[OK, on 5/1/14 I tried to make this section fly, but it is just too slippery and I think my existing Appendix P opening section is much better.
// But later I came back to this approach combined with a perturbation iterative approach (5/5/14)]

P.1 Eddy Current Analysis

There are whole books on the subject of eddy currents, and the web reveals a proliferation of papers and theses on eddy current applications. However, simple analytic examples of eddy current problems are hard to find. The general theory is quite complicated, and we present here a simplified approach which suits are limited purposes, and then we work through two specific sample problems.

Consider this figure, where the free-hand lines generically represent E and B field lines,

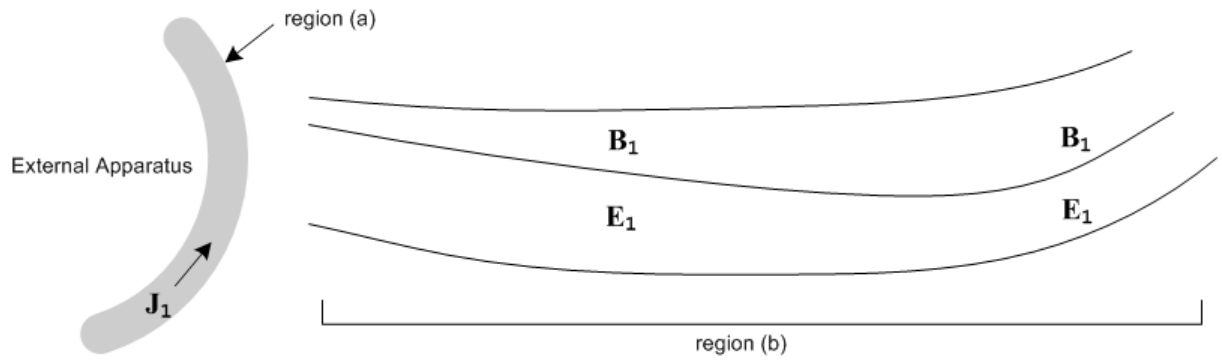


Fig P.1

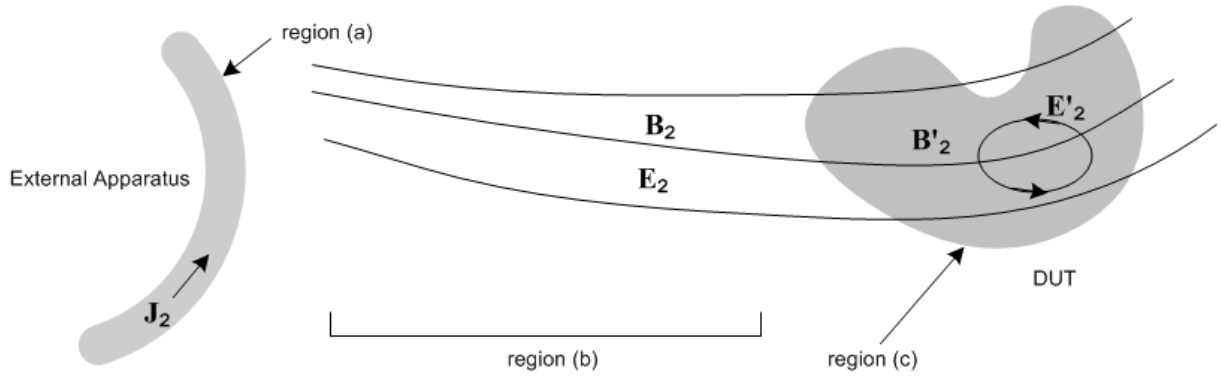
An external apparatus has current density $\mathbf{J_1}$ flowing in some wires and creates magnetic field $\mathbf{B_1}$ and there is some associated electric field $\mathbf{E_1}$ created as well. The Maxwell curl equations for Fig P.1 are:

$$\text{curl } \mathbf{E_1} = -j\omega\mathbf{B_1}$$
$$\text{curl } \mathbf{B_1} = \mu_a\mathbf{J_1} \qquad \text{// region (a)}$$

$$\text{curl } \mathbf{E_1} = -j\omega\mathbf{B_1}$$
$$\text{curl } \mathbf{B_1} = j\omega\mu_b\varepsilon_b\mathbf{E_1} \qquad \text{// region (b)} \qquad \text{(P.1.1)}$$

In region (a) we ignore the displacement current inside the wires. In region (b) $\mathbf{J_1} = 0$ and there is then some displacement current $j\omega\varepsilon\mathbf{E_1}$.

We now bring in a Device Under Test (DUT) to obtain a new picture:



The Maxwell curl equations are now:

$$\text{curl } \mathbf{E}_2 = -j\omega \mathbf{B}_2$$
$$\text{curl } \mathbf{B}_2 = \mu_a \mathbf{J}_2 \qquad \text{// region (a)}$$

$$\text{curl } \mathbf{E}_2 = -j\omega \mathbf{B}_2$$
$$\text{curl } \mathbf{B}_2 = j\omega \mu_b \varepsilon_b \mathbf{E}_2 \qquad \text{// region (b)}$$

$$\text{curl } \mathbf{E'}_2 = -j\omega \mathbf{B'}_2$$
$$\text{curl } \mathbf{B'}_2 = \mu_c \mathbf{J}_2' \qquad \text{// region (c)} \qquad \text{(P.1.2)}$$

In regions (a) and (c) we ignore the displacement currents (the DUT is a good conductor). The quantity $\mathbf{J'}_2$ is the eddy current in the DUT, and $\mathbf{E'}_2$ is its associated electric field.

For arbitrarily large ω, the solution of the problem of the fields inside the DUT is complicated and one must solve a vector Helmholtz equation. Setting $\mathbf{J'}_2 = \sigma_c \mathbf{E'}_2$ we can write for region (c)

$$\text{curl curl } \mathbf{E'}_2 = -j\omega \text{ curl } \mathbf{B'}_2 = -j\omega\mu_c(\sigma_c \mathbf{E'}_2) \qquad \text{(P.1.3)}$$

But we know that

$$\text{curl curl } \mathbf{E'}_2 = \text{grad div } \mathbf{E'}_2 - \nabla^2 \mathbf{E'}_2 = -\nabla^2 \mathbf{E'}_2 \qquad \text{(P.1.4)}$$

since div $\mathbf{E'}_2 = 0$ inside the DUT ($\rho=0$). Combining (P.1.3) and (P.1.4) we get

$$-\nabla^2 \mathbf{E'}_2 = -j\omega\mu_c\sigma_c \mathbf{E'}_2$$

or

$$(\nabla^2 + \beta_c^2)\mathbf{E'}_2 = 0 \qquad \text{where } \beta_c^2 = -j\omega\mu_c\sigma_c$$

and this is the equation encountered in *****. Of course to obtain a full solution, we have to consider the Helmholtz equations in each region and match boundary conditions at all interfaces, a formidable problem. With respect to the DUT, we have both an interior and an exterior problem to worry about.

If ω is small, we can take a different approach to solving the region (c) problem. We first define $\mathbf{B}_{\mathbf{eddy}}$ as the difference between $\mathbf{B}_1$ of the first drawing and $\mathbf{B'}_2$ of the second drawing, and we rename $\mathbf{J'}_2$ and $\mathbf{E'}_2$

$$\mathbf{B}_{\mathbf{eddy}} \equiv \mathbf{B'}_2 - \mathbf{B}_1 \qquad \mathbf{J}_{\mathbf{eddy}} \equiv \mathbf{J'}_2 \qquad \mathbf{E}_{\mathbf{eddy}} \equiv \mathbf{E'}_2$$

The region (c) curl equations of (P.1.2) are then

$$\text{curl } \mathbf{E}_{\mathbf{eddy}} = -j\omega(\mathbf{B}_1 + \mathbf{B}_{\mathbf{eddy}})$$

$$\text{curl } \mathbf{J}_{\mathbf{eddy}} = -j\omega\sigma_c(\mathbf{B}_1 + \mathbf{B}_{\mathbf{eddy}})$$

$$\text{curl } (\mathbf{B}_1 + \mathbf{B}_{\mathbf{eddy}}) = \mu_c\mathbf{J}_{\mathbf{eddy}} \qquad \text{// region (c)}$$

If ω is small, the middle equation implies that $\mathbf{J}_{\mathbf{eddy}}$ is small. If this is the case, we expect that $\mathbf{B}_{\mathbf{eddy}}$ will also be small. In this case, we expect that the eddy currents don't alter $\mathbf{J}_1$ very much when the DUT is added to Fig 1 to get Fig 2, so we then set $\mathbf{J}_2 = \mathbf{J}_1$ in the second drawing. If $\mathbf{B}_{\mathbf{eddy}}$ is small, we then have roughly $\mathbf{B'}_2 = \mathbf{B}_1 + \mathbf{B}_{\mathbf{eddy}} \approx \mathbf{B}_1$. The above equations are then

$$\text{curl } \mathbf{J}_{\mathbf{eddy}} \approx -j\omega\sigma\mathbf{B}_1$$

$$\text{curl } \mathbf{B}_1 \approx \mu_c\mathbf{J}_{\mathbf{eddy}} \qquad \text{// region (c)}$$

It is the first of these two equations that is our main interest. The second equation says that $\mathbf{B}_1$ has a small curl inside the DUT due to the eddy current there. One should not interpret this second equation as saying that the small eddy current somehow *generates* the large field $\mathbf{B}_1$. After all, in region (b) curl $\mathbf{B}_1 = 0$ and we have a large $\mathbf{B}_1$ there.