

Two Simple Models for Analytical Calculation of Eddy Currents in Thin Conducting Plates

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This is a good paper and verifies my circular plate calculation.

Abstract—Two simple models are proposed for the analytical calculation of eddy currents induced by a time-varying magnetic field in thin conducting plates of various shapes. In the first model, it is assumed that the current paths are determined exclusively by the shape of the plate. This assumption makes evident a shape factor that characterizes the shape of the plate under investigation and is independent of the dimensions of the plate. Thus, several important parameters, such as the equivalent resistance of the plate, the total circulating current, etc., are expressed analytically as a function of this shape factor. The application of the model makes possible the estimation of the total eddy current, circulating in plates of any shape, but the accuracy of this estimate is strongly affected by the shape of the plate. This model is then improved for plates having symmetrical shapes by taking into consideration the principle of minimum energy dissipation. The results, obtained by the improved model for the total circulating current, are in good agreement with the results obtained numerically as well as with analytical results obtained by the use of variational methods.

Index Terms— Eddy currents, modeling, plates, resistance, shape.

I. INTRODUCTION

DURING the last two decades, electromagnetic field computation and modeling of eddy currents have received increasing attention. Eddy currents are involved in various domains of applied electricity. They play an important role in many industrial applications [1], for example in the design of electromagnetic devices, magnetic fusion devices, magnetic levitation vehicles, etc.

The phenomena concerning an electromagnetic field problem are described mathematically by Maxwell equations. Consequently, the problem is defined in general by a partial differential equation with appropriate boundary and initial conditions. In most cases, a numerical method is adopted for the solution of such a problem. Since the computer power has increased impressively, the numerical methods become more and more attractive. It is possible with the help of these methods to solve complicated electromagnetic problems even in three dimensions, for which some decades ago, it was difficult to obtain a solution. There are, of course, many limitations and inconveniences associated with numerical methods. In general, they consume considerable central processing unit (CPU) time. Besides, the solution process of any problem must be repeated for any set of input data. That may be required even if one set

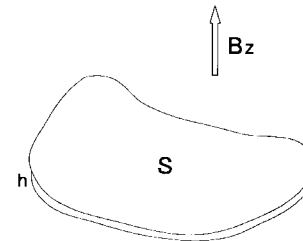


Fig. 1. Schematic diagram of the plate.

is a minor perturbation on a previous set. Thus, an analytical solution to an electromagnetic field problem always poses a great challenge being powerful and rapid. It is also very important since it helps understand better the physical behavior of electromagnetic fields. However, to obtain an analytical solution for a three-dimensional (3-D) application may be extremely difficult and in many cases, impossible. Nevertheless, a large class of problems of practical interest exists for which a two-dimensional (2-D) electromagnetic analysis is sufficient to achieve quite satisfactory results and interpretations.

In the present paper, we propose to solve analytically a 2-D problem: Calculate the eddy current induced by an uniform time-varying magnetic field, either harmonic or transient, in thin conductive plates of various shapes. Our intention is to combine the simplest possible modeling approach with the maximum attainable accuracy.

II. DESCRIPTION OF THE PROBLEM

A conductive plate of thickness h and faces with arbitrary shape of surface S is placed in a time-varying magnetic field (Fig. 1). This field, of magnetic induction $B_z(t)$ is uniform in space and perpendicular to the faces of the plate. Its time variation may be either harmonic, according to

$$B_z(t) = B_{z0} \cos \omega t \quad (1)$$

or transient, for example

$$B_z(t) = B_{z0} e^{-t/\tau} \quad (2)$$

where B_{z0} is the magnetic induction at $t = 0$, ω is the angular frequency and τ is a decay time constant. The magnetic flux variation through the surface S of the plate induces eddy currents in the plate. We assume that the plate lies in the XY plane, while the magnetic field is parallel to the Z axis. In the following, the index z will be omitted for simplicity.

The objective is to calculate analytically the total circulating current, $I = I(t)$, as well as some characteristic electro-

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magnetic parameters of the problem. For this purpose, the following assumptions are made.

- 1) The applied pulse of flux passing through the plate is not attenuated by the eddy-currents, i.e., the penetration depth δ of the time-varying magnetic field is sufficiently greater than the plate thickness h . This penetration depth is given as

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}} \quad (3)$$

for the time-harmonic field and as

$$\delta = \sqrt{\frac{\tau}{\mu\sigma}} \quad (4)$$

for the transient field where μ and σ are the permeability and conductivity of the material of the plate, respectively. We then consider the current uniformly distributed over the thickness h of the plate.

- 2) The magnetic field produced by the eddy current is small compared to the applied magnetic field, and consequently, it may be neglected. The present analysis is, therefore, restricted to the so-called *resistance limited* case. This choice will be justified below.

The governing idea is to adopt a model as simple as possible. In a first approach, it is assumed that the current paths are determined exclusively by the geometry, i.e., the shape of the plate. This idea has also been used in previously published work [2] where we treated the problem of a thin rectangular plate with length l , width a , and thickness h placed in a time-varying magnetic field according to (2). This problem is resolved by assuming that the eddy current circulates around the center of the plate in rectangular paths, one within the other. From the above assumptions [1) and 2)], in the above work we adopted only the first one, concerning the penetration depth. Therefore, the magnetic field produced by the eddy current is not neglected in this work [assumption 2)]. So, we have also considered the conductive plate as an equivalent electric circuit with an inductance L and resistance R . This can be described by

$$L \frac{di(t)}{dt} + Ri(t) = \mathcal{E}(t) \quad (5)$$

where $\mathcal{E}(t)$ is the electromotive force, induced by the magnetic flux variation through the surface S of the plate. The following analytical expressions for the equivalent resistance, inductance, and emf were derived and given in [2]:

$$R = \frac{4}{\sigma h} \frac{l^2 + a^2}{la} \quad (6)$$

$$L = \frac{2\mu}{3\pi} \left[l \left(1 + \ln \frac{l + \sqrt{l^2 + a^2}}{a} \right) + a \left(1 + \ln \frac{a + \sqrt{l^2 + a^2}}{l} \right) + \frac{l^2 - a^2}{\sqrt{l^2 + a^2}} \ln \frac{a(l + \sqrt{l^2 + a^2})}{l(a + \sqrt{l^2 + a^2})} \right] \quad (7)$$

$$\mathcal{E}(t) = \frac{1}{2} S \frac{B_0}{\tau} e^{-t/\tau}. \quad (8)$$

The analytical solution of (5) yields the total circulating current in the plate as a function of time

$$I(t) = \frac{SB_{0z}}{2(R\tau - L)} (e^{-t/\tau} - e^{-Rt/L}). \quad (9)$$

The assumption of perfect rectangular paths instead of rectangular paths with rounded corners results in a slight overestimation of L and R . However, we verify from (9) that the overestimations in L and R are almost mutually compensated. So, the agreement between analytical and experimental results for the current are very satisfactory [2]. The maximum discrepancy of 4% is observed when the current reaches its maximum value. At that instant, the term $L(dI/dt)$ in (5) vanishes, so there is no more compensation between the terms. That leads us to consider in the present work the resistance limited case [assumption 2)] in order to make clear the deficiencies of the adopted model, since in that case we have no more mutual compensations of discrepancies between these terms. Once the precision of our model has been clarified, we then would proceed to the following step: To solve the problem without adopting assumption 2), i.e., by taking into consideration the magnetic field produced by the eddy current. Obviously, now we need to calculate analytically, besides the equivalent resistance of the plate, an equivalent inductance as we have done in the rectangular plate mentioned above [2]. That is possible for plates of various shapes if we approach the current path by straight segments, but this requires increased time and effort. The analytical calculation of the equivalent inductance of plates having relatively simple shapes could be the subject of future work. The present analysis will be restricted to the resistance limited case of a thin plate.

III. THE GEOMETRIC MODEL

In general, the plate may have an arbitrary shape with its perimeter forming an arbitrary curve (Fig. 1). We can approach this curve by a broken line, so that we have an irregular polygon with n sides. Since the time-varying magnetic field is uniform in space and the plate is thin, we may assume that the eddy current circulates around the center of mass K of the plate (Fig. 2). If we join now the point K with the vertices of the polygon, we divide the plate into n triangular elements. As we have already reported, we consider in a first approach of the problem that the current distribution in the plate is determined exclusively by the geometry, i.e., the shape of the plate. The current flow is, therefore, parallel to the sides of the plate, i.e., the current circulates in paths that form polygons, one inside the other as illustrated in Fig. 2. Obviously, the shape of each polygon is similar to the shape of the plate. We note by a_i the basis of the triangular element i , by b_i its height, by r_i the distance of a given current path from the point K , and by w_i the length of the portion of this current path within the element i (Fig. 3).

The Maxwell equation, $\oint \vec{E} \cdot d\vec{l} = -d\Phi/dt$ is written for a given current path apply Faraday to each triangle

$$\sum_{i=1}^n E_i w_i = -\frac{d}{dt} \left(B \sum_{i=1}^n \frac{1}{2} r_i w_i \right). \quad (10)$$

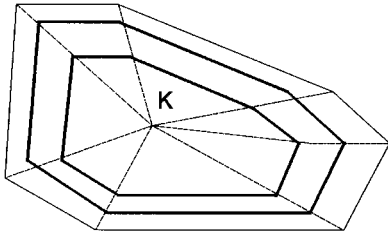


Fig. 2. The division of the plate into triangular elements, showing two typical current paths.

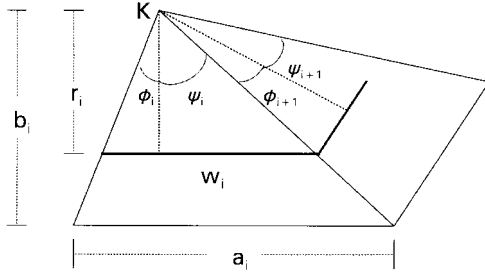


Fig. 3. Characteristic dimensions and angles in a triangular element.

In the above equation the implicit assumption, as we have already reported, is that B is spatially uniform. If we use Ohm's law, $\vec{E} = \vec{j}/\sigma$, we may rewrite (10) into the form

$$\sum_{i=1}^n j_i w_i = -\frac{1}{2} \sigma \dot{B} \sum_{i=1}^n r_i w_i. \quad (11)$$

The total current circulating the plate (around K) is given by

$$I = \int_0^{b_i} j_i(r_i) h dr_i \quad (12)$$

where $i = 1, 2, \dots, n$. Considering now two adjacent elements, we verify from Fig. 3 that

$$\begin{aligned} \frac{r_i}{r_{i+1}} &= \frac{\cos \psi_i}{\cos \phi_{i+1}} \\ &= \frac{b_i}{b_{i+1}}. \end{aligned} \quad (13)$$

For any elements i and k of arbitrary position, we obtain from (13)

$$\begin{aligned} \frac{r_i}{r_k} &= \frac{r_i}{r_{i+1}} \cdot \frac{r_{i+1}}{r_{i+2}} \dots \frac{r_{k-1}}{r_k} \\ &= \frac{b_i}{b_{i+1}} \cdot \frac{b_{i+1}}{b_{i+2}} \dots \frac{b_{k-1}}{b_k} \\ &= \frac{b_i}{b_k}. \end{aligned} \quad (14)$$

Taking into consideration (14), (12) is written

$$\begin{aligned} I &= \int_0^{b_i} j_i h dr_i \\ &= \int_0^{b_k} j_k h dr_k \\ &= \int_0^{b_i} j_k h \frac{b_k}{b_i} dr_i. \end{aligned} \quad (15)$$

Equation (15) yields the following relation between the current densities j_i and j_k :

$$j_i(r_i) = \frac{b_k}{b_i} j_k(r_k). \quad (16)$$

We can express the current densities j_i , $i = 1, 2, \dots, n$ in (11) as function of a given current density j_k by means of (16). Furthermore, the distances r_i , $i = 1, 2, \dots, n$ from the point K may be written as $r_i = (b_i/b_k) r_k$ according to (14). So, we verify from Fig. 3 and (14) that

$$\begin{aligned} w_i &= \frac{a_i}{b_i} r_i \\ &= \frac{a_i}{b_k} r_k. \end{aligned} \quad (17)$$

Equation (11) is then written as

$$j_k r_k \sum_{i=1}^n \frac{a_i}{b_i} = -\frac{1}{2} \sigma \dot{B} \left(\sum_{i=1}^n a_i b_i \right) \frac{r_k^2}{b_k^2}. \quad (18)$$

From (18), we obtain an expression for any current density j_k , $k = 1, 2, \dots, n$ at a given instant t as a function of the distance r_k from the point K

$$j_k(r_k, t) = -\frac{1}{2} \sigma \dot{B}(t) \frac{\sum_{i=1}^n a_i b_i}{\sum_{i=1}^n \frac{a_i}{b_i}} \frac{r_k}{b_k^2}. \quad (19)$$

Inserting the above expression for j_k in (15), we deduce the total current circulating in the plate

$$I(t) = -\frac{1}{4} \sigma h \frac{\sum_{i=1}^n a_i b_i}{\sum_{i=1}^n \frac{a_i}{b_i}} \dot{B}(t). \quad (20)$$

Let us calculate now the total power dissipated in the plate at a given instant t . The elementary power dissipated in an elementary volume dV_k of the triangular element k is

$$\begin{aligned} dP_k &= \frac{j_k^2}{\sigma} dV_k \\ &= \frac{j_k^2}{\sigma} h w_k dr_k. \end{aligned} \quad (21)$$

By inserting expression (19) for j_k and expression (17) for w_k into (21) and integrating from $r_k = 0$ to b_k , we obtain the power dissipated in the triangular element k

$$P_k = \frac{1}{16} \sigma h \dot{B}^2 \frac{\left(\sum_{i=1}^n a_i b_i \right)^2}{\left(\sum_{i=1}^n \frac{a_i}{b_i} \right)^2} \frac{a_k}{b_k}. \quad (22)$$

Obviously, the total power dissipated in the plate is

$$P = \sum_{k=1}^n P_k$$

$$= \frac{1}{16} \sigma h \dot{B}^2 \frac{\left(\sum_{i=1}^n a_i b_i \right)^2}{\left(\sum_{i=1}^n \frac{a_i}{b_i} \right)^2} \sum_{k=1}^n \frac{a_k}{b_k} \quad (23)$$

or

$$P(t) = \frac{1}{16} \sigma h \frac{\left(\sum_{i=1}^n a_i b_i \right)^2}{\sum_{i=1}^n \frac{a_i}{b_i}} \dot{B}^2(t). \quad (24)$$

The equivalent resistance R of the plate is now determined by the fact that the total power dissipated in the plate is also given by $P = I^2 R$ or using (20) for the total current I by

$$P(t) = \frac{1}{16} \sigma^2 h^2 \frac{\left(\sum_{i=1}^n a_i b_i \right)^2}{\left(\sum_{i=1}^n \frac{a_i}{b_i} \right)^2} \dot{B}^2(t) R. \quad (25)$$

Comparing (24) and (25) yields

$$R = \frac{1}{\sigma h} \sum_{i=1}^n \frac{a_i}{b_i}. \quad (26)$$

From (26) we observe that the equivalent resistance of the plate is independent of the surface of the face of the plate. It depends only on the resistivity of the material, the thickness of the plate, and a dimensionless factor that may be defined as

$$f = \sum_{i=1}^n \frac{a_i}{b_i}. \quad (27)$$

This characteristic parameter depends exclusively on the shape of the face exposed to the variable magnetic flux, which is independent of the dimensions of the plate and may be considered as a shape factor. Consequently, plates with geometrically similar faces are characterized by the same shape factor. Thus, (26) may be written

$$R = \frac{1}{\sigma h} f. \quad (28)$$

From the above equation we conclude that, plates with similar faces have the same equivalent resistance, if they have the same thickness and consist from the same material.

The shape factor appears in all the above formulas, concerning the current density, the total current, and the power dissipated in the plate. Furthermore, we note that, the surface

S of the face of the plate also appears in these formulas

he never defines S as area
but I think S = area of plate

$$S = \frac{1}{2} \sum_{i=1}^n a_i b_i. \quad (29)$$

Therefore, the current density (19), the total circulating eddy current (20), and the power dissipated in the plate (24) may all be expressed as functions of the plate surface and the shape factor

$$j_k(r_k, t) = -\sigma \dot{B}(t) \frac{S}{f} \frac{r_k}{b_k^2} \quad (30)$$

$$I(t) = -\frac{1}{2} \sigma h \frac{S}{f} \dot{B}(t) \quad (31)$$

$$P(t) = \frac{1}{4} \sigma h \frac{S^2}{f} \dot{B}^2(t). \quad (32)$$

For the equivalent electromotive force, $\mathcal{E}(t) = I(t) R$, we obtain from (28) and (31)

$$\mathcal{E}(t) = -\frac{1}{2} S \dot{B}(t). \quad (33)$$

So, we verify (8) which was derived through a different formulation in [2].

We can apply and test now the model to some conducting plates with simple geometric shapes.

A. Circular Plate

Let r_0 be the radius of the plate. From (27), the shape factor is

$$f = 2\pi. \quad (34)$$

Consequently, the equivalent resistance will be

$$R = \frac{2\pi}{\sigma h}. \quad (35)$$

According to (30)–(32) and taking into consideration (34), we obtain, for the current density, the total circulating eddy current, and the power dissipated in the plate, the following expressions:

$$j(r, t) = -\sigma \dot{B}(t) \frac{S}{2\pi} \frac{r}{r_0^2}$$

$$= -\frac{1}{2} \sigma \dot{B}(t) r \quad (36)$$

$$I(t) = -\frac{1}{2} \sigma h \frac{S}{2\pi} \dot{B}(t)$$

$$= -\frac{1}{4} \sigma h r_0^2 \dot{B}(t) \quad (37)$$

$$P(t) = \frac{1}{4} \sigma h \frac{S^2}{2\pi} \dot{B}^2(t)$$

$$= \frac{\pi}{8} \sigma h r_0^4 \dot{B}^2(t). \quad (38)$$

these all agree with my circular plate solution!

B. Regular Polygon

We consider now that the faces of the plate have the shape of a regular polygon with n sides ($n = 3, 4, 5, \dots$). Let a be the side of the polygon and b its distance from the point K , that coincides with the geometric center. Equation (27) yields the shape factor for a regular polygon

$$\begin{aligned} f &= n \frac{a}{b} \\ &= 2n \tan\left(\frac{\pi}{n}\right). \end{aligned} \quad (39)$$

Consequently, the equivalent resistance is

$$R = \frac{2n}{\sigma h} \tan\left(\frac{\pi}{n}\right) \quad (40)$$

while for the current density, we obtain from (30)

$$\begin{aligned} j(r, t) &= -\frac{\sigma S \dot{B}(t)}{2n \tan\left(\frac{\pi}{n}\right)} \frac{r}{b^2} \\ &= -\frac{1}{2} \sigma \dot{B}(t) r. \end{aligned} \quad (41)$$

The total current and the power dissipated in the plate are given as a function of the surface S of the plate by (31) and (32) if we use the value of the shape factor f given by (39). These quantities, may be expressed also as a function of one of the characteristic dimensions of the polygon, for example the side a or its distance b from the center. We then have

$$\begin{aligned} I(t) &= -\frac{1}{4} \sigma h b^2 \dot{B}(t) \\ &= -\frac{1}{16} \sigma h a^2 \cot^2\left(\frac{\pi}{n}\right) \dot{B}(t) \end{aligned} \quad (42)$$

$$\begin{aligned} P(t) &= \frac{n}{8} \sigma h b^4 \tan\left(\frac{\pi}{n}\right) \dot{B}^2(t) \\ &= \frac{n}{128} \sigma h a^4 \cot^3\left(\frac{\pi}{n}\right) \dot{B}^2(t). \end{aligned} \quad (43)$$

We may remark that, if we consider the circular plate as a regular polygon with an infinite number of sides ($n \rightarrow \infty$), we again have formulas (34)–(38).

C. Rectangular Plate

Let us consider now that the shape of the plate is rectangular with length l and width a . If we note by γ the ratio l/a (i.e., the aspect ratio of the rectangle), we obtain from (27) for the shape factor

$$f = 4 \frac{\gamma^2 + 1}{\gamma}. \quad (44)$$

Therefore, the equivalent resistance of the plate is given by

$$\begin{aligned} R &= \frac{4}{\sigma h} \frac{\gamma^2 + 1}{\gamma} \\ &= \frac{4}{\sigma h} \frac{l^2 + a^2}{la}. \end{aligned} \quad (45)$$

i.e., we find again (6). From (30)–(32), we obtain the formulas derived in [2] for the current densities, the total circulating

current, and the power dissipated in the plate, respectively

$$\begin{aligned} j_x(y) &= \sigma \dot{B} \frac{\gamma^2}{\gamma^2 + 1} y \\ &= \sigma \dot{B} \frac{l^2}{l^2 + a^2} y \\ j_y(x) &= -\sigma \dot{B} \frac{1}{\gamma^2 + 1} x \\ &= -\sigma \dot{B} \frac{a^2}{l^2 + a^2} x \end{aligned} \quad (46)$$

$$\begin{aligned} I(t) &= -\frac{1}{8} \sigma h \frac{\gamma}{\gamma^2 + 1} S \dot{B}(t) \\ &= -\frac{1}{8} \sigma h \frac{l^2 a^2}{l^2 + a^2} \dot{B}(t) \end{aligned} \quad (47)$$

$$\begin{aligned} P(t) &= \frac{1}{16} \sigma h \frac{\gamma}{\gamma^2 + 1} S^2 \dot{B}^2(t) \\ &= \frac{1}{16} \sigma h \frac{l^3 a^3}{l^2 + a^2} \dot{B}^2(t). \end{aligned} \quad (48)$$

In the present model, we have taken into consideration only the geometry, i.e., the shape of the conductor, and we have expressed some electromagnetic parameters of interest as a function of purely geometric parameters. However, in this way, we introduce an error which depends on whether or not the current path suggested by the shape of the conductor is also energetically the shortest one. The only case where that is true is the case of the circular plate. Consequently, the formulas derived for a circular plate, (34)–(38) are exact. In any other case, we will have a discrepancy that strongly depends on the conductor shape. We will clarify this in Section V, where we compare the analytical results with the numerical ones.

IV. THE IMPROVED MODEL

Up to now, we have assumed that the current paths at any position of the plate have the shape of the plate. In order to obtain an exact analytical solution, we must determine at any position the exact shape of the current path. That is a sufficiently difficult task and requires too much time and effort. Furthermore, it is not certain that the resulting analytical formulas will be sufficiently simple, so that they have a practical interest. The shape of the current paths as a function of the position for plates with various shapes have been approached in [3] by preselecting the form of the stream function and by using variational methods. We will refer to that procedure more extensively in Section V.

We propose now a method to improve the geometric model developed in the previous section, so that we arrive to a sufficiently precise and, at the same time, simple model. This improvement is possible, for plates with symmetric shape under the following assumptions.

- 1) Take into consideration the fact that, the current distribution in a conductor is such that the energy dissipated is a minimum. **how support this idea?**
- 2) Retain the idea adopted in the geometric model for current paths parallel to the sides of the plate. This has two advantages: i) The analytical computations are performed easier, and the formulas maintain their

simplicity. ii) In the case where the secondary magnetic field, produced by the eddy current is not negligible, it is possible although quite tedious to calculate analytically the equivalent inductance of the plate. We have performed this calculation for a rectangular plate in [2] as we have already mentioned in Section II and derived (7) for the inductance. On the contrary, if we consider the real-curved current paths, it is prohibitively difficult and, in many cases impossible, to calculate analytically the inductance of the plate.

The above assumptions will permit us to improve the precision of the previous formulas and at the same time exploit their simplicity. However, our analysis will be limited to conductors of symmetrical shape.

A. Regular Polygon

Through a given area A of the plate, the flux variation is $\dot{B}A$. If the shape of this area is a regular polygon with n sides, a unit charge moving on the circumference to encircle this area must cover a distance equal to the perimeter C of the polygon

$$C = 2\sqrt{nA \tan\left(\frac{\pi}{n}\right)}. \quad (49)$$

Instead, if the shape of the surface is circular, the unit charge moving on the circumference in order to encircle the same area (i.e., a flux variation $\dot{B}A$) covers a distance equal to the perimeter C_0 of the circle

$$C_0 = 2\sqrt{\pi A}. \quad (50)$$

It is, of course, known that, between all the possible paths that cover the same area A , the circular is the shortest one.

Taking into consideration these remarks, the above assumptions 1) and 2) seem to be incompatible because they imply minimum energy dissipation and at the same time, polygonal, i. e., not circular, current paths. In order to compromise these assumptions, we introduce an effective conductivity σ_{eff} having the following property: the real conductivity σ multiplied by the length C of the idealized current path, i.e., the perimeter of the polygon that surrounds the area A is equal to the product of the effective conductivity σ_{eff} by the length of the minimum current path C_0 , i.e., the perimeter of an equivalent circle. So we have $\sigma C = \sigma_{eff} C_0$ or

$$\sigma_{eff} = \sigma \frac{C}{C_0}. \quad (51)$$

If we use now (49) and (50), we obtain from (51) the effective conductivity for a plate, having the shape of a regular polygon

$$\sigma_{eff} = \sigma \sqrt{\frac{n}{\pi} \tan\left(\frac{\pi}{n}\right)}. \quad (52)$$

It is obvious that, if we consider for the plate the effective conductivity given by (52) instead of the real one and repeat the procedure of the previous section, we again find, for the parameters of interest, formulas that are similar to those

obtained by the geometric model, but, with σ_{eff} instead of σ . Equation (41) for the current density at a distance r from the center of the polygon, may then be written

$$\begin{aligned} j(r, t) &= -\frac{1}{2} \sigma_{eff} \dot{B}(t) r \\ &= -\frac{1}{2} \sigma \dot{B}(t) \sqrt{\frac{n}{\pi} \tan\left(\frac{\pi}{n}\right)} r. \end{aligned} \quad (53)$$

We observe that we may preserve the real conductivity in the formulas if we introduce an improved shape factor f_0 , so that $\sigma_{eff} f_0 = \sigma$ or $f_0 = f(C_0/C)$. Thus, taking into consideration (39), (49), and (50) we obtain the improved shape factor

$$f_0 = 2\sqrt{n\pi \tan\left(\frac{\pi}{n}\right)}. \quad (54)$$

Consequently, the equivalent resistance is

$$R = \frac{2}{\sigma h} \sqrt{n\pi \tan\left(\frac{\pi}{n}\right)}. \quad (55)$$

The formulas for the total current and the power dissipated in the plate may be written

$$I(t) = -\frac{1}{4} \sigma h \frac{S \dot{B}(t)}{\sqrt{n\pi \tan\left(\frac{\pi}{n}\right)}} \quad (56)$$

$$P(t) = \frac{1}{8} \sigma h \frac{S^2 \dot{B}^2(t)}{\sqrt{n\pi \tan\left(\frac{\pi}{n}\right)}}. \quad (57)$$

We note again that for $n \rightarrow \infty$ we obtain the corresponding formulas for a circular plate.

B. Rectangular Plate

We consider now the case of a rectangular plate with length l and width a . Through a given area A , we have a flux variation $\dot{B}A$. If the shape of the area is rectangular, a unit charge moving on the circumference in order to surround this area must cover a distance equal to the perimeter C of the rectangle, i.e.,

$$C = 2(\gamma + 1) \sqrt{\frac{A}{\gamma}} \quad (58)$$

where $\gamma = l/a$ is the aspect ratio.

In the case of the regular polygon, the shortest path that a unit charge may travel in order to surround a given area A is the perimeter of a circle. That is always valid but, in the present case, we must also take into consideration the elongated shape of the rectangle. Therefore, we consider the perimeter of an equivalent ellipse instead of the perimeter of an equivalent circle. We suppose that a unit charge, in order to surround a given area A and at the same time have the dissipated energy minimum, must cover a distance C_e equal to the perimeter of an ellipse of surface A . If γ_e is the ratio of the semi-axis of

that ellipse, we have [4]

$$C_e = (\gamma_e + 1) \left[1 + \frac{1}{4} \left(\frac{\gamma_e - 1}{\gamma_e + 1} \right)^2 + \frac{1}{64} \left(\frac{\gamma_e - 1}{\gamma_e + 1} \right)^4 + \frac{1}{256} \left(\frac{\gamma_e - 1}{\gamma_e + 1} \right)^6 + \dots \right] \sqrt{\frac{\pi A}{\gamma_e}}$$

or, approximately

$$C_e \approx \left[\frac{3}{2} (\gamma_e + 1) - \sqrt{\gamma_e} \right] \sqrt{\frac{\pi A}{\gamma_e}}. \quad (59)$$

In order to compromise between rectangular current paths and minimum energy dissipation, we introduce, as we have done in the case of the regular polygon, an effective conductivity σ_{eff} so that $\sigma_{eff} C_e = \sigma C$ or

$$\sigma_{eff} = \sigma \frac{C}{C_e} \quad (60)$$

where C and C_e are given by (58) and (59), respectively. However, in order to calculate C_e by means of (59), we must determine previously the ratio γ_e of the semi-axis of that ellipse. Obviously, the perimeter C_e must be greater than the perimeter C_0 of an equivalent circle, but smaller than the perimeter of an equivalent ellipse with ratio of semi-axis equal to γ , i.e., the aspect ratio of the rectangle under investigation. Thus, it is evident that if we consider the circle as an ellipse with ratio of semi-axis γ_0 equal to one, then γ_e lies between γ_0 and γ , i.e., $\gamma_0 < \gamma_e < \gamma$. A method to determine γ_e is given in detail in the Appendix, where γ_e is obtained as one of the roots of a quadratic equation. Using now (58) and (59), we obtain from (60) the following expression for the effective conductivity of a rectangular plate:

$$\sigma_{eff} = \sigma \frac{2}{\sqrt{\pi}} \frac{\sqrt{\gamma_e}}{1.5(\gamma_e + 1) - \gamma_e} \frac{\gamma + 1}{\sqrt{\gamma}}. \quad (61)$$

The formulas for the parameters of interest derived in Section III-C are still valid if we replace σ by σ_{eff} . However, we may introduce an improved shape factor f_e , as in the case of the regular polygon, so that we preserve in our formulas the real conductivity instead of the effective one. This improved shape factor, $f_e = f(C_e/C)$, where f is given by (44), is expressed as

$$f_e = 2\sqrt{\pi} \left(\frac{3}{2} \frac{\gamma_e + 1}{\sqrt{\gamma_e}} - 1 \right) \frac{\gamma^2 + 1}{(\gamma + 1)\sqrt{\gamma}}. \quad (62)$$

The equivalent resistance of the rectangular plate is now given by

$$R = \frac{2\sqrt{\pi}}{\sigma h} \left(\frac{3}{2} \frac{\gamma_e + 1}{\sqrt{\gamma_e}} - 1 \right) \frac{\gamma^2 + 1}{(\gamma + 1)\sqrt{\gamma}}. \quad (63)$$

The expressions yielding the total circulating current and the power dissipated in the plate may be written

$$I(t) = -\frac{1}{4\sqrt{\pi}} \frac{\sqrt{\gamma_e}}{1.5(\gamma_e + 1) - \sqrt{\gamma_e}} \frac{(\gamma + 1)\sqrt{\gamma}}{\gamma^2 + 1} \sigma h S \dot{B}(t) \quad (64)$$

$$P(t) = \frac{1}{8\sqrt{\pi}} \frac{\sqrt{\gamma_e}}{1.5(\gamma_e + 1) - \sqrt{\gamma_e}} \frac{(\gamma + 1)\sqrt{\gamma}}{\gamma^2 + 1} \sigma h S^2 \dot{B}^2(t). \quad (65)$$

Before we proceed to the evaluation of the models (the geometric and the improved one), we should note the following.

- 1) A square plate may be treated either as a polygon with $n = 4$ or as a rectangle with $\gamma = 1$. We observe that the formulas derived for a regular polygon with $n = 4$ coincide perfectly with the corresponding formulas derived for a rectangle with $\gamma = 1$.
- 2) The shape factor of a circular plate 2π seems to be, under the assumptions introduced in our models, a physical limit. Therefore, it is not possible for a plate of any shape to have a shape factor less than 2π .

V. EVALUATION OF THE MODELS

We are ready now to evaluate our models. We compare the analytical results obtained for plates of various shapes with:

- a) numerical results;
- b) analytical results, obtained by the use of variational methods;
- c) the current distribution obtained numerically and analytically.

A. Numerical Solution and Results

To obtain a numerical solution of the problem, we proceed as follows. We transform Faraday's law, $\vec{\nabla} \times \vec{E} = -\partial \vec{B}/\partial t$, by using Ohm's law, $\vec{E} = \vec{j}/\sigma$, into

$$\vec{\nabla} \times \vec{j} = -\sigma \frac{\partial \vec{B}}{\partial t}. \quad (66)$$

Since $\text{div } \vec{j} = 0$, we introduce the current vector potential $\vec{T}$, defined as an interesting idea!

$$\vec{j} = \vec{\nabla} \times \vec{T}. \quad (67)$$

Taking into consideration that the plate is thin and combining (66) and (67), we finally obtain

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \sigma \dot{B} \quad (68)$$

where T is the component of $\vec{T}$ normal to the plate. Thus instead of a vector quantity, we consider a scalar one, and the 3-D problem is transformed into a 2-D problem. This scalar quantity is the so-called stream function, frequently used for thin plates [5], [6].

Equation (68) has been solved numerically for plates of various shapes. For the numerical analysis, a finite-difference mesh has been considered. The discretization of (68) on the mesh has been performed by using central difference approach for the second derivatives. The resulting set of algebraic equations has been solved with the Gauss-Seidel method [7] by taking into consideration the boundary condition $T = 0$ at the edge of the plate. The number of grid points used in all computations was larger than 800, and it can get as big as 8000 in some of the cases we have examined. This is more

than adequate to practically ensure independence of the results from the exact number of grid points.

The data for our sample problem are the following. A conductive plate, with electrical conductivity $\sigma = 1.10^7$ S/m, surface $S = 0.1$ m², and thickness $h = 2$ mm, is placed in a uniform time-varying magnetic field, either harmonic (1) or transient (2) perpendicular to the faces of the plate. For the computations, we have considered $\dot{B} = 10$ T/s. This would be the maximum value that $\dot{B}$ can take, if we consider for example $B_0 = 0.1$ T and $\omega = 100$ rad/s in the case of the time-harmonic field (1), or $B_0 = 0.05$ T and $\tau = 5$ ms in the case of the transient field (2). In any case, the percentage discrepancy between numerical and analytical results for any set of (B_0, ω) or (B_0, τ) is not affected by the time evolution of $B(t)$ and is exactly the same at any instant. Thus, the results presented in Table II are valid for any combination of (B_0, ω) or (B_0, τ) that satisfy $\dot{B} = B_0\omega = 10$ T/s or $\dot{B} = B_0/\tau = 10$ T/s, respectively, provided that the values of ω or τ are such that the resulting penetration depth is sufficiently greater than the plate thickness. For example, for $\omega = 100$ rad/s we have from (3) $\delta \approx 40$ mm, while for $\tau = 5$ ms we have from (4) $\delta \approx 20$ mm.

We performed our computations for conductive plates with the following shapes: a) circular, b) regular polygons with $n = 3$ (equilateral triangle), $n = 4$ (square), $n = 6$ (hexagon), $n = 8$ (octagon), $n = 12$ (dodecagon), and c) rectangles with aspect ratio $\gamma = 1.0$ (square), 2.0, 3.0, 4.0, and 5.0. The dimensions of each plate have been adjusted so that the surface of the face subjected to the variable magnetic flux is $S = 0.1$ m² in all cases.

In Table I, we give the shape factors for the plates considered, resulting from both models, the geometric and the improved one, i.e., by (34), (39), (44), (54), and (62).

The total circulating current in each plate has been calculated i) by using the geometric model, ii) by using the improved model, and iii) numerically, and the results are given in Table II. We also give the absolute discrepancy ϵ between numerical and analytical results for each case, i.e., $\epsilon = |\text{numerical-analytical}|/\text{numerical}$ (%). Obviously the numerical results depend on the finite-difference mesh considered and we, therefore, include here the maximum discrepancy.

As we observe from Table II in the case of the geometric model, the discrepancy between analytical and numerical results depends strongly on the shape of the plate. So, we have a discrepancy of 25% for the equilateral triangular plate, but only 1.5% for the dodecagon and 3.8% for a rectangle with aspect ratio $l/a = 5.0$. On the contrary, the application of the improved model yields a maximum discrepancy of about 4%. As far as the power dissipated in the plate is concerned, the application of the geometric model yields a discrepancy between analytical and numerical results of about 17% for the equilateral triangular plate or a rectangular plate with aspect ratio $l/a = 5.0$. An 11% discrepancy is observed for a square plate and less than 2.5% for an octagonal plate. The application of the improved model yields for the power losses a discrepancy of 12.5% in the case of the more elongated rectangle ($l/a = 5.0$) but less than 0.5% in the case of a square or an octagonal plate. A more detailed comparison

TABLE I
SHAPE FACTORS

REGULAR POLYGON			RECTANGULAR PLATE		
n	GEOMETRIC MODEL	IMPROVED MODEL	γ	GEOMETRIC MODEL	IMPROVED MODEL
3	10.3923	8.0806	1.0	8.0000	7.0898
4	8.0000	7.0898	2.0	10.0000	8.8623
6	6.9282	6.5978	3.0	13.3333	12.0142
8	6.6274	6.4530	4.0	17.0000	16.6685
12	6.4308	6.3566	5.0	20.8000	19.6070
...	...	...	CIRCULAR PLATE 6.2832		
∞	6.2832	6.2832			

TABLE II
TOTAL EDDY CURRENT (IN A) AND DISCREPANCIES (%)

REGULAR POLYGONS					
n	GEOMETRIC MODEL		IMPROVED MODEL		NUMERICAL
3	962.25	25.0	1237.53	3.5	1283.00
4	1250.00	15.1	1410.47	4.2	1472.42
6	1443.38	7.4	1515.65	2.7	1558.07
8	1508.88	4.3	1549.67	1.7	1576.95
12	1555.02	1.5	1573.18	0.3	1578.68
CIRCULAR PLATE : 1591.55					
RECTANGULAR PLATES					
γ	GEOMETRIC MODEL		IMPROVED MODEL		NUMERICAL
1.0	1250.00	15.1	1410.47	4.2	1472.42
2.0	1000.00	12.2	1128.38	0.9	1138.52
3.0	750.00	8.3	832.35	1.8	817.84
4.0	588.24	5.5	638.22	2.5	622.57
5.0	480.77	3.8	510.02	2.1	499.58

between analytical and numerical results for the dissipated power in rectangular plates is given in Table IV, where we compare also with the analytical results obtained by the use of variational methods.

In conclusion, the geometric model gives an estimation of the total eddy current circulating in a conductive plate of any shape as well as of the corresponding power losses. In some cases, these estimations may have a sufficient precision. On the contrary, the improved model gives a satisfactory precision for any case, but it is limited to conductive plates of symmetric shape. In any case, the estimations of the eddy current and the power losses given by the geometric model provide a lower bound for the total circulating current and the power dissipated in the conductor, respectively. Obviously, the estimate for the equivalent resistance, obtained by the geometric model,

provides an upper bound for the equivalent resistance of the plate.

B. Comparison with Analytical Formulas Obtained by Using Variational Methods

In an earlier publication [3], the eddy currents in thin conducting plates were calculated by the use of variational methods. The procedure minimizes a functional, related to the power losses, so that the stream function can be determined. A detailed description of this procedure is also given in [8].

Two variational methods have been used in [3], namely the Ritz and the Kantorovich methods. The Ritz method preselects the form of the stream function for a plate of given shape, i.e., chooses a trial function of x, y multiplied by appropriate coefficients. These coefficients are then determined by minimizing the functional. The Kantorovich method separates the stream function to the product of two functions. One depending only on x , the other only on y . The form of the function depending on y is preselected. The function depending on x is then determined by solving the Euler equation, resulting from the minimization of the functional.

We compare now the formulas obtained by the improved model for the total current and the power dissipated in rectangular plates with the corresponding formulas obtained in [3] by the use of variational methods.

We have derived in Section IV the following formula for the total current circulating in a rectangular plate (64):

$$I = -\frac{1}{4\sqrt{\pi}} \frac{\sqrt{\gamma_e}}{1.5(\gamma_e + 1) - \sqrt{\gamma_e}} \frac{(\gamma + 1)\sqrt{\gamma}}{\gamma^2 + 1} \sigma h S \dot{B}. \quad (64)$$

If we use our notation, we obtain from [3] the following formulas for the total current:

$$I_{VR} = -0.14642 \frac{\gamma}{\gamma^2 + 1} \sigma h S \dot{B} \quad (69)$$

$$I_{VK} = -\frac{1}{8\gamma} \left[1 - \frac{1}{\cosh\left(\frac{\sqrt{10}}{2} \gamma\right)} \right] \sigma h S \dot{B} \quad (70)$$

where the indexes VR and VK indicate Variational-Ritz and Variational-Kantorovich, respectively. We observe from (64), (69), and (70) that the total current is expressed by the term $\sigma h S \dot{B}$ multiplied by a coefficient depending on the method considered and the aspect ratio $\gamma = l/a$ of the rectangle. The values of these coefficients for $\gamma = 1.0, 2.0, 3.0, 4.0$, and 5.0 are compared in the Table III, where we have reported also the corresponding values resulting from the numerical computations, i.e., the ratio of the total current divided by the term $\sigma h S \dot{B}$.

We follow now the same procedure for the power losses in rectangular plates. In Section IV, we have derived the following formula (65):

$$P = \frac{1}{8\sqrt{\pi}} \frac{\sqrt{\gamma_e}}{1.5(\gamma_e + 1) - \sqrt{\gamma_e}} \frac{(\gamma + 1)\sqrt{\gamma}}{\gamma^2 + 1} \sigma h S^2 \dot{B}^2. \quad (65)$$

TABLE III
COMPARISON OF "CURRENT" COEFFICIENTS

γ	IMPROVED MODEL	VARIATIONAL RITZ	VARIATIONAL KANTOROVICH	NUMERICAL
1.0	0.070 524	0.073 211	0.075 654	0.073 621
2.0	0.056 419	0.058 569	0.057 218	0.056 926
3.0	0.041 617	0.043 927	0.040 941	0.040 892
4.0	0.031 911	0.034 452	0.031 138	0.031 129
5.0	0.025 501	0.028 158	0.024 982	0.024 979

TABLE IV
COMPARISON OF "POWER" COEFFICIENTS

γ	IMPROVED MODEL	VARIATIONAL RITZ	VARIATIONAL KANTOROVICH	NUMERICAL
1.0	0.035 262	0.035 125	0.034 909	0.035 172
2.0	0.028 209	0.028 100	0.028 538	0.028 594
3.0	0.020 809	0.021 075	0.021 923	0.021 947
4.0	0.015 956	0.016 530	0.017 539	0.017 553
5.0	0.012 751	0.013 510	0.014 558	0.014 567

If we use our notation, the corresponding formulas derived in [3] are written

$$R_{VR} = 0.07025 \frac{\gamma}{\gamma^2 + 1} \sigma h S^2 \dot{B}^2 \quad (71)$$

$$R_{VK} = \frac{1}{12\gamma} \left[1 - \frac{2}{\sqrt{10}} \frac{1}{\gamma} \tanh\left(\frac{\sqrt{10}}{2} \gamma\right) \right] \sigma h S^2 \dot{B}^2. \quad (72)$$

We observe again from (65), (71), and (72) that the power is expressed by the term $\sigma h S^2 \dot{B}^2$ multiplied by a coefficient depending on the method considered and the ratio γ . The values of these coefficients for $\gamma = 1.0, 2.0, 3.0, 4.0$, and 5.0 are compared in the Table IV with the corresponding numerical values, i.e., the ratio of the dissipated power divided by the term $\sigma h S^2 \dot{B}^2$.

From Tables III and IV, we may observe the following. Our analytical results for the total current, obtained by application of the improved model, are substantially better than the results obtained by the Ritz method, except for the case of the square plate. However, they are less accurate than the results obtained by the Kantorovich method (Table III). As far as the power dissipated in the plate is concerned, our results are in very good agreement with those obtained by the two variational methods or numerically for small values of γ ($\gamma \leq 2$). For $\gamma > 2$, we have a discrepancy that increases with γ . The results obtained by the Ritz method manifest the same behavior, although the increased discrepancy with γ is smoother. Only the Kantorovich method yields results in very good agreement with the numerical ones for any value of γ considered (Table IV).

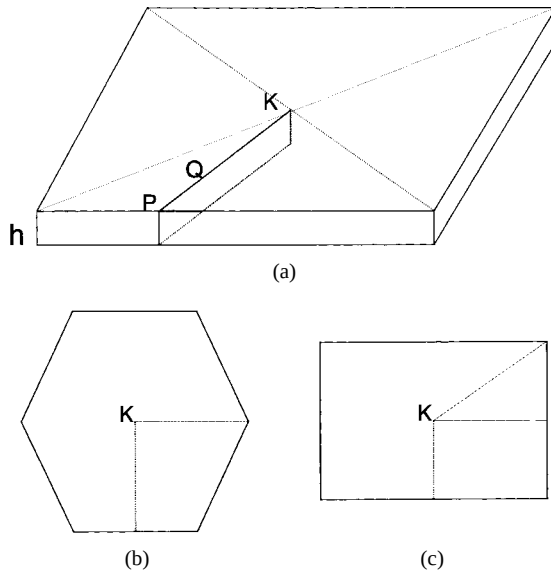


Fig. 4. Geometrical characteristics of the current distribution computations.

C. Comparison of Current Distributions Obtained Numerically and Analytically

The analytical models provide idealized current paths having the shape of the plate under investigation. In reality, the current flow inside the plate does not follow exactly the shape of the plate. Therefore, it is interesting to compare the current distribution inside the plate given by the improved model with that obtained numerically.

The eddy current circulates around the center of mass K of the plate that coincides with the geometric center for plates having symmetrical shapes. Therefore, the current circulating through any section with area $(KP)h$ as the shaded one in Fig. 4(a) is independent of the position of the point P at the edge of the plate and equal to the total eddy current. Furthermore, the current is uniformly distributed over the thickness h of the plate, according to the assumption 1) in Section II. The emerging question now has to do with the current distribution between the center and the edge of the plate. If we consider a point Q on the line KP between K and P [Fig. 4(a)], the current circulating through the section with area $(KQ)h$ depends on two parameters: The distance KQ of the point Q from the center of the plate and the orientation of the line KQP . The current distribution over sections lying in some characteristic directions has been computed analytically (improved model) and numerically.

In fact, we have computed the current circulating between the points K and Q , i.e., through the section with area $(KQ)h$ as a function of the distance KQ . In order to report on the same diagram, the current distribution over some characteristic orientations for a given plate, we have considered the normalized distance $(KQ)/(KP)$ instead of KQ . Thus, $(KQ)/(KP)$ varies from zero to one as Q moves from K to P .

In the case of plates having the shape of a regular polygon, we have considered the current distribution over two characteristic directions [Fig. 4(b)]: One perpendicular to the side of the polygon and the other in the direction joining the center K of the plate with a given vertex. For rectangular

plates, we have computed the current distribution over three characteristic directions [Fig. 4(c)]: Perpendicular to the longer side, perpendicular to the shorter side, and lying on the diagonal.

The results of the computations are shown in Fig. 5 for plates having the shape of a regular polygon and in Fig. 6 for rectangular plates. The current distribution given by the improved model (solid line in Figs. 5 and 6) is the same over any direction considered. This is an expected result because we have assumed current paths similar to the shape of the plate. Therefore, the current density depends on one space variable only. On the other hand, the current distributions obtained numerically (dashed lines in Figs. 5 and 6) are different for each of the directions considered. This is so because in reality, the current density depends on two space variables. However, the current distribution obtained by the improved model is in very good agreement with the numerical results for the current distribution over the shortest distance between the center and the edge of the plate and represents in most cases, a mean value for the current distribution over all directions. In the case of elongated rectangles, the numerical results for the current distribution over the direction perpendicular to the shorter side of the plate have a special characteristic which is worth mentioning. The fraction of the current circulating between the center K and the mid-distance from the shorter side is practically negligible. As we can verify from Fig. 6 (with $\gamma = 5$), the current circulating between the center and a normalized distance 0.6 is less than the 1/20 of the total circulating current. Thus, the main fraction of the current circulates in the outer part (between 0.6 and 1.0), where we observe a rapid rise of the curve. This can explain why for rectangles with $\gamma = 5$ we have the maximum discrepancy (12.5%) for the dissipated power, between improved model and numerical results (Section V-A). The same is true for comparisons between the improved model and analytical results obtained by the use of a variational method (Section V-B).

VI. CONCLUSIONS

In the present work, we have proposed two simple models for the analytical calculation of eddy currents in thin conductive plates. In a first step, we have developed a purely geometric model, i.e., we have assumed that the current paths in the plate are determined exclusively by the shape of the plate. This assumption made apparent a dimensionless *shape factor* that is characteristic for the shape of the plate considered and is independent of the dimensions of the plate. Thus, the parameters of interest, i.e., current densities, equivalent resistance of the plate, total circulating current, and power dissipated in the plate, are expressed as a function of this shape factor. By using this model, we can estimate the total eddy current circulating in plates of any shape, but the precision of the estimation depends strongly on the shape of the plate. However, the precision may be satisfactory for some particular cases. Anyway, this estimation constitutes a lower bound for the total circulating current.

In a second step, we have improved the geometric model by taking into consideration the fact that the current distribution

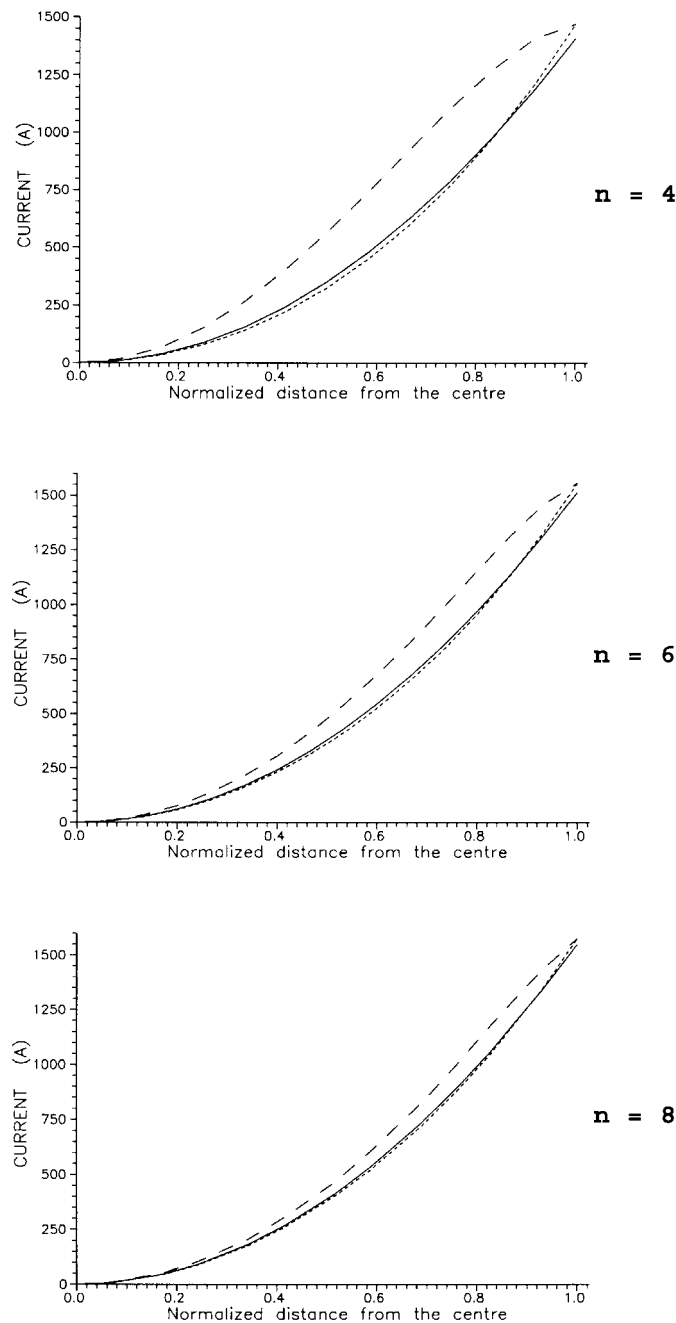


Fig. 5. Current distribution in plates having the shape of regular polygon. — Improved model. — — — Numerical in diagonal direction. Numerical in direction perpendicular to the side.

This would explain uniform J_z in a DC wire!

in the plate is such that the dissipated energy is minimum. The application of this improved model yields a maximum discrepancy between analytical and numerical results for the total circulating current of the order of 4%, but the model is restricted to plates with symmetrical shape.

We have compared also our analytical results for plates with rectangular shape to analytical results obtained by using variational methods, namely the Ritz and Kantorovich methods. We have verified that our results concerning the total circulating current are better than those obtained by the Ritz method but are less accurate than the results obtained by the Kantorovich method. As far as the power losses are

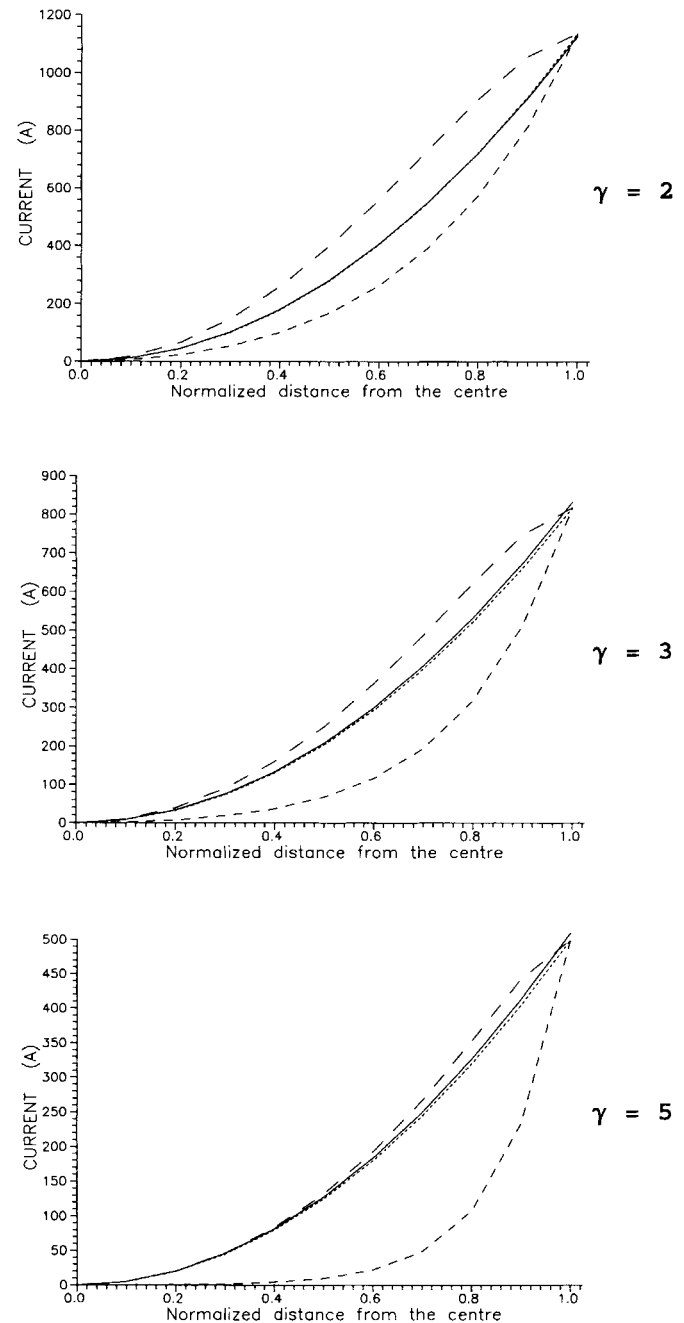


Fig. 6. Current distribution in rectangular plates. — Improved model. — — — Numerical in diagonal direction. - - - - Numerical in direction perpendicular to the shorter side. Numerical in direction perpendicular to the longer side.

concerned, our results are accurate for regular polygons or less elongated rectangles but less accurate than those obtained by the variational methods for elongated rectangles. This is due mainly to the fact that, in our models, we have approximated the real-curved current paths by straight segments. Thus, we consider in each current path segment a mean value for the current density, i.e., we assume that it depends on only one space variable (x , y , or r), instead of two. Consequently, the mean current densities are less accurate when the current path segments considered are strongly unequal as in the case of elongated rectangles. In that case, our models yield an estimate

for the current densities less accurate than the variational methods where curved current paths have been considered that better represent the real current paths. This deficiency in our models is less evident in the case of the total circulating current that depends linearly on the current densities but more apparent in the case of the power losses that are proportional to the square of the current densities. However, the use of variational methods requires guessing a different stream function for each shape of plate and needs lengthier calculations. On the contrary, our method is simpler and needs fewer calculations. It is rather more general than the variational, since the need to analyze each shape of plate separately is avoided, and we may treat a group of shapes (for example polygons) simultaneously.

Despite the fact that our models are based on geometrical assumptions rather than physical ones, they achieve in most cases quite satisfactory results and at the same time, combine effectively electromagnetics and geometry. This geometric character of our models offers a further advantage: it allows an extension (obviously under some supplementary assumptions) to cases where the secondary magnetic flux is not negligible. This is due to the fact that we approach the curved current paths by idealized ones consisting of straight segments. Thus, we have the possibility of calculating analytically the magnetic flux through the surface of the plate due to the eddy current. Consequently, we may calculate analytically the equivalent inductance of the plate. This is planned for some plates of chosen shape in future work. Another extension of the present paper would be to modify the improved model that is now limited to plates with symmetrical shape so that it is applicable to plates of any shape.

APPENDIX

We seek the ratio γ_e of the semi-axis of an ellipse having the following property. Its perimeter C_e , given by (59) as a function of γ_e , must approximate the length of the shortest (energetically) current path among all the possible current paths surrounding the same area A in a rectangle with aspect ratio γ . If we consider an ellipse with area A and ratio of semi-axis γ , its perimeter C_1 represents a probable current path in the rectangle under investigation, but this path is not energetically the shortest one. On the other hand, if we consider a circle with area A , its perimeter C_0 has the shortest path, but it is not appropriate as current path in the rectangle because of the elongated shape of the rectangle. Consequently, the perimeter C_e must be greater than C_0 given by (50) but smaller than C_1 given as a function of γ by a relation similar to (59), i.e.,

$$C_1 \approx \left[\frac{3}{2}(\gamma + 1) - \sqrt{\gamma} \right] \sqrt{\frac{\pi A}{\gamma}}. \quad (73)$$

Thus, we assume that C_e is given by the relation

$$C_e = \lambda C_1 + (1 - \lambda) C_0 \quad (74)$$

where λ and $1 - \lambda$ are weighting factors, determined with the following grounds.

The required ellipse with ratio of semi-axis γ_e will be closer to an ellipse with ratio of semi-axis γ , i.e., equal to the

TABLE V

γ	β	γ_e	γ_e/γ
1.0	2.0000	1.0000	1.0000
2.0	2.3301	1.7628	0.8814
3.0	2.9821	2.5970	0.8657
4.0	3.7600	3.4720	0.8680
5.0	4.6018	4.3732	0.8746

aspect ratio of the rectangle under investigation, depending on whether this rectangle is elongated or not. On the other hand, this ellipse will be again closer to an ellipse with ratio of semi-axis $\gamma_0 = 1$, i.e., a circle, depending on how well this rectangle approaches a square ($\gamma = 1$). Thus, we assume that a proportionality exists between λ and γ and similarly between $1 - \lambda$ and γ_0 , i.e., $\lambda \propto \gamma$ and $(1 - \lambda) \propto \gamma_0$, or

$$\frac{\lambda}{1 - \lambda} = \frac{\gamma}{\gamma_0}. \quad (75)$$

From (75), we obtain

$$\lambda = \frac{\gamma}{\gamma + \gamma_0} \quad 1 - \lambda = \frac{\gamma_0}{\gamma + \gamma_0}. \quad (76)$$

The fact that $\gamma_0 = 1$ allows us to write the above relations as

$$\lambda = \frac{\gamma}{\gamma + 1} \quad 1 - \lambda = \frac{1}{\gamma + 1}. \quad (77)$$

Consequently, (74) is now written

$$C_e = \frac{\gamma}{\gamma + 1} C_1 + \frac{1}{\gamma + 1} C_0. \quad (78)$$

If we take into consideration (50), (59), and (73), we put (78) into the form

$$\begin{aligned} & \left[\frac{3}{2}(\gamma_e + 1) - \sqrt{\gamma_e} \right] \sqrt{\frac{\pi A}{\gamma_e}} \\ &= \frac{\gamma}{\gamma + 1} \left[\frac{3}{2}(\gamma + 1) - \sqrt{\gamma} \right] \sqrt{\frac{\pi A}{\gamma}} + \frac{2}{\gamma + 1} \sqrt{\pi A}. \end{aligned} \quad (79)$$

From (79), we obtain the following relation between γ_e and γ :

$$\frac{\gamma_e + 1}{\sqrt{\gamma_e}} = \sqrt{\gamma} + \frac{2}{\gamma + 1}. \quad (80)$$

Equation (80) results to a quadratic equation for γ_e

$$\gamma_e^2 - \beta \gamma_e + 1 = 0 \quad (81)$$

where, for brevity, we set

$$\beta = \left(\sqrt{\gamma} + \frac{2}{\gamma + 1} \right)^2 - 2. \quad (82)$$

The solution of the quadratic (81) yields γ_e

$$\gamma_e = \frac{\beta \pm \sqrt{\beta^2 - 4}}{2}. \quad (83)$$

The root with the minus sign is rejected because the resulting value for γ_e is less than unity. So, we retain only the root with the plus sign.

In Table V, we give, for $\gamma = 1.0, 2.0, 3.0, 4.0$, and 5.0 , the resulting values for β and γ_e as well as the ratio γ_e/γ . We remark that, for $\gamma > 1$, the ratio γ_e/γ is approximately equal to $\sqrt{\pi}/2$ (0.8862). Thus, we may use the relationship $\gamma_e \approx \gamma\sqrt{\pi}/2$ for a rapid estimation of γ_e instead of solving (81).

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