\theta) = 0$$

or

$$E_r(a,\theta) = -\mu\, E_\theta(a,\theta)B_3(x=a\cos\theta) \qquad // J_r(a,\theta) = 0 \text{ BC in terms of fields}$$

So we end up with a boundary condition at r = a that involves both E_r and E_θ fields which certainly is not very pleasant.

The F Idea.

One idea would be to define a new field:

$$F(r,\theta) \equiv E_r(r,\theta) + \mu E_\theta(r,\theta)B(x=r\cos\theta) = E_r(r,\theta) + \mu E_\theta(r,\theta)B(\mathbf{x})$$

Then maybe I can come up with an ODE for F, and it then has a clean BC. Let's try this. Start again with

$$\partial_r(rE_\theta) - \partial_\theta E_r = -j\omega B(\mathbf{x})\, r \qquad \text{PDE\#1}$$

$$\partial_r(rE_r) + \partial_\theta E_\theta = 0 \qquad \text{PDE\#2}$$

Now replace $E_r(r,\theta) = F(r,\theta) - \mu E_\theta(r,\theta)B(\mathbf{x})$:

$$\partial_r(rE_\theta) - \partial_\theta[F(r,\theta) - \mu E_\theta(r,\theta)B(\mathbf{x})] = -j\omega B(\mathbf{x})\, r \qquad \text{PDE\#1}$$

$$\partial_r(r[F(r,\theta) - \mu E_\theta(r,\theta)B(\mathbf{x})]) + \partial_\theta E_\theta(r,\theta) = 0 \qquad \text{PDE\#2}$$

Write them out

$$\partial_r(rE_\theta) - \partial_\theta F + \mu\partial_\theta[E_\theta B(\mathbf{x})] = -j\omega B(\mathbf{x})\, r$$

$$\partial_r(rF) - \mu\, \partial_r [E_\theta B(\mathbf{x})] + \partial_\theta E_\theta = 0$$

But now I don't see how to "separate" these to obtain an ODE for F. So give up on the F idea.

Resume trying to solve: go to partial waves

Here I try partial waves on the PDE (*) above (has no assumptions), and I get a half-baked solution pathway. Things don't really work right in the details, but at least it is something possibly viable.

So OK, let's forget the BC for the moment and go back to our PDE (*) from above

$$\partial_r[r\partial_r(rE_\theta)] + \partial_\theta^2 E_\theta = -j\omega\, \partial_r \{ [B_0 - \alpha\, r\cos\theta]\, r^2\} \qquad (*)$$

I am then forced to go into partial waves as in Appendix D, where $\partial_\theta = jm$. Then

$$\partial_r[r\partial_r(rE_\theta)] - m^2 E_\theta = -j\omega\, \partial_r \{ [B_0 - \alpha\, r\cos\theta]\, r^2\} \qquad (*)$$

where now E_r means $E_r(r,m)$. What does Maple have to say about this ODE? Write

$-j\omega B_0 = c$ as above

$j\omega\alpha\cos\theta = d$ = a new symbol. Then we have

$\partial_r[r\partial_r(rE_\theta)] - m^2\ E_\theta = \partial_r\ \{\ [c - dr]\ r^2\} = 2rc\text{-}3dr^2$

or

$\partial_r[r\partial_r(rE_\theta)] - m^2\ E_\theta\ \ = 2rc\text{-}3dr^2$

$\partial_r[r\partial_r(ru)] - m^2\ u\ \ = 2rc\text{-}3dr^2$

Amazingly, Maple has a solution for this equation that is not too bad:

```
eq2 := diff(r*diff(r*u(r),r),r) - m^2*u(r) = 2*c*r - 3*d*r^2;
```

$$eq2 := u(r) + r\left(\frac{\partial}{\partial r}u(r)\right) + r\left(2\left(\frac{\partial}{\partial r}u(r)\right) + r\left(\frac{\partial^2}{\partial r^2}u(r)\right)\right) - m^2\,u(r) = 2\,c\,r - 3\,d\,r^2$$

```
dsolve(eq2);
```

$$u(r) = \frac{r\,(-2\,c\,m^2 + 18\,c + 3\,d\,r\,m^2 - 12\,d\,r)}{(m^2 - 4)\,(m^2 - 9)} + _C1\,r^{(m-1)} + _C2\,r^{(-m-1)}$$

.

Unfortunately this solution is singular when m = ±2 or ± 3 so that seems to ruin things. But Maple has solutions in these cases as well:

$$u(r) = -\frac{3}{5}d\,r^2 + \frac{1}{2}r\,c\,\ln(r) - \frac{1}{8}c\,r + _C1\,r + \frac{_C2}{r^3} \qquad m = \pm 2$$

$$u(r) = -\frac{1}{60}r\,(24\,c + 30\,d\,\ln(r)\,r - 5\,d\,r) + _C1\,r^2 + \frac{_C2}{r^4} \qquad m = \pm 3$$

So in principle we have a partial wave series solution with lots of unknown constants. The boundary condition for $E_r(r,m)$ has products of functions of θ

$E_r(a,\theta) = -\ \mu\ E_\theta(a,\theta)B_3(x=a\cos\theta)$

and that does not work well in the partial waves. [but see later]

I could have set $B_0 = 0$ to have a simpler problem, which causes c = 0 in the above. The results are then a lot simpler. For general m we get

$$u(r) = 3\frac{d\,r^2}{m^2 - 9} + _C1\,r^{(-m-1)} + _C2\,r^{(m-1)}$$

and for m = ±3 we get

$$= u(r) = -\frac{1}{2} d \ln(r) r^2 + \frac{1}{12} d r^2 + _C1 r^2 + \frac{_C2}{r^4}$$

For m = 0 get

$$= u(r) = -\frac{1}{3} d r^2 + \frac{_C1}{r} + \frac{_C2 \ln(r)}{r}$$

For m = ±1 get

$$q := u(r) = -\frac{3}{8} d r^2 + _C1 + \frac{_C2}{r^2}$$

Here then is my summary

m = 0:	$u_0(r) = -(1/3)dr^2$	$u_m = E_\theta(r,m)$
m = ±1:	$u_{\pm 1}(r) = -(3/8)dr^2 + A_{\pm 1}$	
m = ±3:	$u_{\pm 3}(r) = A_{\pm 3}\, r^2$	
other m	$u_{\pm m}(r) = 3d/(m^2-9)^* r^2 + A_{\pm m}\, r^{\lvert m\rvert -1}$	

d = jωαcosθ

It is interesting that there is a m=0 contribution in this case of having just a field gradient where the average field B_0 on the circular plate is zero.

I would perhaps take for the solution just the m = 0 and 1 terms to get some idea of things. But I don't see why the series should be fast converging. That would all still be a lot of work to do. I think the result would be

$E_\theta(r,\theta) = Ar^2 + [B + Cr^2]\cos\theta$	A = -(1/3)d	C = -(3/8)d
$\partial_\theta E_\theta = -[B + Cr^2]\sin\theta$	d = jωαcosθ	

The two PDE's are now

$\partial_r(rE_\theta) - \partial_\theta E_r = -j\omega\, B(\mathbf{x})\, r$ PDE#1

$\partial_r(rE_r) + \partial_\theta E_\theta = 0$ PDE#2

Then from the second we get

$\partial_r(rE_r) = [B + Cr^2]\sin\theta$

and Maple says the solution of this is

$$Er(r) = \sin(\theta)\, B + \frac{1}{3}\sin(\theta)\, C\, r^2 + \frac{C1}{r}$$

which says

$E_r(r,\theta) = B \sin\theta + (1/3)C\sin\theta\ r^2$

So given $B_0 = 0$ and a pure gradient field described by α, if we keep only the m = 0 and m = ±1 terms we get

$E_\theta(r,\theta) = Ar^2 + [B + Cr^2]\cos\theta$ $\quad A = -(1/3)d \quad C = -(3/8)d$

$E_r(r,\theta) = B \sin\theta + (1/3)C\sin\theta\ r^2$ $\quad d = j\omega\alpha\cos\theta$

The boundary condition is then

$E_r(a,\theta) + \mu E_\theta(a,\theta)B_3(x=a\cos\theta) = 0$ $\quad B_3(x=a\cos\theta) = [B_0 - \alpha r\cos\theta] = -\alpha r\cos\theta$

or

$E_r(a,\theta) - \mu\alpha\cos\theta\, E_\theta(a,\theta) = 0$

or

$Aa^2 + [B + Ca^2]\cos\theta - \mu\alpha\cos\theta\,[B\sin\theta + (1/3)C\sin\theta\ a^2] = 0$

Now insert the known values

$-(1/3)d\,a^2 + [B - (3/8)d\,a^2]\cos\theta - \mu\alpha\cos\theta\,[B\sin\theta - (1/3)(3/8)d\sin\theta\ a^2] = 0$

$-(1/3)j\omega\alpha\cos\theta\,a^2 + [B - (3/8)j\omega\alpha\cos\theta\,a^2]\cos\theta - \mu\alpha\cos\theta\,[B\sin\theta - (1/3)(3/8)j\omega\alpha\cos\theta\sin\theta\ a^2] = 0$

Write as

$\cos\theta\ [-(1/3)j\omega\alpha\,a^2 + B]$

$+ \cos^2\theta\,[-(3/8)j\omega\alpha\,a^2]$

$+ \cos\theta\sin\theta\,[-\mu\alpha\,B]$

$+ \cos^2\theta\sin\theta\,[+(1/8)\mu\alpha\,j\omega\alpha a^2]$

Things obviously don't work very wall. One item is

$B = -(1/3)j\omega\alpha\,a^2$

and then I have to argue the other terms away based on "small α". So here we are:

$E_\theta(r,\theta) = Ar^2 + [B + Cr^2]\cos\theta$ $A = -(1/3)d$ $C = -(3/8)d$

$E_r(r,\theta) = B\sin\theta + (1/3)C\sin\theta\ r^2$ $d = j\omega\alpha\cos\theta$

or

$E_\theta(r,\theta) = -(1/3)\ j\omega\alpha\cos\theta\ r^2 + [-(1/3)\ j\omega\alpha\ a^2 \ -(3/8)\ j\omega\alpha\cos\theta r^2]\cos\theta$

$E_r(r,\theta) = -(1/3)\ j\omega\alpha\ a^2\sin\theta + (1/3)[\ -(3/8)\ j\omega\alpha\cos\theta]\sin\theta\ r^2$

or

$E_\theta(r,\theta) = -(1/3)\ j\omega\alpha\cos\theta\ r^2\ -(1/3)\ j\omega\alpha\ a^2\cos\theta\ -(3/8)\ j\omega\alpha r^2\cos^2\theta$

$E_r(r,\theta) = -(1/3)\ j\omega\alpha\ a^2\sin\theta - (1/8)\ j\omega\alpha\ \cos\theta\sin\theta\ r^2$

or

$E_\theta(r,\theta) = j\omega\alpha\ \{\ -(1/3)\cos\theta\ r^2\ -(1/3)\ a^2\cos\theta\ -(3/8)\ r^2\cos^2\theta\ \}$

$E_r(r,\theta) = j\omega\alpha\ \{-(1/3)\ a^2\sin\theta - (1/8)\ \cos\theta\sin\theta\ r^2\}$

Obviously this is a wrong result because I had to fudge a lot of things, but it does suggest a few things:

- the fields go to 0 if $\alpha = 0$, as expected
- fields are proportional to ω, as I would expect
- fields have power r^2 here and there and reasonably simple angular dependence.

Conclusion: This sort of confirms my guess after reading some literature that eddy current problems are quite difficult to solve, even in the simplest of situations beyond the uniform B case. That two methods paper did not venture beyond a uniform B field.

(h) Direct Helmholtz equation approach using App D results

I think you could approach this circular disk eddy current problem along the lines of Appendix D, where the z direction is more or less ignored. I think this means setting $\beta_d = 0$, meaning infinitely long wavelength. You do the partial waves and you get down to the box (D.1.20) with the three Helm and the one divergence equation. You then solve these equations and you arrive at (D.2.21) in the circular thin plate, where as usual we have K_m and a_m undetermined in each partial wave. So far there is no connection to any *magnetic* fields, such as the one that is causing our "action" in the plate.

At this point I might say the following: for this thin plate, $J_z = 0$ because it has nowhere to go and because that just seems reasonable for a "thin" plate where we don't really activate any currents in that direction. I think it also reasonable to say the $J_z = \sigma E_z$ so that $E_z = 0$ as well. Then especially noting the appearance of $1/\beta_d$ in the E_z equation, it just seems that you need to set $K_m = 0$, something new to me. There is just no way around this, at least for the J_z part. Certainly we can TRY it!

First summary of the E field solutions (D.2.21)

$$E_z(r,m) = -j(\beta'/\beta_d)\frac{K_m}{2} J_m(x) \qquad x = \beta' r \qquad \textit{(D.1.27)}$$

$$E_r(r,m) = a_m x^{-1} J_m(x) + \frac{K_m}{2} J_{m+1}(x) \qquad \beta'^2 = \beta^2 - \beta_d^2 \qquad \textit{(D.2.11)}$$

$$jE_\theta(r,m) = -a_m x^{-1} J_m(x) + \left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x) \qquad \textit{(D.2.15)}$$

Then we get

$$E_r(r,m) = a_m x^{-1} J_m(x) \qquad x = \beta r \qquad \beta'^2 = \beta^2 \qquad \textit{(D.2.11)} \qquad \text{(D.2.21)}$$

$$jE_\theta(r,m) = -a_m x^{-1} J_m(x) + \frac{a_m}{m} J_{m+1}(x) \qquad \textit{(D.2.15)}$$

Remember that this includes div E = 0 and all the Helmholtz stuff.

Magnetic Ohm's Law

Now maybe look again at Magnetic Ohm's law in the x,y plane of this thin circular plate. I go back to Appendix N.4 where also we had B in the z direction (not necessarily uniform. I write the motional Lorentz and solve to get

$$J_x = \sigma (E_x + \omega_c \tau E_y) \frac{1}{1+(\omega_c\tau)^2} \qquad \omega_c \equiv (qB/m)$$

$$J_y = \sigma (E_y - \omega_c \tau E_x) \frac{1}{1+(\omega_c\tau)^2} \qquad \mathbf{B} = B\hat{\mathbf{z}} \qquad \sigma = (nq^2\tau/m)$$

$$J_z = \sigma E_z \ . \qquad \sigma = (nq^2\tau/m) \qquad \text{(N.4.10)}$$

Now let's throw out the tiny correction ratio, and install arguments:

$$J_x = \sigma (E_x + \omega_c \tau E_y)$$

$$J_y = \sigma (E_y - \omega_c \tau E_x)$$

I don't think I ever wrote this down above for Problem 1 maybe because it was not needed. Now since no cross products etc, I can translate this so the x,y = 1,2 = r,θ to get

$$J_r = \sigma (E_r + \omega_c \tau E_\theta)$$

$$J_\theta = \sigma (E_\theta - \omega_c \tau E_r)$$

and this is our magnetic Ohm's Law in the plane of our disk. But now there are no θ function products, so we can take this right to m-space to get

$$J_r(r,m) = \sigma (E_r(r,m) + \omega_c\tau\, E_\theta(r,m))$$

$$J_\theta(r,m) = \sigma (E_\theta(r,m) - \omega_c\tau\, E_r(r,m))$$

Boundary Condition

Then we have this clean boundary condition

$$J_r(a,m) = 0$$

or

$$E_r(a,m) + \omega_c\tau\, E_\theta(a,m) = 0$$

Boundary Condition Cannot be Met in the $K_{\underline{m}}$= 0 scenario

What happens now if I use this with the expressions above:

$$E_r(r,m) = a_m\, x^{-1}\, J_m(x) \qquad x = \beta r \qquad \beta'^2 = \beta^2 \qquad \textit{(D.2.11)} \qquad \text{(D.2.21)}$$

$$jE_\theta(r,m) = -\, a_m\, x^{-1}\, J_m(x) + \frac{a_m}{m}\, J_{m+1}(x) \qquad \textit{(D.2.15)}$$

Here we go:

$$E_r(a,m) + \omega_c\tau\, E_\theta(a,m) = 0$$

$$a_m\, x_a^{-1}\, J_m(x_a) + \omega_c\tau\, j\, [a_m\, x_a^{-1}\, J_m(x_a) - \frac{a_m}{m}\, J_{m+1}(x_a)\,] = 0$$

Unfortunately (and but of course), a_m cancels out and we get

$$x_a^{-1}\, J_m(x_a) + \omega_c\tau\, j\, [x_a^{-1}\, J_m(x_a) - (1/m)\, J_{m+1}(x_a)\,] = 0$$

$$x_a^{-1}\, J_m(x_a)\, [\,1 + j\omega_c\tau] = (1/m)\, J_{m+1}(x_a)$$

So our solution has failed! I think this probably means that the E_z= 0 assumption is no good. Maybe there is some E_z and some J_z and it is pumping charge onto the top and bottom surfaces of the plate! That give the problem a little bit of an escape hatch. I guess another option is surface charge on the rim of the disk. This then links us to the "external problem".

Try stupid $L_{\underline{m}}$ idea, goes nowhere

So go back to this point:

First summary of the E field solutions (D.2.21)

$$E_z(r,m) = -j(\beta'/\beta_d)\frac{K_m}{2} J_m(x) \qquad x = \beta' r \qquad (D.1.27)$$

$$E_r(r,m) = a_m x^{-1} J_m(x) + \frac{K_m}{2} J_{m+1}(x) \qquad \beta'^2 = \beta^2 - \beta_d^2 \qquad (D.2.11)$$

$$jE_\theta(r,m) = -a_m x^{-1} J_m(x) + \left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x) \qquad (D.2.15)$$

It is certainly off-putting to see that β_d sitting there. We might replace

$$\frac{K_m}{2}(1/\beta_d) = \frac{L_m}{2}$$

Then we have

$$E_z(r,m) = -j\beta'\frac{L_m}{2} J_m(x) \qquad x = \beta' r \qquad (D.1.27)$$

$$E_r(r,m) = a_m x^{-1} J_m(x) + \beta_d \frac{L_m}{2} J_{m+1}(x) \qquad \beta'^2 = \beta^2 - \beta_d^2 \qquad (D.2.11)$$

$$jE_\theta(r,m) = -a_m x^{-1} J_m(x) + \left(\beta_d \frac{L_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x) \qquad (D.2.15)$$

But then in our limit that $\beta_d \rightarrow 0$ we get

$$E_z(r,m) = -j\beta'\frac{L_m}{2} J_m(x) \qquad x = \beta' r \qquad (D.1.27)$$

$$E_r(r,m) = a_m x^{-1} J_m(x) \qquad \beta'^2 = \beta^2 - \beta_d^2 \qquad (D.2.11)$$

$$jE_\theta(r,m) = -a_m x^{-1} J_m(x) + \left(\frac{a_m}{m}\right) J_{m+1}(x) \qquad (D.2.15)$$

and we are back to our no good boundary condition problem. So this L_m idea accomplished nothing.

Smythian Forms Comment

If I think in terms of Smythian Forms and this Helm equation, I have already incorporated the θ and r forms, but my z direction is weak.

$$e^{\pm kz}\ [\ J_m(kr),\ N_m(kr)]\ \ e^{\pm im\theta}$$

To what equation do these atoms apply? They are for Laplace, M&S page 14 second grouping. The solutions for scalar Helm start bottom of page 15 and see first grouping on page 16 with q = β' and then this is what App D is all about. Their χ is then β and then $\chi^2-q^2 = \beta_d^2$ so then $\chi^2 = \beta'^2 + \beta_d^2 = \beta^2$ and everyone is happy.

Back to K_m= 0 but allow rim surface charge similar to App D

Let's go back to $K_m = 0$ where we then had

$$E_r(r,m) = a_m\ x^{-1}\ J_m(x) \qquad x = \beta r \qquad \beta'^2 = \beta^2 \qquad \textit{(D.2.11)} \qquad \text{(D.2.21)}$$

$$jE_\theta(r,m) = -\ a_m\ x^{-1}\ J_m(x) + \frac{a_m}{m}\ J_{m+1}(x) \qquad \textit{(D.2.15)}$$

Then suppose we <u>try a charge pumping condition on the disk perimeter surface</u>. Something like in lines doc.

$$E_r(r=a,m) = (j\omega/\sigma)\ N_m$$

Then we get

$$a_m\ x_a^{-1}\ J_m(x_a) = (j\omega/\sigma)\ N_m$$

and this then relates a_m to N_m. OK, let that lie for a moment and lets then look at the corresponding B fields,

$$B_z(r,m) = (\beta'/\omega)\left(\frac{K_m}{2} + \frac{a_m}{m}\right)J_m(x)$$

$$B_r(r,m) = j(\beta'/\omega)\left\{ + \left(m\frac{K_m}{2}\frac{\beta'}{\beta_d} - \frac{\beta_d}{\beta'}a_m\right) x^{-1}J_m(x) + \frac{\beta_d}{\beta'}\left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x) \right\}$$

$$B_\theta(r,m) = (\beta'/\omega)\left\{ - \left(m\frac{K_m}{2}\frac{\beta'}{\beta_d} - \frac{\beta_d}{\beta'}a_m\right) x^{-1}J_m(x) + \frac{K_m}{2}\left(\frac{\beta_d}{\beta'} + \frac{\beta'}{\beta_d}\right) J_{m+1}(x) \right\}, \qquad \textit{(D.4.8)}$$

I set $K_m = 0$ here right off the bat to get

$$B_z(r,m) = (\beta'/\omega)\left(\frac{a_m}{m}\right)J_m(x)$$

$$B_r(r,m) = j(\beta'/\omega)\left\{ + \left(- \frac{\beta_d}{\beta'}a_m\right) x^{-1}J_m(x) + \frac{\beta_d}{\beta'}\left(\frac{a_m}{m}\right) J_{m+1}(x) \right\}$$

$$B_\theta(r,m) = (\beta'/\omega)\left\{ - \left(- \frac{\beta_d}{\beta'}a_m\right) x^{-1}J_m(x) \right\}, \qquad \textit{(D.4.8)}$$

Then I also let $\beta_d = 0$ and we then have

$$B_z(r,m) = (\beta'/\omega)\left(\frac{a_m}{m}\right)J_m(x)$$
$$B_r(r,m) = 0$$
$$B_\theta(r,m) = 0 \qquad (D.4.8)$$

Warning:

I think the B_z field here is the Lenz'z Law B_z field associated with the E fields earlier, and is not the applied B_z external field, so forcing it to be that external field might cause trouble.

OK, maybe this determines the a_m from the applied B field, and then we have everything. We then have

$$B_z(r,m) = (1/2\pi)\int_{-\pi}^{\pi} d\theta\, B_z(r,\theta)\, e^{-jm\theta}$$

OK, let's try our particular B field of interest [see warning above!]

$$B_z(r,\theta) = \alpha\, r\cos\theta = \alpha x$$

which is a linear in x field which is zero at x = 0 = center of disk. Then

$$B_z(r,m) = (1/2\pi)\alpha r \int_{-\pi}^{\pi} d\theta\, \cos\theta\, e^{-jm\theta} = (1/2\pi)\alpha r \int_{-\pi}^{\pi} d\theta\, \cos\theta\cos(m\theta) = (1/2\pi)\alpha r\, \pi\, \delta_{m,1}$$

$$= (1/2)\alpha\, r\, \delta_{m,\pm 1}$$

That was easy. I just showed that the transform of cos(mθ) is this

$$(1/2\pi)\int_{-\pi}^{\pi} d\theta\, \cos\theta\, e^{-jm\theta} = (1/2\pi)\,\pi\,\delta_{m,\pm 1}$$

So set the above deal equal to

$$B_z(r,m) = (\beta'/\omega)\left(\frac{a_m}{m}\right)J_m(x) = (1/2)\alpha\, r\, \delta_{m,\pm 1}$$

Then

$$\frac{a_m}{m} = (1/2)\alpha\, r\, (\omega/\beta)/J_m(x)\ \delta_{m,\pm 1}$$

But this is of course junk since there is no constant a_m that works. The problem here is that I am here dealing with the B field corresponding to the currents, not the "applied" B field.

curl **E** = -jω**B**

$r^{-1}\partial_r(rE_\theta) - r^{-1}\partial_\theta E_r = -j\omega\,[B_0 - \alpha\, r\cos\theta]$

or

$\partial_r(rE_\theta) - \partial_\theta E_r = -j\omega\,[B_0 - \alpha\, r\cos\theta]\, r$

Transforming this equation says

$\partial_r(rE_\theta) - jmE_r = -j\omega\,[B_0 - \alpha\, r\,(1/2\pi)\,\pi\,\delta_{m,\pm 1}]\, r$

Let's simplify with $B_0 = 0$ and restate

$\partial_r(rE_\theta) - jmE_r = j\omega\alpha\, r\,(1/2)\;\delta_{m,\pm 1}]\, r$

At least this is something vaguely new. It says

$\partial_r(rE_\theta(r,m)) - jE_r(r,m) = j\omega\alpha\, r\,(1/2)\;\delta_{m,\pm 1}]\, r$ PDE#1

$\partial_r(rE_r) + jmE_\theta = 0$ PDE#2 just recalling

which then says

$\partial_r(rE_\theta(r,m)) = jE_r(r,m)$ $m \neq \pm 1$

$\partial_r(rE_\theta(r,1)) - jE_r(r,1) = j\omega\alpha\,(1/2)\, r^2$

How about using this as a condition on our fields:

$E_r(r,m) = a_m\, x^{-1}\, J_m(x)$ $x = \beta r$ $\beta'^2 = \beta^2$ *(D.2.11)* (D.2.21)

$jE_\theta(r,m) = -\, a_m\, x^{-1}\, J_m(x) + \frac{a_m}{m}\, J_{m+1}(x)$ *(D.2.15)*

Then :

$\partial_r(r[a_m\, x^{-1}\, J_m(x)]) - jm(-j)[-\, a_m\, x^{-1}\, J_m(x) + \frac{a_m}{m}\, J_{m+1}(x)] = j\omega\alpha\, r\,(1/2)\;\delta_{m,\pm 1}]\, r$

For m = 0 everything is 0 which is fine.

What about m = 2:

$$\partial_r(r[a_m\ x^{-1}\ J_m(x)]) - m[-\ a_m\ x^{-1}\ J_m(x) + \frac{a_m}{m}\ J_{m+1}(x)] \ = 0$$

Solution is $a_2 = 0$, which is just fine.

Now try m = +1

$$\partial_r(r[a_1\ x^{-1}\ J_1(x)]) - \ [-\ a_1\ x^{-1}\ J_1(x) + \frac{a_1}{1}\ J_2(x)] \ = j\omega\alpha\ r\ (1/2)]\ r$$

But again, you have a fancy function of x on the left, and a simple function of r on the right, how can it work? Again in curl **E** = -jω**B** we must have the TOTAL B, and I show only the applied B and we get a contradiction. But be patient just a little more and grind this into the ground.

$$\partial_r(r[a_1\ x^{-1}\ J_1(x)]) \ = \partial_x(x[a_1\ x^{-1}\ J_1(x)]) \ = \ \partial_x([a_1 J_1(x)]) \ = a_1\ J_1'(x)$$

Then the above says

$$a_1\ [J_1'(x) + \ x^{-1}\ J_1(x) - \ J_2(x)\ ] = \ j\omega\alpha\ r\ (1/2)]\ r$$

or

$$a_1\ [xJ_1'(x) + \ J_1(x) - \ xJ_2(x)\ ] = \ j\omega\alpha\ r\ (1/2)]\ r\ x$$

I can replace

$$xJ_1'(x) = J_1(x) - xJ_2(x)$$

so we then have

$$2a_1\ [xJ_1'(x)] \ = j\omega\alpha\ r\ (1/2)]\ r\ x$$

$$2a_1\ J_1'(x) = j\omega\alpha\ (1/2)]\ r^2$$

and again we just have garbage. Something is very wrong with this approach, I guess it is the "warning" given above.

Try sin(β_dz) in the z direction

I suppose I could assume something like sin(β_dz) instead of $e^{-j\beta dz}$. I think this just shows that you are activating a whole spectrum of modes of the thin disk with your external B field. I was hoping to get a mode in which $\beta_d = 0$

If I were to assume sin(β_dz) instead of $e^{-j\beta dz}$ in Appendix D, I think the rule is this:

wherever β_d appears not squared, replace it with $\beta_d$cot(β_dz).

This arises because these linear B_d factors all come from divE = 0. It easy to show the above idea if you just look at the way ∂_z appears in the div E = 0 equation in App D.

Then you end up with I guess

First summary of the E field solutions (D.2.21)

$E_z(r,m,z) = -j(\beta'/\beta_d)\tan(\beta_d z)\frac{K_m}{2} J_m(x)$ $\quad x = \beta' r$ *(D.1.27)*

$E_r(r,m) = a_m x^{-1} J_m(x) + \frac{K_m}{2} J_{m+1}(x)$ $\quad \beta'^2 = \beta^2 - \beta_d^2$ *(D.2.11)*

$jE_\theta(r,m) = -a_m x^{-1} J_m(x) + \left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x)$ *(D.2.15)*

But now β_d has nothing to do with waves, something else will determine it. If we want $J_z(z=0) = J_z(z=h) = 0$ as our nowhere to go conditions, and if $J_z = \sigma E_z$, then $J_z(z=0)= 0$ is already met, and then we have to say that $\sin(\beta_d h) = 0$ where the bottom of the circular disk is at z = 0 and its height is h. This means $\beta_d h = M\pi$ and we then have a new mode number M floating around. For M = 0 there is no solution because we don't get $E_z = 0$ at z = h since $(\beta'/\beta_d)\tan(\beta_d z) = (\beta'/\beta_d)\sin(\beta_d z) = \beta' z$. So $\beta_d = M(\pi/h)$ and if h is small, then β_d is large.

$\beta_d = M(\pi/h)$ $\quad M = 1,2,3...$

Now we don't have to say $K_m = 0$! That is the new benefit.

Maybe NOW we can apply the boundary conditions? I think this is not a good path.

3. What about that stream function gizmo?

What about that stream function gizmo mentioned in the numeric section of "two methods" . That is after all how they DID their numeric stuff. Here is that plan:

A. Numerical Solution and Results

To obtain a numerical solution of the problem, we proceed as follows. We transform Faraday's law, $\vec{\nabla} \times \vec{E} = -\partial \vec{B}/\partial t$, by using Ohm's law, $\vec{E} = \vec{j}/\sigma$, into

$$\vec{\nabla} \times \vec{j} = -\sigma \frac{\partial \vec{B}}{\partial t}. \tag{66}$$

Since div $\vec{j} = 0$, we introduce the current vector potential $\vec{T}$, defined as

$$\vec{j} = \vec{\nabla} \times \vec{T}. \tag{67}$$

Taking into consideration that the plate is thin and combining (66) and (67), we finally obtain

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \sigma \dot{B} \tag{68}$$

where T is the component of $\vec{T}$ normal to the plate. Thus instead of a vector quantity, we consider a scalar one, and the 3-D problem is transformed into a 2-D problem. This scalar quantity is the so-called *stream function,* frequently used for thin plates [5], [6].

Equation (68) has been solved numerically for plates of various shapes. For the numerical analysis, a finite-difference mesh has been considered. The discretization of (68) on the mesh has been performed by using central difference approach for the second derivatives. The resulting set of algebraic equations has been solved with the Gauss–Seidel method [7] by taking into consideration the boundary condition $T = 0$ at the edge of the plate. The number of grid points used in all computations was larger than 800, and it can get as big as 8000 in some of the cases we have examined. This is more

If this is correct, then this whole thing is boiled down to a simple scalar Laplace equation, holy cow, that is sure easier than what I have been doing here. The two refs are these:

[5] A. Krawczyk and J. A. Tegopoulos, *Numerical Modeling of Eddy Currents.* Oxford, U.K.: Clarendon, 1993.

[6] T. Morisue, "Analysis of a coupled problem: The FELIX cantilevered beam," *IEEE Trans. Magn.,* vol. 26, no. 2, pp. 540–543, 1990.

I found another PDF that also uses this T idea. Maybe a little fiddle on this tomorrow.

April 5, 2014 (Sat)

Many things to say concerning all the above work. My main approach was to make use of these two Maxwell equations each of which is a PDE:

curl **E** = -jω **B**
div **E** = 0

My plan was to use for **B** the driving external field and ignore any Lenz Field. So I was hoping to just solve these equations for **E** and be done with it! But when you write it all out, it is just not obvious how you invert that curl equation.

Inverse curl operator idea

Belatedly today I recall a PDF I got about the "inverse curl operator". I looked at it just now, and yes, it does seem to give a general formula for the solution and it does so in any curvilinear orthogonal system! Author notes that the inversion is not unique

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

$$\vec{A} = (\vec{\nabla}\times)^{-1}\vec{B} + \vec{\nabla}\phi$$

where ϕ is an arbitrary scalar function.

So perhaps I could use his formula to find that

$\mathbf{E_1}$ = curl^{-1} [-jω**B**]
E = $\mathbf{E_1}$ + grad ψ
div **E** = 0 but div grad = $\nabla\bullet\nabla = \nabla^2$

so you would end up with

div { curl^{-1} [-jω**B**] } + $\nabla^2\psi$ = 0

and you still have to find ψ, so it would be a long road.

General Remarks about What We Know

Let's go back to our starting point

curl **E** = -jω **B**
div **E** = 0

If we can use **J** = σ**E**, then the second equation implies div **J** = 0. But I well know that in magnetic circumstances, this is not quite true, such as Hall effect and all that stuff. But I think I regard it as

basically true for ω ranges of my interest. But maybe its non-truth in the exact is important in my current problem of interest?

Normally you use the *other* Maxwell equations to show that div **J** = 0 :

curl **H** = ∂_t**D** + **J**
div **D** = ρ
D = ε**E**

Then taking div of the first equation and using div curl = 0 you end up with div **J** = -$\partial_t\rho$.

So inside our metal we do know that ρ = 0 (for all practical purposes) and so div **J** = 0. Maybe this is a separate condition that I have not been using, so I have missing information in my solutions.

Why people use potentials

Now, in lines doc and in King, we know that div **B** = 0 and that allows us to define **A** such that **B** = curl **A** and in the lines doc situation we get to think only about A_z so it is easier to work with a scalar function like A_z than a vector function like **B**.

Similarly, since curl **E** = 0 in electrostatics, we can define **E** = -grad φ and then div **E** = ρ/ε_0 gives us the Poisson equation for φ, and again, we are dealing with a scalar potential instead of a vector function. Think what a mess "potential theory" would be if you tried to only work with **E**.

Theory of the Stream Function T_z

OK, so in a similar vein, we have this plan with the stream function T_z. Since inside a metal we know that div **J** = 0, we can write **J** = curl **T** where **T** is a "vector current potential". In cylindricals for example, we get

$$\mathbf{J} = \text{curl } \mathbf{T} = \hat{\mathbf{r}}\,[\, r^{-1}\partial_\theta T_z - \partial_z T_\theta] + \hat{\boldsymbol{\theta}}\,[\partial_z T_r - \partial_r T_z] + \hat{\mathbf{z}}\,[\, r^{-1}\partial_r(rT_\theta) - r^{-1}\partial_\theta T_r\,]$$

and then we have

$$J_r = r^{-1}\partial_\theta T_z - \partial_z T_\theta$$

$$J_\theta = \partial_z T_r - \partial_r T_z$$

$$J_z = r^{-1}\partial_r(rT_\theta) - r^{-1}\partial_\theta T_r$$

Now consider again

curl **E** = -jω **B**

and suppose we go ahead and assume **J** = σ**E** so this says

curl **J** = (-jωσ) **B**

Then make use of ∇ x (∇ x **A**) = $\nabla(\nabla \bullet A) - \nabla^2 \mathbf{A}$ to write

curl **J** = curl curl **T** = grad(div **T**) - $\nabla^2 \mathbf{T}$

Then we have this equation

grad(div **T**) - $\nabla^2 \mathbf{T}$ = $(-j\omega\sigma)$ **B** (*)

This does not look all that nice, but let's keep going a bit. We have

div **T** = $r^{-1}\partial_r(rT_r) + r^{-1}\partial_\theta T_\theta + \partial_z T_z$

Now I will shoot from the hip, but will read soon. Perhaps you now argue that T_r and T_θ << T_z so you can ignore the first two "transverse terms", just as I keep doing in the King world. Maybe this is an ansatz. You are then saying that

div **T** $\approx$ $\partial_z T_z$

Now take the z component of (*) to get

$\partial_z [\ 0 + 0 + \partial_z T_z] - \nabla^2 T_z = (-j\omega\sigma)\ B_z$

or

$\partial_z^2 T_z - (\nabla_{2D}^2 T_z + \partial_z^2 T_z = (-j\omega\sigma)\ B_z$

or

$-\nabla_{2D}^2 T_z = (-j\omega\sigma)\ B_z$

or

$\nabla_{2D}^2 T_z = \sigma [j\omega\ B_z] = \sigma \dot{B}_z$

which is a 2D Poisson equation for T_z !

Meanwhile we have

$J_r = r^{-1}\partial_\theta T_z - \partial_z T_\theta \approx r^{-1}\partial_\theta T_z$

$J_\theta = \partial_z T_r - \partial_r T_z \approx -\partial_r T_z$

$J_z = r^{-1}\partial_r(rT_\theta) - r^{-1}\partial_\theta T_r \approx 0$

Write the first two in more detail

$J_r(r,\theta) = r^{-1}\partial_\theta T_z(r,\theta)$
$J_\theta(r,\theta) = -\partial_r T_z(r,\theta)$

Now consider our boundary condition at the disk edge! $J_r(r=a,\theta) = 0$. Suppose we did partial waves here so the first equation says

$$J_r(r,m) = r^{-1} jm\, T_z(r,m)$$

Then at least for $m \neq 0$, we find that $J_r(a,m) = 0 \Rightarrow T_z(r,m) = 0$!! I think the m=0 component of T is just the constant part, and the constant part of any "potential" can be ignored so we set $T_z(r,0) = 0$ by fiat, and then we have this very nice fact:

$$J_r(a,\theta) = 0 \Rightarrow T_z(a,\theta) = 0$$

We then have this relatively simple 2D potential theory problem:

$$\nabla_{2D}^2 T_z(r,\theta) = \sigma [j\omega B_z] = \sigma \dot{B}_z \qquad T_z(a,\theta) = 0$$

which is like a 2D Poisson problem where $\dot{B}_z$ is like the charge density. If we can solve this problem, then we have a solution for our currents and E fields since $\mathbf{J} = \sigma\mathbf{E}$ **:**

$$J_r = r^{-1}\partial_\theta T_z$$
$$J_\theta = -\partial_r T_z$$

So the Poisson problem I just derived matches the two methods PDF,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \sigma \dot{B} \qquad (68)$$

This general stream function method can be applied to any shaped thin plate, not just a circular one.

Comment: The above discussion *assumes* at the start that $\mathbf{J} = \sigma\mathbf{E}$. That is how J entered the discussion since we started with curl $\mathbf{E} = -j\omega\mathbf{B}$.

Application of stream method to the uniform B field circular plate

Since we know we will get a symmetric solution, we can write

$$\nabla_{2D}^2 T_z(r,\theta) = r^{-1}\partial_r(r\partial_r T_z)$$

Our problem is then

$$r^{-1}\partial_r(r\partial_r T_z) = \sigma \dot{B}_z \qquad T_z(a,\theta) = 0$$

So integrate to get

$$\partial_r(r\partial_r T_z) = r\sigma \dot{B}_z = r\sigma j\omega B_z \quad \Rightarrow \quad \partial_r(r\partial_r T_z) - r\sigma j\omega B_z = 0$$

$$r\partial_r T_z = \sigma \dot{B}_z (1/2)r^2 + C_1$$

$$\partial_r T_z = \sigma \dot{B}_z (1/2)r + C_1/r$$

$$T_z(r) = \sigma \dot{B}_z (1/4)r^2 + C_1 \ln(r) + C_2$$

$$J_r = r^{-1}\partial_\theta T_z = 0$$

$$J_\theta = -\partial_r T_z = -(1/2)\sigma \dot{B}_z r - C_1/r$$

Since J_θ = finite at disk center, must have $C_1 = 0$ and our solution is then

$$J_r = 0$$

$$J_\theta = -(1/2)\sigma \dot{B}_z r$$

and this agrees with $J_\theta(r) = \sigma(-\dot{B}_z/2) r$ which I found by other means in Problem 1 above! Very excellent! Meanwhile, we can do this:

$$T_z(r) = \sigma \dot{B}_z (1/4)r^2 + C_2 = \sigma \dot{B}_z (1/4)(r^2-a^2) = j\omega\sigma B_z (1/4)(r^2-a^2) \qquad (***)$$

so that $T_z(a) = 0$.

Try Eddy Problem 2 with this stream method!

Since no longer have symmetry, we have

$$\nabla_{2D}^2 T_z(r,\theta) = \sigma[j\omega B_z] = \sigma \dot{B}_z \qquad T_z(a,\theta) = 0$$

or

$$r^{-1}\partial_r(r\partial_r T_z) + r^{-2}\partial_\theta^2 T_z = \sigma \dot{B}_z(r,\theta) = j\omega\sigma B_z = j\omega\sigma[B_0 - \alpha\, r\cos\theta] \qquad T_z(a,\theta) = 0$$

or

$$\partial_r(r\partial_r T_z) + r^{-1}\partial_\theta^2 T_z = \sigma \dot{B}_z(r,\theta) = j\omega\sigma B_z r = j\omega\sigma[B_0 - \alpha\, r\cos\theta]r \qquad T_z(a,\theta) = 0$$

Now take this to partial waves as per App D to get

$$r^{-1}\partial_r(r\partial_r T_z(r,m)) - r^{-2}m^2 T_z(r,m) = \sigma \dot{B}_z(r,m) = j\omega\sigma B_z(r,m)$$

This is a very simple ODE driven by the right side, I like it! Now we have this assumed B field (in ω space)

$$B_z(r,\theta) = B_0 - \alpha\, r\cos\theta \qquad \text{note that } \dim(\alpha) = B/m$$

so that

$$B_z(r,m) = (1/2\pi)\int_{-\pi}^{\pi} d\theta\, B_z(r,\theta)\, e^{-jm\theta} = (1/2\pi)\int_{-\pi}^{\pi} d\theta\, [B_0 - \alpha\, r\cos\theta]\, e^{-jm\theta}$$

$$= B_0\ (1/2\pi)\int_{-\pi}^{\pi} d\theta\, e^{-jm\theta} - \alpha\, r\ (1/2\pi)\int_{-\pi}^{\pi} d\theta\, \cos\theta\, e^{-jm\theta}$$

$$= B_0\ (1/2\pi)\int_{-\pi}^{\pi} d\theta\, \cos(m\theta) - \alpha\, r\, (1/2\pi)\int_{-\pi}^{\pi} d\theta\, \cos\theta\, \cos(m\theta)$$

$$= B_0\ (1/\pi)\int_{0}^{\pi} d\theta\, \cos(m\theta) - \alpha\, r\ (1/\pi)\int_{0}^{\pi} d\theta\, \cos\theta\, \cos(m\theta)$$

But Spiegel page 96 says

$$\int_0^{\pi} d\theta\, \cos(n\theta)\cos(m\theta) = 0 \text{ if } m \neq n; \quad = \pi/2 \text{ if } m = n \neq 0; \quad = \pi \text{ if } m = n = 0$$

So I find that

$$B_z(r,m) = B_0\ (1/\pi)\int_{0}^{\pi} d\theta\, \cos(0\theta)\cos(m\theta) - \alpha\, r\, (1/\pi)\int_{0}^{\pi} d\theta\, \cos(1\theta)\cos(m\theta)$$

$$= \delta_{m,0} B_0\ (1/\pi)\,\pi - \alpha\, r\ (1/\pi)\delta_{m,\pm 1}\,\pi/2$$

$$= \delta_{m,0}\, B_0 - (\alpha/2)\, r\, \delta_{m,\pm 1}$$

So we then have

$$r^{-1}\partial_r(r\partial_r T_z(r,m)) - r^{-2}m^2 T_z(r,m) = j\omega\sigma[\delta_{m,0}\, B_0 - (\alpha/2)\, r\, \delta_{m,\pm 1}] \qquad T_z(a,m) = 0$$

Now for $m \neq 0$ and $m \neq \pm 1$, we know that $T_z(r,m) = 0$ from potential theory of Laplace's equation where the function has no source and vanishes on a closed boundary. Great! So this leaves us with these two equations for non-zero solutions:

$$r^{-1}\partial_r(r\partial_r T_z(r,0)) = j\omega\sigma B_0 \qquad \text{EQ1}$$

$$r^{-1}\partial_r(r\partial_r T_z(r,\pm 1)) - r^{-2}T_z(r,\pm 1) = -j\omega\sigma\, r\, (\alpha/2) = -A\, r \qquad \text{EQ2}$$

I just solved EQ1 above and found that

$$T_z(r,0) = j\omega\,\sigma\,B_0\,(1/4)r^2 + C_1\,\ln(r) + C_2$$

We cannot have ln(r) so $C_1 = 0$ and then

$$T_z(r,0) = j\omega\,\sigma\,B_0\,(1/4)r^2 + C_2$$

The boundary condition $T_z(a,0) = 0$ says

$$C_2 = -\,j\omega\,\sigma\,B_0\,(1/4)a^2$$

and therefore

$$T_z(r,0) = j\omega\,\sigma\,B_0\,(1/4)(r^2-a^2)$$

EQ2 looks Besslesque to me. Write as

$$r^{-1}\partial_r(r\partial_r T_z(r,\pm1)) - r^{-2}T_z(r,\pm1) = -A\,r$$

$$\partial_r(r\partial_r u) - (1/r)u = -Ar^2 \qquad A = j\omega\sigma\,(\alpha/2)$$

Hand this off to Maple. It is NOT Bessel after all:

```
eq := Diff(r*Diff(u(r),r),r)- (1/r)*u(r)- A*r^2 = 0;
```

$$eq := \left(\frac{\partial}{\partial r} r \left(\frac{\partial}{\partial r} u(r)\right)\right) - \frac{u(r)}{r} - A\,r^2 = 0$$

```
dsolve(value(eq)): expand(%,r);
```

$$u(r) = \frac{1}{8}A\,r^3 + _C1\,r - \frac{_C1}{r} + _C2\,r + \frac{_C2}{r}$$

So our solution is then this:

$$T_z(r,\pm1) = -\,(1/8)Ar^3 + C_1(r-1/r) + C_2(r+1/r)$$

$$= -\,(1/8)Ar^3 + r\,(C_1+C_2) + (1/r)(-C_1+C_2)$$

To be finite at r = 0 we have to pick $C_1 = C_2$ and then

$$T_z(r,\pm1) = -\,(1/8)Ar^3 + C_3\,r = j\omega\sigma(\alpha/16)\,r^3 + C_3 r \qquad A = j\omega\sigma\,(\alpha/2)$$

Wow, I like it. The boundary condition is

$$0 = T_z(a,\pm 1) = -j\omega\sigma(\alpha/16)\, a^3 + C_3 a \qquad \Rightarrow \qquad C_3 = +j\omega\sigma(\alpha/16)\, a^2$$

so then

$$T_z(r,\pm 1) = -(1/8)Ar^3 + C_3 r = -j\omega\sigma(\alpha/16)\, r^3 + j\omega\sigma(\alpha/16)\, a^2 r$$

$$= -j\omega\sigma(\alpha/16) r(r^2-a^2) \qquad \text{ok pass 3}$$

So here is a summary of our partial wave solutions for the vector current potential component T_z :

$$T_z(r,0) = j\omega\,\sigma\, B_0\, (1/4)(r^2-a^2)$$

$$T_z(r,\pm 1) = -j\omega\sigma(\alpha/16) r(r^2-a^2)$$

$$T_z(r,m) = 0 \qquad \text{for all other values of m}$$

Very good, now get back to θ space

$$T_z(r,\theta) = T_z(r,m)\, e^{jm\theta} = T_z(r,0) + T_z(r,1)\, e^{j\theta} + T_z(r,-1)\, e^{-j\theta}$$

$$= T_z(r,0) + T_z(r,1)\, 2\cos\theta$$

$$= j\omega\,\sigma\, B_0\, (1/4)(r^2-a^2) - j\omega\sigma(\alpha/8) r(r^2-a^2)\cos\theta$$

$$= j\omega\sigma\, [B_0\, (1/4)(r^2-a^2) - (\alpha/8) r(r^2-a^2)\cos\theta]$$

$$= j\omega\sigma\, [B_0\, (1/4)(r^2-a^2) - (\alpha/8)(r^3-a^2 r)\cos\theta]$$

Note that for α = 0, this $T_z(r,\theta)$ agrees with (***) above found for Problem 1! So the B_0 term is certainly correct.

Wow, a genuine answer. Then we get

$$J_r = r^{-1}\partial_\theta T_z = -r^{-1}\, j\omega\sigma(\alpha/8) r(r^2-a^2)\,[-\sin\theta] = +j\omega\sigma(\alpha/8)\,(r^2-a^2)\sin\theta$$

$$J_\theta = -\partial_r T_z = -\partial_r \{ j\omega\sigma\, [B_0\, (1/4)(r^2-a^2) - (\alpha/8)(r^3-a^2 r)\cos\theta]\}$$

$$= -j\omega\sigma \{ B_0\, (1/2)\, r - (\alpha/8)(3r^2-a^2)\cos\theta \}$$

To summarize,

$$J_r(r,\theta) = j\omega\sigma\, [+ (\alpha/8)\,(r^2-a^2)\sin\theta\,]$$

$J_\theta(r,\theta) = j\omega\sigma\,[-(1/2)B_0 r + (\alpha/8)\cos\theta\,[\,3r^2 - a^2\,]$

and

$E_r(r,\theta) = j\omega\,[\,+(\alpha/8)\,(r^2-a^2)\sin\theta\,]$

$E_\theta(r,\theta) = j\omega\,[-(1/2)B_0 r + (\alpha/8)\cos\theta\,[\,3r^2 - a^2\,]$

Now I want to **run a lot of checks,** since I will no doubt publish this result.

- both currents proportional to jω as we expect
- $J_r(a,\theta) = 0$
- when α = 0, we duplicate the uniform B result
- Verify that $T_z(r,\theta)$ satisfies the stream PDE: First, here is that PDE after I enter it as eq in Maple

$$eq := \left(\frac{\partial}{\partial r} r \left(\frac{\partial}{\partial r} T(r,\theta)\right)\right) + \frac{\frac{\partial^2}{\partial \theta^2} T(r,\theta)}{r} - j\,\omega\,\sigma\,(B0 - \alpha\, r\cos(\theta))\, r = 0$$

This matches a result above

$\partial_r(r\partial_r T_z) + r^{-1}\partial_\theta{}^2 T_z = \sigma\,\dot{B}_z(r,\theta) = j\omega\sigma B_z r = j\omega\sigma[B_0 - \alpha\, r\cos\theta]r$ $\qquad$ $T_z(a,\theta) = 0$

I then enter T from above and see if I get 0=0 for the equation

```
T := r ->j*omega*sigma*( B0*(1/4)*(r^2-a^2)-(alpha/8)*r*(r^2-a^2)*cos(theta));
```

$$T := r \to j\,\omega\,\sigma\left(\frac{1}{4} B0\,(r^2-a^2) - \frac{1}{8}\alpha\, r\,(r^2-a^2)\cos(\theta)\right)$$

```
eq;
```

$$\left(\frac{\partial}{\partial r} r\left(\frac{\partial}{\partial r} j\,\omega\,\sigma\left(\frac{1}{4} B0\,(r^2-a^2) - \frac{1}{8}\alpha\, r\,(r^2-a^2)\cos(\theta)\right)\right)\right) + \frac{1}{8} j\,\omega\,\sigma\,\alpha\,(r^2-a^2)\cos(\theta) - j\,\omega\,\sigma\,(B0 - \alpha\, r\cos(\theta))\, r = 0$$

```
value(%);
```

$$\omega\,\sigma\left(\frac{1}{2} B0\, r - \frac{1}{8}\alpha\,(r^2-a^2)\cos(\theta) - \frac{1}{4}\alpha\, r^2\cos(\theta)\right) + j\, r\,\omega\,\sigma\left(\frac{1}{2} B0 - \frac{3}{4}\alpha\, r\cos(\theta)\right) + \frac{1}{8} j\,\omega\,\sigma\,\alpha\,(r^2-a^2)\cos(\theta)$$
$$- j\,\omega\,\sigma\,(B0 - \alpha\, r\cos(\theta))\, r = 0$$

```
simplify(%);
```

$$0 = 0$$

- Now I want to look at my big two PDE's of yesterday which were

$\partial_r(rE_\theta) - \partial_\theta E_r = -j\omega\,[B_0 - \alpha\, r\cos\theta]\, r$	PDE#1	curl **B** = -jω**E**
$\partial_r(rE_r) + \partial_\theta E_\theta = 0$	PDE#2	div **E** = 0

Here is the code and it works! First here is all the entry

```
restart: alias(I=I,j=sqrt(-1));
```

```
pde1 := Diff(r*Eth(r,theta),r) - Diff(Er(r,theta),theta) = -j*omega*(B0-alpha*r*cos(theta))*r;
```

$$pde1 := \left(\frac{\partial}{\partial r} r\,\mathrm{Eth}(r,\theta)\right) - \left(\frac{\partial}{\partial \theta}\mathrm{Er}(r,\theta)\right) = -j\,\omega\,(B0 - \alpha\, r\cos(\theta))\,r$$

```
pde2 := Diff(r*Er(r,theta),r) + Diff(Eth(r,theta),theta) = 0;
```

$$pde2 := \left(\frac{\partial}{\partial r} r\,\mathrm{Er}(r,\theta)\right) + \left(\frac{\partial}{\partial \theta}\mathrm{Eth}(r,\theta)\right) = 0$$

```
Er := (r,theta) -> j*omega*(alpha/8)*(r^2-a^2)*sin(theta);
```

$$Er := (r,\theta) \to \frac{1}{8} j\,\omega\,\alpha\,(r^2 - a^2)\sin(\theta)$$

```
Eth := (r,theta) -> j*omega* ( -B0*r/2 + (alpha/8)*cos(theta)*(3*r^2-a^2) );
```

$$Eth := (r,\theta) \to j\,\omega\left(-\frac{1}{2} B0\, r + \frac{1}{8}\alpha\cos(\theta)\,(3\,r^2 - a^2)\right)$$

Then here is evaluation of the two PDE's

```
pde1;
```

$$\left(\frac{\partial}{\partial r} j\,r\,\omega\left(-\frac{1}{2} B0\, r + \frac{1}{8}\alpha\cos(\theta)\,(3\,r^2 - a^2)\right)\right) - \left(\frac{\partial}{\partial \theta}\left(\frac{1}{8} j\,\omega\,\alpha\,(r^2 - a^2)\sin(\theta)\right)\right) = -j\,\omega\,(B0 - \alpha\, r\cos(\theta))\,r$$

```
value(%);
```

$$j\,\omega\left(-\frac{1}{2} B0\, r + \frac{1}{8}\alpha\cos(\theta)\,(3\,r^2 - a^2)\right) + j\,r\,\omega\left(-\frac{1}{2} B0 + \frac{3}{4}\alpha\, r\cos(\theta)\right) - \frac{1}{8} j\,\omega\,\alpha\,(r^2 - a^2)\cos(\theta) = -j\,\omega\,(B0 - \alpha\, r\cos(\theta))\,r$$

```
simplify(%);
```

$$-j\,\omega\, B0\, r + j\,\omega\,\alpha\cos(\theta)\,r^2 = -j\,\omega\, B0\, r + j\,\omega\,\alpha\cos(\theta)\,r^2$$

```
rhs(%)-lhs(%);
```

$$0$$

Then

```
pde2;
```

$$\left(\frac{\partial}{\partial r}\left(\frac{1}{8} j\,r\,\omega\,\alpha\,(r^2 - a^2)\sin(\theta)\right)\right) + \left(\frac{\partial}{\partial \theta} j\,\omega\left(-\frac{1}{2} B0\, r + \frac{1}{8}\alpha\cos(\theta)\,(3\,r^2 - a^2)\right)\right) = 0$$

```
value(%);
```

$$\frac{1}{8} j\,\omega\,\alpha\,(r^2 - a^2)\sin(\theta) + \frac{1}{4} j\,r^2\,\omega\,\alpha\sin(\theta) - \frac{1}{8} j\,\omega\,\alpha\sin(\theta)\,(3\,r^2 - a^2) = 0$$

```
simplify(%);
```

$$0 = 0$$

and all seems well. Here is an alternative verification using Maple's fancy diff ops:

```
> restart: alias(I=I,j=sqrt(-1)): with(linalg):
Warning, new definition for norm
Warning, new definition for trace
> Er := j*omega*(alpha/8)*(r^2-a^2)*sin(theta);
```

$$Er := \frac{1}{8} j\,\omega\,\alpha\,(r^2 - a^2)\sin(\theta)$$

```
> Eth := j*omega* ( -B0*r/2 + (alpha/8)*cos(theta)*(3*r^2-a^2) );
```

$$Eth := j\,\omega\left(-\frac{1}{2}B0\,r + \frac{1}{8}\alpha\cos(\theta)\,(3\,r^2 - a^2)\right)$$

```
> E := [Er,Eth,0];
```

$$E := \left[\frac{1}{8} j\,\omega\,\alpha\,(r^2 - a^2)\sin(\theta), j\,\omega\left(-\frac{1}{2}B0\,r + \frac{1}{8}\alpha\cos(\theta)\,(3\,r^2 - a^2)\right), 0\right]$$

```
> c := [r,theta,phi];
```

$$c := [r, \theta, \phi]$$

```
> curl(E,c,coords=cylindrical): simplify(%);
```

$$[0, 0, -j\,\omega\,B0 + j\,\omega\,\alpha\cos(\theta)\,r]$$

```
> diverge(E,c,coords=cylindrical);
```

$$0$$

So, here is our Problem 2 solution,

$J_r(r,\theta) = j\omega\sigma\,[\, + (\alpha/8)\,(r^2 - a^2)\sin\theta\,]$

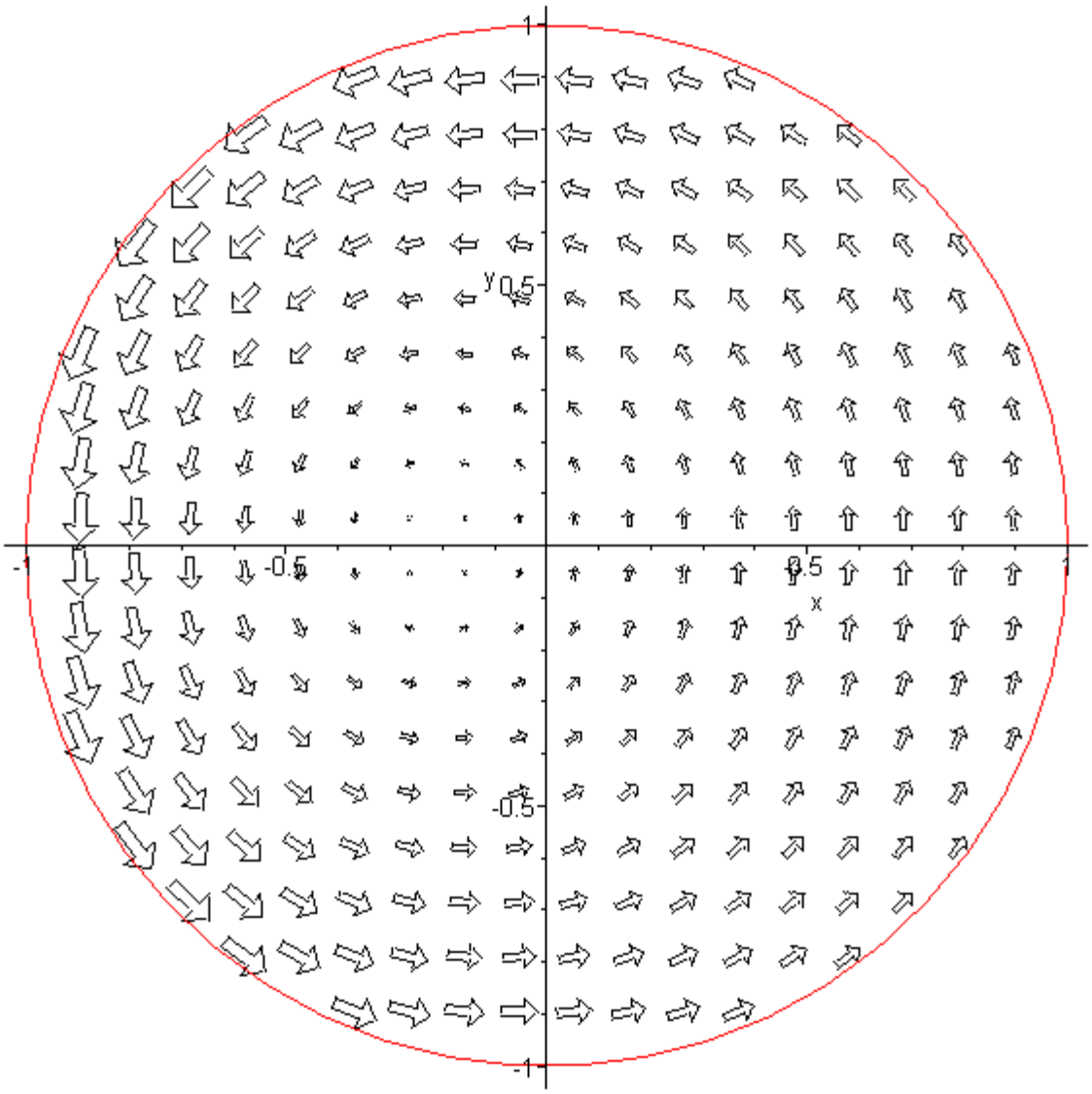
$J_\theta(r,\theta) = j\omega\sigma\,[-(1/2)B_0 r + (\alpha/8)\cos\theta\,[\,3r^2 - a^2\,]$

We can of course write

$J_x = J_r\,\hat{\mathbf{r}}\bullet\hat{\mathbf{x}} + J_\theta\hat{\boldsymbol{\theta}}\bullet\hat{\mathbf{x}} = J_r\cos\theta - J_\theta\sin\theta$

$J_y = J_r\,\hat{\mathbf{r}}\bullet\hat{\mathbf{y}} + J_\theta\hat{\boldsymbol{\theta}}\bullet\hat{\mathbf{y}} = J_r\sin\theta + J_\theta\cos\theta$

OK, here is some new plotting code. I replicate the old plot of $\alpha = 0$, but here is a plot with $B_0 = 1$ and $\alpha = 1$ (in file "problem2.mws")



This is what I have long been looking for!!! Several days now. It shows the proximity effect induced by having a perp B field with a gradient! The current is larger where the field is larger! I would like to get some field lines going here!! You can see that the field lines are no longer circles centered at the center. And you can see that current is tangent at the circle edge. Wow, finally!

Each question spawns five new questions:

What about the force on the above plate?	skip this item
What about the Lenz B field?	DONE BELOW
What about the T components that I neglected? Was that justified?	DONE BELOW
Can you make a skin effect comment here?	DONE BELOW
Why was I able to ignore Magnetic Ohm's Law in my solution?	DONE BELOW

Magnetic Ohm's Law and the Stream Problem 2 Solution

In Cartesians we know from App N that the MOL solution is this

$$J_x = \sigma\,(E_x + \omega_c\tau\, E_y)\,\frac{1}{1+(\omega_c\tau)^2} \qquad \omega_c \equiv (qB/m)$$

$$J_y = \sigma\,(E_y - \omega_c\tau\, E_x)\,\frac{1}{1+(\omega_c\tau)^2} \qquad \mathbf{B} = B\hat{\mathbf{z}} \qquad \sigma = (nq^2\tau/m)$$

$$J_z = \sigma E_z \ . \qquad \sigma = (nq^2\tau/m) \qquad \text{(N.4.10)}$$

which I now convert to cylindricals,

$$J_r = \sigma\,(E_r + \omega_c\tau\, E_\theta)\,\frac{1}{1+(\omega_c\tau)^2} \qquad \omega_c \equiv (qB/m)$$

$$J_\theta = \sigma\,(E_\theta - \omega_c\tau\, E_r)\,\frac{1}{1+(\omega_c\tau)^2} \qquad \mathbf{B} = B\hat{\mathbf{z}} \qquad \sigma = (nq^2\tau/m)$$

$$J_z = \sigma E_z \ . \qquad \sigma = (nq^2\tau/m) \qquad \text{(N.4.10)}$$

In an analysis in which MOL was respected, you would have the BC that $J_r(a,\theta) = 0$ so that

$$E_r(a,\theta) + \omega_c\tau\, E_\theta(a,\theta) = 0 \qquad \text{precision BC}$$

The MOL shown above of course *violates* the regular Ohm's law unless $\omega_c\tau << 1$. Since the stream method *assumes* regular Ohm's Law, we may conclude that the stream method is only valid when $\omega_c\tau << 1$. Recall that our stream solutions above were.

$$E_r(r,\theta) = j\omega\,[\,+(\alpha/8)\,(r^2-a^2)\sin\theta\,] \qquad E_r(a,\theta) = 0$$

$$E_\theta(r,\theta) = j\omega\,[-(1/2)B_0 r\, + (\alpha/8)\cos\theta\,[\,3r^2 - a^2\,] \qquad E_\theta(a,\theta) = j\omega\,[-(1/2)B_0 a\, + (\alpha/8)\cos\theta\, a^2\,]$$

Thus, you cannot expect the "precision BC" above to be respected by these fields since these fields were derived in a way that *ignored* the MOL. Indeed the precision BC is violated to order $\omega_c\tau$, no surprise.

Stream Method for E ?

Could we redo the stream method entirely in terms of the E field and not use ROL (regular)? Let's try in with copy paste and edit:

Since div **E** = 0 inside the plate, we try

$$\mathbf{E} = \text{curl}\,\mathbf{T} = \hat{\mathbf{r}}\,[\,r^{-1}\partial_\theta T_z - \partial_z T_\theta] + \hat{\boldsymbol{\theta}}\,[\partial_z T_r - \partial_r T_z] + \hat{\mathbf{z}}\,[\,r^{-1}\partial_r(rT_\theta) - r^{-1}\partial_\theta T_r\,]$$

and then we have

$$E_r = r^{-1}\partial_\theta T_z - \partial_z T_\theta$$

$E_\theta = \partial_z T_r - \partial_r T_z$

$E_z = r^{-1}\partial_r(rT_\theta) - r^{-1}\partial_\theta T_r$

We again argue that T_r and T_θ << T_z and we end up with

$\nabla_{2D}^2 T_z = \sigma [j\omega B_z] = \sigma \dot{B}_z$

But now we cannot use the boundary condition $T_z(a,\theta) = 0$! See derivation above which is then flawed because

$E_r(a,\theta) = 0 \Rightarrow T_z(a,\theta) = 0$ is useless

It is useless because we only know that $J_r = 0$ at the edge, not that $E_r = 0$. Recall in the Hall effect that in fact we have $J_r = 0$ at the edge but $E_r \neq 0$. So this stream approach does not fly!

The **Conclusion** is this: The stream method is valid as long as $\omega_c\tau$ << 1

Why can the transverse T components be neglected?

In the stream method we use ROL to start with curl **J** = (-jωσ) **B** and **J** = curl **T .** We can then write out the Maxwell components to get

$J_r = r^{-1}\partial_\theta T_z - \partial_z T_\theta$
$J_\theta = \partial_z T_r - \partial_r T_z$
$J_z = r^{-1}\partial_r(rT_\theta) - r^{-1}\partial_\theta T_r$

If the plate is thin, we argue that everything is constant in the z direction so then $T_i = T_i(r,\theta)$ only. Then we can discard ∂_z wherever it appears, and we then have

$J_r = r^{-1}\partial_\theta T_z$
$J_\theta = - \partial_r T_z$
$J_z = r^{-1}\partial_r(rT_\theta) - r^{-1}\partial_\theta T_r$

We then get "everything we want" from the first two equations. In order to "be happy with" the third equation, we then argue that $T_r \approx$ constant and $T_\theta \approx$ constant/r so that $J_z \approx 0$. Alternatively, we could just hope that $T_r \approx 0$ and $T_\theta \approx 0$ so again $J_z \approx 0$.

If we maintain these components and look back at our derivation, we have

curl curl **T** = (-jωσ) **B** (*)

so we really do have three equations here. If we don't assume smallness on the transverse components, we get three hugely complicated equations involving double derivatives. Just for Maple practice, I will have it show these three equations, see "stream mess.mws"

```
eq1 := curlcurlT[1] = -j*omega*sigma*Br:expand(%,r);
```

$$\frac{\frac{\partial}{\partial\theta}\mathrm{Tth}(r,\theta,z)}{r^2}+\frac{\frac{\partial^2}{\partial\theta\,\partial r}\mathrm{Tth}(r,\theta,z)}{r}-\frac{\frac{\partial^2}{\partial\theta^2}\mathrm{Tr}(r,\theta,z)}{r^2}-\left(\frac{\partial^2}{\partial z^2}\mathrm{Tr}(r,\theta,z)\right)+\left(\frac{\partial^2}{\partial z\,\partial r}\mathrm{Tz}(r,\theta,z)\right)=-j\,\omega\,\sigma\,Br$$

```
eq2 := curlcurlT[2] = -j*omega*sigma*Bth:expand(%,r);
```

$$\frac{\frac{\partial^2}{\partial z\,\partial\theta}\mathrm{Tz}(r,\theta,z)}{r}-\left(\frac{\partial^2}{\partial z^2}\mathrm{Tth}(r,\theta,z)\right)+\frac{\mathrm{Tth}(r,\theta,z)}{r^2}-\frac{\frac{\partial}{\partial r}\mathrm{Tth}(r,\theta,z)}{r}-\frac{\frac{\partial}{\partial\theta}\mathrm{Tr}(r,\theta,z)}{r^2}-\left(\frac{\partial^2}{\partial r^2}\mathrm{Tth}(r,\theta,z)\right)+\frac{\frac{\partial^2}{\partial\theta\,\partial r}\mathrm{Tr}(r,\theta,z)}{r}=-j\,\omega\,\sigma\,Bth$$

```
eq3 := curlcurlT[3] = -j*omega*sigma*Bz:expand(%,r);
```

$$\frac{\frac{\partial}{\partial z}\mathrm{Tr}(r,\theta,z)}{r}-\frac{\frac{\partial}{\partial r}\mathrm{Tz}(r,\theta,z)}{r}+\left(\frac{\partial^2}{\partial z\,\partial r}\mathrm{Tr}(r,\theta,z)\right)-\left(\frac{\partial^2}{\partial r^2}\mathrm{Tz}(r,\theta,z)\right)-\frac{\frac{\partial^2}{\partial\theta^2}\mathrm{Tz}(r,\theta,z)}{r^2}+\frac{\frac{\partial^2}{\partial z\,\partial\theta}\mathrm{Tth}(r,\theta,z)}{r}=-j\,\omega\,\sigma\,Bz$$

If I assume that the components do not depend on z, the results are at least simpler,

```
eq1 := curlcurlT[1] = -j*omega*sigma*Br:expand(%,r);
```

$$\frac{\frac{\partial}{\partial\theta}\mathrm{Tth}(r,\theta)}{r^2}+\frac{\frac{\partial^2}{\partial\theta\,\partial r}\mathrm{Tth}(r,\theta)}{r}-\frac{\frac{\partial^2}{\partial\theta^2}\mathrm{Tr}(r,\theta)}{r^2}=-j\,\omega\,\sigma\,Br$$

```
eq2 := curlcurlT[2] = -j*omega*sigma*Bth:expand(%,r);
```

$$\frac{\mathrm{Tth}(r,\theta)}{r^2}-\frac{\frac{\partial}{\partial r}\mathrm{Tth}(r,\theta)}{r}-\frac{\frac{\partial}{\partial\theta}\mathrm{Tr}(r,\theta)}{r^2}-\left(\frac{\partial^2}{\partial r^2}\mathrm{Tth}(r,\theta)\right)+\frac{\frac{\partial^2}{\partial\theta\,\partial r}\mathrm{Tr}(r,\theta)}{r}=-j\,\omega\,\sigma\,Bth$$

```
eq3 := curlcurlT[3] = -j*omega*sigma*Bz:expand(%,r);
```

$$-\frac{\frac{\partial}{\partial r}\mathrm{Tz}(r,\theta)}{r}-\left(\frac{\partial^2}{\partial r^2}\mathrm{Tz}(r,\theta)\right)-\frac{\frac{\partial^2}{\partial\theta^2}\mathrm{Tz}(r,\theta)}{r^2}=-j\,\omega\,\sigma\,Bz$$

The last equation is basically that of the stream theory which says $\nabla_{2D}{}^2 T_z = \sigma\,[j\omega\,B_z]$. If we have no sources for the first two equations, then reasonable solutions to these equations are $T_r = T_\theta = 0$. Notice that the first two equations do not contain T_z anywhere. But for a thick plate, these Laplace-like equations could have non-zero solutions even if no source on the right. I imagine these somehow as "modes" of the solid resonant cylinder that are excited by the applied B field, and this is not a subject I want to pursue.

Thinking of my sine business earlier and BC's at top and bottom of the plate, perhaps you can use the thin plate theory as long as the plate thickness is much smaller than your ω wavelength! If you cannot fit a

half sine wave vertically, you cannot have a mode. Inside a solid copper cylinder we have the Helm equation and it says $k = \beta = 2\pi/\lambda$ where

$\beta = (j - 1)\sqrt{\omega\mu\sigma/2}$ real part $k = -\sqrt{\omega\mu\sigma/2} = 2\pi/\lambda$

$\lambda = 2\pi/\sqrt{\omega\mu\sigma/2} = 2\pi\sqrt{2/\omega\mu\sigma} = 2\pi\,\delta$

So the wavelength is the skin depth. So I would guess that my result above applies as long as the plate thickness is much smaller that δ. At 100 Hz we have δ = 0.66 cm for copper. If plate is thinner than this, perhaps you cannot get the J_z mode going, ball park.

Eddy Current Testing

I just did some reading on a very popular topic **"eddy current testing"** or ECT. Lots of papers on this practical non-destructive testing topic! They are typically looking for cracks that are maybe up to 15 mm below the surface, or less. They note that (1) lower ω increases skin depth so get stronger penetrating signal, but (2) lower ω reduces scattering from a defect. (3) lower ω reduces resolution of the scattering data. Therefore, for a given depth there is some optimal ω that maximizes the scattered defect signal. One paper looking for deep defects at 15 mm (1.5 cm) used 16Hz to 64Hz. But another paper looking at smaller depths used 1.5 KHz. Airplane wheel system with SQUID sensitive detectors uses 100-300Hz.

Skin Effect and Problem 2

As noted above, I think you need the plate thickness to be << δ to shut down z direction action. But as for the transverse field and current pattern, it is true that it is smaller at the center, but I think that is just because of Faraday's emf being small on small loops near the center since flux is small etc. Recall

$E_r(r,\theta) = j\omega\,[\,+(\alpha/8)\,(r^2-a^2)\sin\theta\,]$ $\qquad E_r(a,\theta) = 0$

$E_\theta(r,\theta) = j\omega\,[-(1/2)B_0 r + (\alpha/8)\cos\theta\,[\,3r^2 - a^2\,]$ $\qquad E_\theta(a,\theta) = j\omega\,[-(1/2)B_0 a + (\alpha/8)\cos\theta\, a^2\,]$

There is no radial damping that is related to ω here! Of course signal increases linearly with ω. So I claim there is no transverse skin effect visible in these solutions. The B field always penetrates in the z direction all the way!

What about the Lenz's Law B field?

Suppose we start with our driving B field having some extent larger than the plate. We induce the eddy currents. The eddy currents then create a dipole-like Lenz B field in response and this field then extends out beyond the external driving field's radius. I guess you could make some kind of simple estimate of the size of this field. Maybe compute the total circulating current I and assume you have a ring of that current which has radius a and then look up the field made by such a ring.

What is the magnetic field on axis of a current ring radius a? See B&B p 150

$B_z = (1/2)\mu\, Ia^2 / (z^2+a^2)^{3/2}$ on axis distance z from the ring

At the center of the ring we then get

$B_z(z=0) = (1/2)\mu\, I a^2/a^3 = (1/2)\,\mu I/a$ dim(RHS) = hen/m *amp/m = amp-hen/m^2 = OK

Let's go back to the symmetric Problem 1 where

$I = -(1/4)\, h\, \sigma a^2 \dot{B}_z = -(1/4)\, h\, \sigma a^2 j\omega B_z$

Then we get

$B_z^{Lenz} = -(1/4)\, h\, \sigma a^2 j\omega B_o * (1/2)\, \mu\, a^2 / (z^2+a^2)^{3/2}$

$$= \{ -j\omega\, (1/8)\, \sigma\mu h a \frac{a^3}{(z^2+a^2)^{3/2}} \}\ B_o$$

$\dim(\omega\sigma\mu h a) = \sec^{-1}$ mho m^{-1} hen m^{-1} m^2 = $\sec^{-1}$ mho hen = $\sec^{-1}$ sec = 1

So we have a dimensionless fraction sitting there, so I can try some numbers. Maple

$\omega\, (1/8)\, \sigma\mu h a$ = ?

```
2*Pi*f*(1/8)*sigma0*mu0*(.01)*(.001):evalf(%);
```

.0005734240159 f

So if f = 60 Hz, this dimensionless quantity is .0006*60 = .036 = 3.6%

So there is my first-ever estimate of the Lenz field which I am ignoring at 60 Hz. Fine.

Stream theory for rectangular x,y z:

$\mathbf{J} = \text{curl}\ \mathbf{T} = \hat{\mathbf{x}}\,(\partial_y T_z - \partial_z T_y) + \hat{\mathbf{y}}\,(\partial_z T_x - \partial_x T_z) + \hat{\mathbf{z}}\,(\partial_x T_y - \partial_y T_x)$

$J_x = \partial_y T_z - \partial_z T_y$
$J_y = \partial_z T_x - \partial_x T_z$
$J_z = \partial_x T_y - \partial_y T_x$

curl $\mathbf{J}$ = $(-j\omega\sigma)$ $\mathbf{B}$

curl $\mathbf{J}$ = curl curl $\mathbf{T}$ = grad(div $\mathbf{T}$) - $\nabla^2\mathbf{T}$

grad(div $\mathbf{T}$) - $\nabla^2\mathbf{T}$ = $(-j\omega\sigma)$ $\mathbf{B}$

$T_x, T_y << T_z$

$$\text{div } \mathbf{T} \approx \partial_z T_z$$

$$\partial_z [\, 0 + 0 + \partial_z T_z] \; - \nabla^2 T_z \;\; = \;\; (-j\omega\sigma)\, B_z$$

$$\nabla_{2D}^2 T_z \;\; = \sigma\, [j\omega\, B_z] \;\; = \sigma\, \dot{B}_z$$

OK, same as before.

Let's put the upper and lower edges at y = ±a and call it the center plane of cylinder. Then we have

$T_z(x,\pm a) = 0$ since no current there.

So here is our problem:

$(\partial_x^2 + \partial_y^2)\, T_z(x,y) = j\omega\sigma[B_0 + \alpha y]$ $T_z(x,\pm a) = 0$ field stronger at the top

Maybe throw in $T_z(0,y) = T_z(L,y)$ at the left and right ends.

The general solution is

$$\varphi(x) = \int_R dx'\, g(x|x')\, q(x') \; - \int_\sigma dS_\xi\, f(\xi)\, \partial_{\xi n} g(x|\xi) \qquad \text{// Stakgold (6.81) .} \qquad (1.5.11)$$

But since f(ξ) = 0 on the entire rectangular boundary, we just get

$$\varphi(x) = \int_R dx'\, g(x|x')\, q(x')$$

The Green's function will be some mess like this

$$g(x,y|x',y') = \Sigma_m \Sigma_{n \;\; odd} \cos(\, n\pi y/2a)\, \cos(\, n\pi y'/2a) \sin(m\pi x/L)\, \sin(m\pi x'/L) A_{n,m}$$

based on this idea

$$g(\mathbf{x}|\boldsymbol{\xi}) \;\; = \Sigma_n\, \varphi_{\lambda n}(\mathbf{x}) \bar{\varphi}_{\lambda n}(\boldsymbol{\xi}) \,/\, \lambda_n$$

So yes, I could do this problem and keep some finite number of terms and try to make a plot.

It would be nice to have a tool that could just do a numerical solution! Maple can only do this with ODE not PDE. I looked around, $1000 for a package, various people have them. I could write my own. This is not for today!

[This section Added 4/30/14.] In my App P version 2 writeup I have these two PDE's

$$\partial_r(rE_\theta) - \partial_\theta E_r = -\, j\omega[B_0 - \alpha\, r\cos\theta]r \;\; \equiv \;\; p(r,\theta)$$

$$\partial_{\mathbf{r}}(rE_{\mathbf{r}}) + \partial_{\boldsymbol{\theta}}E_{\boldsymbol{\theta}} = 0 \qquad \text{(P.3.5)}$$

Why can't I do the usual partial wave analysis where $\partial_{\boldsymbol{\theta}} \rightarrow$ jm as in App D and then we have

$$\partial_{\mathbf{r}}[rE_{\boldsymbol{\theta}}(r,m)] - jmE_{\mathbf{r}}(r,m) = p(r,m)$$

$$\partial_{\mathbf{r}}[rE_{\mathbf{r}}(r,m)] + jmE_{\boldsymbol{\theta}}(r,m) = 0$$

where p(r,m) has only m = 0 and m = ±1 components. I could then solve the second

$$E_{\boldsymbol{\theta}}(r,m) = (1/jm)\,\partial_{\mathbf{r}}[rE_{\mathbf{r}}(r,m)]$$

and plug this into the first equation,

$$(1/jm)\,\partial_{\mathbf{r}}[r\,\partial_{\mathbf{r}}[rE_{\mathbf{r}}(r,m)]] - jmE_{\mathbf{r}}(r,m) = p(r,m)$$

But then this is really a Helmholtz equations as per Appendix D. In any event, I already went down this path in section (g) above. No doubt it can be made to work, but the stream function method is very much simpler!