Appendix Q Rewrite PhL 8.30.14

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I just realized that I take frequency limits in this appendix taking R,L,G,C all as constants, but that is totally the wrong thing to do in the real world!! So I want to go through App Q and make fixes as needed.

(a) Properties of k(ω)

The opening (Q.1) section is OK because no limits on ω are taken, no assumption is made about whether the four parameters are constants or functions of ω.

Fact (Q.2) is the high ω limit, and this is going to require work.

1. Notes on a model for R,G,L,C

In the original appendix, I treat R,G,L,C as constants, independent of ω.

I know from my Belden appendix that this is a very bad assumption at higher frequencies certainly. So I guess my first task is to come up with "models" for the four parameters. I avoided doing this up to now, but I suppose I really should do this right!

Model for C: At all frequencies, I will treat C as a constant. That leaves G,R,L to worry about.

Model for G: The exact model is this: [ε'_d is the real part of ε_d]

$$G = (\sigma_{eff}/\varepsilon_d)\, C \qquad \text{where } \sigma_{eff} = (\sigma_d + \omega\varepsilon'_d \tan_L)$$

Then

$$G = (1/\varepsilon_d)C\,[\sigma_d + \omega\varepsilon'_d \tan_L] = G_{dc} + (\varepsilon_d'/\varepsilon_d)\tan_L C\,\omega \qquad G_{dc} = (\sigma_d/\varepsilon_d)\, C$$

Should I maintain all this detail? Any good transmission line should have G_{dc} very small, such as is the case with Belden 8281. G only becomes an issue at higher ω. One thing I CAN do is absorb $(\varepsilon_d'/\varepsilon_d)$ into $\tan_L$ to have fewer symbols. Then

$G = G_{dc} + (\tan_L C)\,\omega$

So the above line would be a find model for G.

Model for L and R: In App R I deal specifically with the coaxial cable. I first work at DC and there I compute the external inductance L_e using the K formula. I don't think L_e changes with ω very much, what changes is the distribution of the currents inside the two conductors and probably outside you can regard things as crudely constant. My whole Chap 4 is oriented to high ω anyway, so I think $L_e = \frac{\mu_0}{4\pi} K$ is a fine model for L_e at all ω. Meanwhile, I compute L_i for each conductor at DC. But then if you look at Fig R.2, you see that, at all frequencies ω, you really could neglect L_i relative to L_e as a rough approximation. You could make some kind of combined model if you wanted. My high ω model is this for R and L_i

$$R_1 = \frac{1}{\sigma(2\pi a_1)\delta} \qquad (2.4.18)$$

$$L_{1i} = (1/\omega)\,R_1 = \frac{1}{\omega\sigma(2\pi a_1)\delta} = \frac{1}{\omega\sigma(2\pi a_1)\sqrt{2/\omega\mu_0\sigma}} = \frac{1}{2\pi a_1}\sqrt{\frac{\mu_0}{2\sigma\omega}}\ . \qquad (2.4.19)$$

and similar equations for conductor #2. Let's write the above again as

$$L_{1i} = \frac{1}{2\pi a_1}\sqrt{\frac{\mu_0}{2\sigma}}\,\omega^{-1/2} \qquad \text{decreases with } \omega$$

$$R_1 = \omega L_{1i} = \frac{1}{2\pi a_1}\sqrt{\frac{\mu_0}{2\sigma}}\,\omega^{+1/2} \qquad \text{increases with } \omega$$

I think my generalizations of these equations to an arbitrary conductor #1 would be this

$$L_{1i} = \frac{1}{p_1}\sqrt{\frac{\mu_0}{2\sigma}}\,\omega^{-1/2} \qquad \text{decreases with } \omega$$

$$R_1 = \omega L_{1i} = \frac{1}{p_1}\sqrt{\frac{\mu_0}{2\sigma}}\,\omega^{+1/2} \qquad \text{increases with } \omega$$

where p_1 is the active perimeter of the cross section of conductor C_1. Then adding we get

$$L_i = \left(\frac{1}{p_1} + \frac{1}{p_2}\right)\sqrt{\frac{\mu_0}{2\sigma}}\,\omega^{-1/2} \qquad \text{decreases with } \omega$$

$$R = \left(\frac{1}{p_1} + \frac{1}{p_2}\right)\sqrt{\frac{\mu_0}{2\sigma}}\,\omega^{+1/2} \qquad \text{increases with } \omega \qquad \kappa = \left(\frac{1}{p_1} + \frac{1}{p_2}\right)\sqrt{\frac{\mu_0}{2\sigma}}$$

dim check: dim(κ) = m^{-1} * (hen/m * ohm-m)$^{1/2}$ = m^{-1} * (ohm-sec* ohm)$^{1/2}$ = **ohm/m * sec**$^{1/2}$ for κ
Then dim(L_i) = ohm/m * $sec^{1/2}$ $sec^{1/2}$ = ohm-sec/m = henry/m = **correct**
Then dim(R) = ohm/m * $sec^{1/2}$ $sec^{-1/2}$ = ohm/m = **correct**

So I think this is a good model for L_i and R at high ω. How high should ω be for this model to be viable? One approach is to say that δ << a_1 and δ << a_2 where the a_i are the effective radii of the two conductors. That means, using $\delta \equiv \sqrt{2/\omega\mu\sigma}$,

$\sqrt{2/\omega\mu\sigma} \ll a_i$

$2/\omega\mu_0\sigma \ll a_i^2$

$2 \ll (\mu_0\sigma a_i^2)\omega$

$1 \ll (\mu_0\sigma a_i^2/2)\omega$

$1/\omega \ll (\mu_0\sigma a_i^2/2)$

$\omega \gg 2/(\mu_0\sigma a_i^2) \equiv \omega_i$ ok

What is this for the Belden case?

```
omega1 := 2/(mu0*sigma*a1^2);f1 := omega1/(2*pi);
                              ω1 := 176731.1768
                              f1 := 28127.64038
omega2 := 2/(mu0*sigma*a2^2); f2 := omega2/(2*pi);
                              ω2 := 4308.157850
                              f2 := 685.6646175
```

So for Belden we would use our high ω model as long as f >> 28 KHz and f > 685 Hz, where the inner conductor dominates the bound. For a power transmission line we would take the a_2 case above but instead of a_2 = .0025 m, we would use a_2 = 1/2" = 1.27 cm = .0127 m so ω_i will be smaller by the following ratio: $(.0127/.0025)^2 = 5.08^2 = 26$. Then we would have

f_2 = 686 Hz / 26 = 26 Hz

so a power line is in fact in the high ω limit more or less.

I would like some simple way to morph from high ω model to the DC model.

How about this very simple approximation:

$$R(\omega) = R_{dc} + \left(\frac{1}{p_1} + \frac{1}{p_2}\right)\sqrt{\frac{\mu_0}{2\sigma}}\,\omega^{+1/2} = R_{dc} + \kappa\,\omega^{+1/2} \qquad \kappa = \left(\frac{1}{p_1} + \frac{1}{p_2}\right)\sqrt{\frac{\mu_0}{2\sigma}}$$

Probably it is inaccurate at low ω, but it is exact near ω = 0 and exact for ω >> $\omega_{1,2}$ and so maybe it is reasonable for all ω.

Here is another approach

$R(\omega) = R_{dc}\,\theta(\omega<\omega_1) + \kappa\,\omega^{+1/2}\,\theta(\omega\geq\omega_1)$ $\omega_1 = (\omega_1+\omega_2)/2$ perhaps

where ω_1 makes the two sides of the function match:

$R_{dc} = \kappa\,\omega_1^{+1/2}$ => $(R_{dc}/\kappa)^2 = \omega_1$.

My model for $L_i(\omega)$ is this

$L_i(\omega) = L_{i,DC}\,\theta(\omega < \omega_a) + \kappa\,\omega^{-1/2}\,\theta(\omega \leq \omega_a)$ $\omega_a = (\kappa/ L_{i,DC})^2$

where the value ω_a causes the two sides of the function to meet at $L_{i,DC}$.

and then

$L(\omega) = L_e + L_{i,DC}\,\theta(\omega < \omega_a) + \kappa\,\omega^{-1/2}\,\theta(\omega \geq \omega_a)$

For large ω, we get $L(\omega) \approx L_e$ in this model.

For small ω, we get $L(\omega) = L_e + L_{i,DC}$ in this model.

We know that

$L_e = \mu_d K/(4\pi)$

$C = 4\pi\varepsilon_d/K$

So we can express things in terms of C as follows:

$C = 4\pi\varepsilon_d/K$ => $K = 4\pi\varepsilon_d/C$ ok

$L_e = (\mu_d/4\pi)\,K$ => $L_e = (\mu_d/4\pi)\,4\pi\varepsilon_d/C = (\mu_d\varepsilon_d)/C = 1/(v_d^2 C)$ ok

$G_{dc} = 4\pi\sigma_d/K$ $= 4\pi\sigma_d\ C / 4\pi\varepsilon_d = (\sigma_d/\varepsilon_d)C$ ok

or

$L_e = 1/(v_d^2 C)$

$G_{dc} = (\sigma_d/\varepsilon_d)C$

dim(1/RHS1) = m^2/sec^2* far/m = far/sec^2 * m = m/hen = **correct**

dim(RHS2) = mho/m * m/far * far/m = mho/m = **correct**

Summary of my proposed model:

$C(\omega) = C$

$G(\omega) = G_{dc} + (\tan_L C)\,\omega \qquad G_{dc} = (\sigma_d/\varepsilon_d)\,C$

$R(\omega) = R_{dc}\,\theta(\omega<\omega_1) + \kappa\,\omega^{+1/2}\,\theta(\omega\geq\omega_1) \qquad \kappa = \left(\frac{1}{p_1}+\frac{1}{p_2}\right)\sqrt{\frac{\mu_0}{2\sigma}} \qquad \omega_1 = (R_{dc}/\kappa)^2$

$L(\omega) = L_e + L_{idc}\,\theta(\omega < \omega_0) + (\kappa/\sqrt{\omega})\,\theta(\omega \geq \omega_0) \qquad \omega_0 = (\kappa/L_{i,DC})^2$

Notes:

1. For a good transmission line, one has $\sigma_d \sim 10^{-15}$ so G_{dc} is very small and can be neglected except very close to $\omega = 0$, more specifically, for $\omega \leq \sigma_d/(\varepsilon_d \tan_L)$.

2. $L_{idc} = \mu_0/(4\pi)$ for both conductors being round wires. For other cross sectional shapes, this expression is approximately valid. See ****.

3. Parameter p_i is the length of the active perimeter of conductor i, see ****.

4. μ_0 has been assumed inside the conductors.

Now restate using C wherever possible:

$C(\omega) = C$

$G(\omega) = (\sigma_d/\varepsilon_d)\,C + (\tan_L C)\,\omega$

$R(\omega) = R_{dc}\,\theta(\omega<\omega_1) + \kappa\,\omega^{+1/2}\,\theta(\omega\geq\omega_1) \qquad \kappa = \left(\frac{1}{p_1}+\frac{1}{p_2}\right)\sqrt{\frac{\mu_0}{2\sigma}} \qquad \omega_1 = (R_{dc}/\kappa)^2$

$L(\omega) = 1/(v_d^2 C) + L_{idc}\,\theta(\omega < \omega_0) + (\kappa/\sqrt{\omega})\,\theta(\omega \geq \omega_0) \qquad \omega_0 = (\kappa/L_{i,DC})^2$

I want a better symbol for (σ_d/ε_d) which has dimensions:

$\dim(\sigma_d/\varepsilon_d)$ = mho/m // farad/m = mho/m * m/far = mho/far = 1/(far-ohm) = 1/sec

Lets then define

$\omega_d \equiv \sigma_d/\varepsilon_d$ // all DC

Then we have

$C(\omega) = C$

$G(\omega) = \omega_d C + (\tan_L C)\,\omega = C\,(\omega_d + \tan_L \omega)$

$R(\omega) = R_{dc}\,\theta(\omega<\omega_1) + \kappa\,\omega^{+1/2}\,\theta(\omega\geq\omega_1)$ $\qquad \kappa = \left(\frac{1}{p_1} + \frac{1}{p_2}\right)\sqrt{\frac{\mu_0}{2\sigma}}$ $\qquad \omega_1 = (R_{dc}/\kappa)^2$

$L(\omega) = 1/(v_d^2 C) + L_{idc}\,\theta(\omega < \omega_0) + (\kappa/\sqrt{\omega})\,\theta(\omega \geq \omega_0)$ $\qquad \omega_0 = (\kappa/ L_{i,DC})^2$

Right away I want to think about small and high ω limits of these results,

2. Notes on computing large ω limits of things

high ω:

$C(\omega) = C$
$G(\omega) = C\,(\omega_d + \tan_L \omega)$
$R(\omega) = \kappa\,\omega^{+1/2}$
$L(\omega) = 1/(v_d^2 C) + \kappa\,\omega^{-1/2}$

$(R^2+\omega^2L^2) = \kappa^2\,\omega + \omega^2\,[1/(v_d^2 C) + \kappa\,\omega^{-1/2}]^2$

$\approx \omega^2\,1/(v_d^2 C)^2$

$(G^2+\omega^2C^2) = [(\sigma_d/\varepsilon_d)\,C + (\tan_L C)\,\omega]^2+\omega^2C^2 \approx \omega^2C^2(1 + \tan_L^2) \approx \omega^2C^2$

We then know that

$a^2 = |k|^2 = [(R^2+\omega^2L^2)(G^2+\omega^2C^2)]^{1/2}$

$= \omega/(v_d^2 C) * \omega C = \omega^2/v_d^2$

and

$a = |k| = \omega/v_d$ as expected

Let's now have Maple do the high ω limit of things: I call the maple file q2.mws, and here are my results.

```
G := C*(wd + t*u^2);
```

$$G := C\,(wd + t\,u^2)$$

```
R := kappa*u;
```

$$R := \kappa\,u$$

```
L := 1/(vd^2*C) + kappa/u;
```

$$L := \frac{1}{vd^2\,C} + \frac{\kappa}{u}$$

I am using u = $\sqrt{\omega}$. Now I first compute a , I convert it to a series in small u, and then I take that result for small t. Here are some results. First, here is a:

```
a := ((R^2+u^4*L^2)*(G^2+u^4*C^2))^(1/4);
```

$$a := \left(\left(\kappa^2\,u^2 + u^4\left(\frac{1}{vd^2\,C} + \frac{\kappa}{u}\right)^2\right)\left(C^2\,(wd + t\,u^2)^2 + u^4\,C^2\right)\right)^{\left(\frac{1}{4}\right)}$$

Then here is the series for a, which I call a1:

```
> series(a,u=infinity,1): a1 :=  collect(%,u);
```

$$a1 := \left(\frac{C^2\,t^2 + C^2}{vd^4\,C^2}\right)^{\left(\frac{1}{4}\right)} u^2 + \frac{1}{2}\left(\frac{C^2\,t^2 + C^2}{vd^4\,C^2}\right)^{\left(\frac{1}{4}\right)} vd^2\,C\,\kappa\,u + \left(\frac{C^2\,t^2 + C^2}{vd^4\,C^2}\right)^{\left(\frac{1}{4}\right)}\left(\frac{1}{4}\,\frac{\left(2\,\frac{t\,wd}{vd^4} + 2\,\kappa^2\,(C^2\,t^2 + C^2)\right)vd^4\,C^2}{C^2\,t^2 + C^2} - \frac{3}{8}\,vd^4\,C^2\,\kappa^2\right) + O\left(\frac{1}{u}\right)$$

which you see has the form a1 = Au^2 + Bu + C + O(1/u). I then take each of the three leading terms, simplify the term, then get a series for small t,

```
op(1,a1); simplify(%): series(%,t=0,2);
```

$$\left(\frac{C^2 t^2 + C^2}{vd^4 C^2}\right)^{\left(\frac{1}{4}\right)} u^2$$

$$\frac{u^2}{vd} + O(t^2)$$

```
op(2,a1); simplify(%): series(%,t=0,2);
```

$$\frac{1}{2}\left(\frac{C^2 t^2 + C^2}{vd^4 C^2}\right)^{\left(\frac{1}{4}\right)} vd^2 C \kappa u$$

$$\frac{1}{2} vd\, C \kappa u + O(t^2)$$

```
op(3,a1); simplify(%): series(%,t=0,2);
```

$$\left(\frac{C^2 t^2 + C^2}{vd^4 C^2}\right)^{\left(\frac{1}{4}\right)} \left(\frac{1}{4}\frac{\left(2\frac{t\,wd}{vd^4} + 2\kappa^2 (C^2 t^2 + C^2)\right) vd^4 C^2}{C^2 t^2 + C^2} - \frac{3}{8} vd^4 C^2 \kappa^2\right)$$

$$\frac{1}{8} vd^3 C^2 \kappa^2 + \frac{1}{2}\frac{wd}{vd} t + O(t^2)$$

My result for a is then

$$a = (1/v_d)\omega + (1/2)v_d C\kappa \sqrt{\omega} + [\,(1/8)\, v_d^3 (C\kappa)^2 + (1/2)\,(\omega_d/v_d)\, t]$$

It seems save to ignore the term here that is constant in ω, so then

$$|k| = a \approx (1/v_d)\omega + (1/2)v_d C\kappa \sqrt{\omega}$$

and this then gives some idea as to the correction for the expected leading term ω/v_d.

Next, I consider Re(k):

```
> # DO RE(K)
> Rek := sqrt((a^2-c)/2);
```

$$Rek := \frac{1}{2}\sqrt{2\sqrt{\left(\kappa^2 u^2 + u^4\left(\frac{1}{vd^2 C}+\frac{\kappa}{u}\right)^2\right)\left(C^2(wd+t u^2)^2 + u^4 C^2\right)} - 2\kappa u C(wd+t u^2) + 2u^4\left(\frac{1}{vd^2 C}+\frac{\kappa}{u}\right)C}$$

```
> series(Rek,u=infinity,1): Rek1 :=collect(%,u);
```

$$Rek1 := \frac{1}{2}\sqrt{2\frac{\sqrt{C^2 t^2 + C^2}}{vd^2 C} + 2\frac{1}{vd^2}}\,u^2 + \frac{1}{4}\frac{(2\sqrt{C^2 t^2 + C^2}\,\kappa + 2\kappa C - 2\kappa C t)u}{\sqrt{2\frac{\sqrt{C^2 t^2 + C^2}}{vd^2 C} + 2\frac{1}{vd^2}}} + \frac{1}{2}\sqrt{2\frac{\sqrt{C^2 t^2 + C^2}}{vd^2 C} + 2\frac{1}{vd^2}}$$
$$\left(\frac{\sqrt{C^2 t^2 + C^2}\left(\frac{1}{2}\frac{\left(2\frac{t\,wd}{vd^4} + 2\kappa^2(C^2 t^2 + C^2)\right)vd^4 C^2}{C^2 t^2 + C^2} - \frac{1}{2}vd^4 C^2 \kappa^2\right)}{vd^2 C\left(2\frac{\sqrt{C^2 t^2 + C^2}}{vd^2 C} + 2\frac{1}{vd^2}\right)} - \frac{1}{8}\frac{(2\sqrt{C^2 t^2 + C^2}\,\kappa + 2\kappa C - 2\kappa C t)^2}{\left(2\frac{\sqrt{C^2 t^2 + C^2}}{vd^2 C} + 2\frac{1}{vd^2}\right)^2}\right) + O\left(\frac{1}{u}\right)$$

It is a mess, but now again I take each term, simplify it, then take the small t limit. I have to display the ops because sometimes the ops are wrong.

```
> op(1,Rek1); simplify(%): series(%,t=0,3);
```

$$\frac{1}{2}\sqrt{2\frac{\sqrt{C^2 t^2 + C^2}}{vd^2 C} + 2\frac{1}{vd^2}}\,u^2$$

$$\frac{u^2}{vd} + \frac{1}{8}\frac{u^2}{vd}t^2 + O(t^4)$$

```
> op(2,Rek1); simplify(%): series(%,t=0,3);
```

$$\frac{1}{4}\frac{(2\sqrt{C^2 t^2 + C^2}\,\kappa + 2\kappa C - 2\kappa C t)u}{\sqrt{2\frac{\sqrt{C^2 t^2 + C^2}}{vd^2 C} + 2\frac{1}{vd^2}}}$$

$$\frac{1}{2}vd\kappa C u - \frac{1}{4}vd\kappa C u t + \frac{1}{16}vd\kappa C u t^2 + O(t^3)$$

```
> op(3,Rek1); simplify(%): series(%,t=0,3);
```

$$\frac{1}{2}\sqrt{2\frac{\sqrt{C^2 t^2 + C^2}}{vd^2 C} + 2\frac{1}{vd^2}}$$
$$\left(\frac{\sqrt{C^2 t^2 + C^2}\left(\frac{1}{2}\frac{\left(2\frac{t\,wd}{vd^4} + 2\kappa^2(C^2 t^2 + C^2)\right)vd^4 C^2}{C^2 t^2 + C^2} - \frac{1}{2}vd^4 C^2 \kappa^2\right)}{vd^2 C\left(2\frac{\sqrt{C^2 t^2 + C^2}}{vd^2 C} + 2\frac{1}{vd^2}\right)} - \frac{1}{8}\frac{(2\sqrt{C^2 t^2 + C^2}\,\kappa + 2\kappa C - 2\kappa C t)^2}{\left(2\frac{\sqrt{C^2 t^2 + C^2}}{vd^2 C} + 2\frac{1}{vd^2}\right)^2}\right)$$

$$\frac{1}{8}\frac{2\,wd + vd^4 C^2 \kappa^2}{vd}t + O(t^3)$$

Thus, my result is (since t << 1)

$$\mathrm{Re}(k) \approx \omega/v_d + (1/2)v_d\kappa C\sqrt{\omega} + (1/8v_d)(2\omega_d + \kappa^2 v_d^4 C^2)\tan_L + O(1/\sqrt{\omega})$$

$$= A\omega + B\sqrt{\omega} + C + O(1/\sqrt{\omega})$$

The leading term is as expected, but now you see some of the lower terms as well.

I then treat Imk the same way

```
> # DO IM(K)
> Imk := - sqrt((a^2+c)/2);
```

$$Imk := -\frac{1}{2}\sqrt{2\sqrt{\left(\kappa^2 u^2 + u^4\left(\frac{1}{vd^2 C}+\frac{\kappa}{u}\right)^2\right)\left(C^2(wd+t u^2)^2 + u^4 C^2\right)} + 2\kappa u C(wd+t u^2) - 2u^4\left(\frac{1}{vd^2 C}+\frac{\kappa}{u}\right)C}$$

```
> series(Imk,u=infinity,1): Imk1 :=collect(%,u);
```

$$Imk1 := -\frac{1}{2}\sqrt{2\frac{\sqrt{C^2 t^2 + C^2}}{vd^2 C} - 2\frac{1}{vd^2}}\,u^2 - \frac{1}{4}\frac{(2\sqrt{C^2 t^2 + C^2}\,\kappa - 2C\kappa + 2C\kappa t)u}{\sqrt{2\frac{\sqrt{C^2 t^2 + C^2}}{vd^2 C} - 2\frac{1}{vd^2}}} - \frac{1}{2}\sqrt{2\frac{\sqrt{C^2 t^2 + C^2}}{vd^2 C} - 2\frac{1}{vd^2}}$$
$$\left(\frac{\sqrt{C^2 t^2 + C^2}\left(\frac{1}{2}\frac{\left(2\frac{t\,wd}{vd^4} + 2\kappa^2(C^2 t^2 + C^2)\right)vd^4 C^2}{C^2 t^2 + C^2} - \frac{1}{2}vd^4 C^2\kappa^2\right)}{vd^2 C\left(2\frac{\sqrt{C^2 t^2 + C^2}}{vd^2 C} - 2\frac{1}{vd^2}\right)} - \frac{1}{8}\frac{(2\sqrt{C^2 t^2 + C^2}\,\kappa - 2C\kappa + 2C\kappa t)^2}{\left(2\frac{\sqrt{C^2 t^2 + C^2}}{vd^2 C} - 2\frac{1}{vd^2}\right)^2}\right) + O\left(\frac{1}{u}\right)$$

Then here are the series

```
> op(1,Imk1); simplify(%): series(%,t=0,4);
```

$$-\frac{1}{2}\sqrt{2\frac{\sqrt{C^2t^2+C^2}}{vd^2\,C}-2\frac{1}{vd^2}}\,u^2$$

$$-\frac{1}{2}\frac{u^2}{vd}t+\frac{1}{16}\frac{u^2}{vd}t^3+O(t^5)$$

```
> op(2,Imk1); simplify(%): series(%,t=0,4);
```

$$-\frac{1}{4}\frac{(2\sqrt{C^2t^2+C^2}\,\kappa-2\kappa C+2\kappa C t)\,u}{\sqrt{2\frac{\sqrt{C^2t^2+C^2}}{vd^2\,C}-2\frac{1}{vd^2}}}$$

$$-\frac{1}{2}vd\,\kappa C u-\frac{1}{4}vd\,\kappa C u t-\frac{1}{16}vd\,\kappa C u t^2+O(t^3)$$

```
> op(3,Imk1); simplify(%): series(%,t=0,4);
```

$$-\frac{1}{2}\sqrt{2\frac{\sqrt{C^2t^2+C^2}}{vd^2\,C}-2\frac{1}{vd^2}}\left(\frac{\sqrt{C^2t^2+C^2}\left(\frac{1}{2}\frac{\left(2\frac{t\,wd}{vd^4}+2\kappa^2(C^2t^2+C^2)\right)vd^4C^2}{C^2t^2+C^2}-\frac{1}{2}vd^4C^2\kappa^2\right)}{vd^2\,C\left(2\frac{\sqrt{C^2t^2+C^2}}{vd^2\,C}-2\frac{1}{vd^2}\right)}-\frac{1}{8}\frac{(2\sqrt{C^2t^2+C^2}\,\kappa-2\kappa C+2\kappa C t)^2}{\left(2\frac{\sqrt{C^2t^2+C^2}}{vd^2\,C}-2\frac{1}{vd^2}\right)^2}\right)$$

$$\frac{1}{8}\frac{2\,vd^4C^2\kappa^2-4\,wd}{vd}+O(t^2)$$

and then I can read off the result

$$\mathrm{Im}(k) = -(1/2v_d)\,\omega\tan_L - (1/2)v_dC\kappa\sqrt{\omega} - (1/2v_d)(\omega_d - v_d^4C^2\kappa^2/2)$$

This result seems wrong since Im(k) should be negative!

Summary of the large ω limit of k(ω): // This is the new (Q.2) I suppose.

$$\mathrm{Re}(k) \approx \omega/v_d + (1/2)v_d\kappa C\sqrt{\omega} + (1/8v_d)(2\,\omega_d + \kappa^2v_d^4C^2)\tan_L + O(1/\sqrt{\omega})$$

$$\mathrm{Im}(k) = -(1/2v_d)\,\omega\tan_L - (1/2)v_dC\kappa\sqrt{\omega} - (1/2v_d)(\omega_d - v_d^4C^2\kappa^2/2) + O(1/\sqrt{\omega})$$

These are completely new results for Appendix Q. They are completely different from the results I got in my original Appendix Q !!! If I just keep the leading terms,

$\mathrm{Re}(k) \approx (\omega/v_d)$ the usual result

$\mathrm{Im}(k) = -(1/2)(\omega/v_d)\tan_L$ perhaps this is something new.

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3. Notes on computing low-ω limits of things

low ω:

Start again with

$C(\omega) = C$

$G(\omega) = \omega_d C + (\tan_L C)\,\omega \;\; = C\,(\omega_d + \tan_L \omega)$

$R(\omega) \;=\; R_{dc}\,\theta(\omega<\omega_1) + \kappa\,\omega^{+1/2}\,\theta(\omega\geq\omega_1)$ $\qquad \kappa = \left(\frac{1}{p_1} + \frac{1}{p_2}\right)\sqrt{\frac{\mu_0}{2\sigma}}$ $\qquad \omega_1 = (R_{dc}/\kappa)^2$

$L(\omega) = 1/(v_d^2 C) + L_{idc}\,\theta(\omega < \omega_0) \;+\; (\kappa/\sqrt{\omega})\,\theta(\omega \geq \omega_0)$ $\qquad \omega_0 = (\kappa/\,L_{i,DC})^2$

Then for low ω we get

$G(\omega) = C\,(\omega_d + \tan_L \omega)$

$R(\omega) \;=\; R_{dc}$

$L(\omega) = 1/(v_d^2 C) + L_{idc}$

I made a copy and paste version of the Maple file which is Q2 lo omega k. First, I do a:

```
> a := ((R^2+u^4*L^2)*(G^2+u^4*C^2))^(1/4);
```

$$a := \left(\left(Rdc^2 + u^4\left(\frac{1}{vd^2\,C} + Lidc\right)^2\right)\left(C^2\,(wd + t\,u^2)^2 + u^4\,C^2\right)\right)^{\left(\frac{1}{4}\right)}$$

```
> series(a,u=0,1): a1 := collect(%,u);
```

$$a1 := (Rdc^2\,C^2\,wd^2)^{\left(\frac{1}{4}\right)} + \frac{1}{2}\frac{(Rdc^2\,C^2\,wd^2)^{\left(\frac{1}{4}\right)}\,t}{wd}\,u^2 + (Rdc^2\,C^2\,wd^2)^{\left(\frac{1}{4}\right)}\left(\frac{1}{4}\frac{Rdc^2\,(C^2\,t^2 + C^2) + \left(\frac{1}{vd^2\,C} + Lidc\right)^2 C^2\,wd^2}{Rdc^2\,C^2\,wd^2} - \frac{3}{8}\frac{t^2}{wd^2}\right)u^4 + O(u^6)$$

Then the t series,

```
op(1,a1); simplify(%): series(%,t=0,2);
```

$$\frac{\left(\frac{1}{4}\right)}{\left(Rdc^2\, C^2\, wd^2\right)}$$

$$\frac{\sqrt{Rdc}\,\sqrt{C}}{\sqrt{wd}}$$

```
op(3,a1); simplify(%): series\(%,t=0,4);
```

$$\frac{1}{2}\,\frac{\left(\frac{1}{4}\right)}{\left(Rdc^2\, C^2\, wd^2\right)}\,\frac{t}{wd}$$

$$\frac{1}{2}\,\frac{\sqrt{Rdc}\,\sqrt{C}}{\sqrt{wd}}\,t$$

I conclude then that

$$a = \sqrt{R_{dc} C \omega_d} + (1/2)\,\sqrt{R_{dc} C/\omega_d}\; t\,\omega + O(\omega^2)$$

Next comes Re(k):

```
> # DO RE(K)
> Rek := sqrt((a^2-c)/2);
```

$$Rek := \frac{1}{2}\sqrt{2\sqrt{\left(Rdc^2+u^4\left(\frac{1}{vd^2\,C}+Lidc\right)^2\right)\left(C^2\,(wd+t\,u^2)^2+u^4\,C^2\right)}-2\,Rdc\,C\,(wd+t\,u^2)+2\,u^4\left(\frac{1}{vd^2\,C}+Lidc\right)C}$$

```
> series(Rek,u=0,3): Rek1 :=collect(%,u);
```

$$Rek1 := \frac{1}{2}\sqrt{2\,Rdc\,C\,wd\left(\frac{1}{2}\,\frac{Rdc^2\,(C^2\,t^2+C^2)+\left(\frac{1}{vd^2\,C}+Lidc\right)^2 C^2\,wd^2}{Rdc^2\,C^2\,wd^2}-\frac{1}{2}\,\frac{t^2}{wd^2}\right)+2\left(\frac{1}{vd^2\,C}+Lidc\right)C\,u^2+O(u^4)}$$

```
> op(1,Rek1); simplify(%): series(%,t=0,3);
```

$$\frac{1}{2}\sqrt{2\,Rdc\,C\,wd\left(\frac{1}{2}\,\frac{Rdc^2\,(C^2\,t^2+C^2)+\left(\frac{1}{vd^2\,C}+Lidc\right)^2 C^2\,wd^2}{Rdc^2\,C^2\,wd^2}-\frac{1}{2}\,\frac{t^2}{wd^2}\right)+2\left(\frac{1}{vd^2\,C}+Lidc\right)C}$$

$$\frac{1}{2}\,\frac{wd+wd\,Lidc\,vd^2\,C+Rdc\,C\,vd^2}{vd^2\,\sqrt{Rdc}\,\sqrt{C}\,\sqrt{wd}}$$

Here you see that Re(k) = A ω + B ω^2 so I just keep the leading term to get [there is no constant term!]

$$Re(k) = (\omega/2)\,(\omega_d + \omega_d L_{idc} v_d^2 C + R_{dc} C\, v_d^2)\;/\;(\,v_d^2\sqrt{R_{dc} C \omega_d}\,)\;\omega$$

or

$$\mathrm{Re}(k) = (\omega/2)\,(\omega_d + C\, v_d^2[R_{dc} + \omega_d L_{idc}])\ /\ (\, v_d^2\sqrt{R_{dc} C \omega_d}\,)$$

This is similar to my G > 0 result (Q.3) but not the same.

Now for the Imaginary part:

```
> # DO IM(K)
> Imk := - sqrt((a^2+c)/2);
```

$$Imk := -\frac{1}{2}\sqrt{2\sqrt{\left(Rdc^2 + u^4\left(\frac{1}{vd^2\,C} + Lidc\right)^2\right)\left(C^2\,(wd + t\,u^2)^2 + u^4\,C^2\right)} + 2\,Rdc\,C\,(wd + t\,u^2) - 2\,u^4\left(\frac{1}{vd^2\,C} + Lidc\right)C}$$

```
> series(Imk,u=0,3): Imk1 :=collect(%,u);
```

$$Imk1 := -\sqrt{Rdc\,C\,wd} - \frac{1}{2}\frac{\sqrt{Rdc\,C\,wd}\,t}{wd}u^2 + O(u^4)$$

I can then just read off

$$\mathrm{Im}(k) \approx -\sqrt{R_{dc} C\, \omega_d} - (1/2\omega_d)\sqrt{R_{dc} C\, \omega_d}\ \tan_L\ \omega + O(\omega^2)$$

Summary for low ω:

$$\mathrm{Re}(k) = (\omega/2)\,(\omega_d + C\, v_d^2[R_{dc} + \omega_d L_{idc}])\ /\ (\, v_d^2\sqrt{R_{dc} C \omega_d}\,) + \quad O(\omega^2)$$

$$\mathrm{Im}(k) \approx -\sqrt{R_{dc} C\, \omega_d} - (1/2\omega_d)\sqrt{R_{dc} C\, \omega_d}\ \tan_L\ \omega + O(\omega^2)$$

There have the form

$$\mathrm{Re}(k) = A\omega + O(\omega^2)$$
$$\mathrm{Im}(k) = A + B\omega + O(\omega^2)$$

Now recall that $\omega_d \equiv \sigma_d/\varepsilon_d$. For a non-conducting dielectric, we have $\sigma_d = 0$ so $\omega_d = 0$ and then my results are singular. I will run them again for this special case. First

```
> a := ((R^2+u^4*L^2)*(G^2+u^4*C^2))^(1/4);
```

$$a := \left(\left(Rdc^2 + u^4\left(\frac{1}{vd^2\,C} + Lidc\right)^2\right)\left(C^2\,t^2\,u^4 + u^4\,C^2\right)\right)^{\left(\frac{1}{4}\right)}$$

```
> series(a,u=0,1): a1 :=  collect(%,u);
```

$$a1 := \left(Rdc^2\left(C^2\,t^2 + C^2\right)\right)^{\left(\frac{1}{4}\right)} u + O(u^5)$$

```
> op(1,a1); simplify(%): series(%,t=0,3);
```

$$\left(Rdc^2\left(C^2\,t^2 + C^2\right)\right)^{\left(\frac{1}{4}\right)}$$

$$\sqrt{Rdc}\,\sqrt{C} + \frac{1}{4}\sqrt{Rdc}\,\sqrt{C}\,t^2 + O(t^4)$$

So we end up with

$$a = \sqrt{R_{dc}C}\ \sqrt{\omega}\ + O(\omega^{5/2})$$

Next comes Re(k)

```
# DO RE(K)
Rek := sqrt((a^2-c)/2);
```

$$Rek := \frac{1}{2}\sqrt{2\sqrt{\left(Rdc^2 + u^4\left(\frac{1}{vd^2\,C} + Lidc\right)^2\right)\left(C^2\,t^2\,u^4 + u^4\,C^2\right)} - 2\,Rdc\,C\,t\,u^2 + 2\,u^4\left(\frac{1}{vd^2\,C} + Lidc\right)C}$$

```
series(Rek,u=0,3): Rek1 :=collect(%,u);
```

$$Rek1 := \frac{1}{2}\sqrt{2\,Rdc\sqrt{C^2\,t^2 + C^2} - 2\,Rdc\,C\,t}\,u + O(u^3)$$

```
op(1,Rek1); simplify(%): series(%,t=0,3);
```

$$\frac{1}{2}\sqrt{2\,Rdc\sqrt{C^2\,t^2 + C^2} - 2\,Rdc\,C\,t}$$

$$\frac{1}{2}\sqrt{2}\,\sqrt{Rdc}\,\sqrt{C} - \frac{1}{4}\sqrt{2}\,\sqrt{Rdc}\,\sqrt{C}\,t + \frac{1}{16}\sqrt{2}\,\sqrt{Rdc}\,\sqrt{C}\,t^2 + O(t^3)$$

So the result here is

$$Re(k) = \sqrt{R_{dc}C/2}\ \sqrt{\omega}\ + O(\omega^{3/2})$$

Next,

```
# DO IM(K)
Imk := - sqrt((a^2+c)/2);
```

$$Imk := -\frac{1}{2}\sqrt{2\sqrt{\left(Rdc^2+u^4\left(\frac{1}{vd^2\,C}+Lidc\right)^2\right)(C^2\,t^2\,u^4+u^4\,C^2)}+2\,Rdc\,C\,t\,u^2-2\,u^4\left(\frac{1}{vd^2\,C}+Lidc\right)C}$$

```
series(Imk,u=0,3): Imk1 :=collect(%,u);
```

$$Imk1 := -\frac{1}{2}\sqrt{2\,Rdc\sqrt{C^2\,t^2+C^2}+2\,Rdc\,C\,t}\,u+O(u^3)$$

```
op(1,Imk1); simplify(%): series(%,t=0,3);
```

$$-\frac{1}{2}\sqrt{2\,Rdc\sqrt{C^2\,t^2+C^2}+2\,Rdc\,C\,t}$$

$$-\frac{1}{2}\sqrt{2}\sqrt{Rdc}\sqrt{C}-\frac{1}{4}\sqrt{2}\sqrt{Rdc}\sqrt{C}\,t-\frac{1}{16}\sqrt{2}\sqrt{Rdc}\sqrt{C}\,t^2+O(t^3)$$

and the result is

$$Im(k) = -\sqrt{R_{dc}C/2}\,\sqrt{\omega} + O(\omega^{3/2})$$

and then for $\omega_d = 0$ we find that

$$Re(k) = \sqrt{R_{dc}C/2}\,\sqrt{\omega} + O(\omega^{3/2})$$

$$Im(k) = -\sqrt{R_{dc}C/2}\,\sqrt{\omega} + O(\omega^{3/2})$$

and this in fact agrees with (Q.4), though I don't have the coefficient here of $\omega^{3/2}$.

This concludes my efforts to evaluate k(ω) for large ω and then for small ω.

4. More Notes Made 9.2.14

In "math mystery" doc I have pondered the issue of $\omega_d \to 0$ and $G_d \to 0$ and problems this causes in my analysis. The situations in which this limit might arise would be when the dielectric is a perfect vacuum. In the case of an air dielectric, I now realize that air has some finite conductivity,

http://chemistry.about.com/od/moleculescompounds/a/Table-Of-Electrical-Resistivity-And-Conductivity.htm

Ohring, Milton (1995). Engineering materials science, Volume 1 (3rd edition ed.). p. 561.

Pawar, S. D.; Murugavel, P.; Lal, D. M. (2009). "Effect of relative humidity and sea level pressure on electrical conductivity of air over Indian Ocean". Journal of Geophysical Research 114: D02205.

Table of Resistivity and Conductivity at 20°C

Material	ρ (Ω•m) at 20 °C Resistivity	σ (S/m) at 20 °C Conductivity
Silver	1.59×10^{-8}	6.30×10^{7}
Copper	1.68×10^{-8}	5.96×10^{7}
Annealed copper	1.72×10^{-8}	5.80×10^{7}
Gold	2.44×10^{-8}	4.10×10^{7}
Aluminum	2.82×10^{-8}	3.5×10^{7}
Glass	10×10^{10} to 10×10^{14}	10^{-11} to 10^{-15}
Hard rubber	1×10^{13}	10^{-14}
Wood (oven dry)	1×10^{14} to 16	10^{-16} to 10^{-14}
Sulfur	1×10^{15}	10^{-16}
Air	1.3×10^{16} to 3.3×10^{16}	3×10^{-15} to 8×10^{-15}
Paraffin wax	1×10^{17}	10^{-18}
Fused quartz	7.5×10^{17}	1.3×10^{-18}
PET	10×10^{20}	10^{-21}
Teflon	10×10^{22} to 10×10^{24}	10^{-25} to 10^{-23}

I know that PE is about 10^{-15} and air is seen to be in that same ballpark. So even an air dielectric conducts at room temperature to this 10^{-15} extent. You see above that ideal PET and ideal teflon are much better insulators getting to 10^{-24} or so.

I think therefore that when doing a low ω limit for a practical transmission line, even with PE and Belden we would use the $\omega_d > 0$ version of things. In particular for PE or air,

$$\omega_d \equiv (\sigma_d/\varepsilon_d) = 10^{-15} / 8.8541877 \times 10^{-12} = 1000/8.85 = 113$$ for PE dielectric

$$f_d = \omega/2\pi = 18 \text{ Hz}.$$

Thus I get $\omega_d = 113$ and not some super small number like 10^{-10} say.