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Appendix Q: Properties of the functions k(ω) and $Z_0(\omega)$

The two functions of interest are,

$$k = -j\sqrt{\mathbf{zy}} = -j\sqrt{(R+j\omega L)(G+j\omega C)} \qquad (5.3.5) \qquad (K.7)$$

$$Z_0 = \sqrt{\frac{\mathbf{z}}{\mathbf{y}}} = \sqrt{\frac{R+j\omega L}{G+j\omega C}} \ . \qquad (4.11.16) \qquad (K.4)$$

These expressions for k and Z_0 were derived from both the "physics model" of Chapters 4 and 5 and the network model of Appendix K, with equations numbers shown above. The expressions are valid for all ω in the network model, but in the physics model they were derived only in the skin effect regime, though we might have some expectation that the expressions are approximately valid for low ω.

The four parameters R,L,G and C are by definition all real numbers. For example, in the physics model R is defined to be the real part of **z** and ωL the imaginary part. In the k and Z_0 sections below, our first task shall be to compute the real and imaginary parts of k and Z_0. We shall then be interested in the high ω and low ω limits of these expressions. In order to compute these limits properly, we must remember that the four real parameters R,L,G and C are in general functions of ω and cannot be treated as constants. This requires that we construct some kind of model for these parameters as functions of ω, and that is the task of the following section.

Q.1 A Model for R, L, G and C

The model considered here is just an *ad hoc* "reasonable" model which is accurate at high and low frequencies and which bridges the gap in a crude manner just so we have something to work with. We shall assume that C and L_e are the constants appearing in (4.11.34), namely

$C = 4\pi\varepsilon_d/K$	capacitance per meter
$L_e = \frac{\mu_d}{4\pi} K$	external inductance per meter
$G_{dc} = 4\pi\sigma_d/K$	DC conductance per meter

where K is the constant defined also in (4.11.34). The conductance has a dc subscript because σ_d is the DC conductivity of the dielectric.

Model for C

We assume C to be the constant value shown above, so

$$C(\omega) = C = 4\pi\varepsilon_d/K = \text{independent of } \omega \qquad (Q.1.1)$$

Model for G

Recall first from (3.3.2) and (3.3.4) that

$\sigma_{eff} = (\sigma_d + \omega\varepsilon'_d \tan_L)$ where $\tan_L \equiv (\varepsilon''_d/\varepsilon'_d)$ and $\varepsilon_d = \varepsilon'_d - j\varepsilon''_d$.

This σ_{eff} is the effective conductivity of a dielectric whose dielectric constant has an imaginary part. The dielectric burns energy just as if the loss were all simple ohmic loss. From (4.11.34) we then have

$$G = (\sigma_{eff}/\varepsilon_d)C = (\sigma_d + \omega\varepsilon'_d \tan_L)C/\varepsilon_d = (\sigma_d/\varepsilon_d)C + (\varepsilon'_d/\varepsilon_d) \tan_L \omega C$$

$$= G_{dc} + (\varepsilon'_d/\varepsilon_d) \tan_L \omega C$$

where recall that $\tan_L$ is the so-called loss-tangent or dissipation factor of the dielectric. Clearly G has a strong dependence on ω, being a linear function of ω. Normally $(\varepsilon'_d/\varepsilon_d) \approx 1$ since ε''_d and $\tan_L$ are normally very small. Rather than set $(\varepsilon'_d/\varepsilon_d) = 1$, we just absorb this factor into the definition of $\tan_L$ and then write G(ω) in this simpler form,

$$G(\omega) = G_{dc} + (C \tan_L) \omega \qquad \text{where} \qquad G_{dc} = (\sigma_d/\varepsilon_d)C . \qquad \text{(Q.1.2)}$$

Since (σ_d/ε_d) has the dimensions $\sec^{-1}$ it is convenient to rewrite the above G(ω) as

$$G(\omega) = C (\omega_d + \tan_L \omega) \qquad \omega_d \equiv (\sigma_d/\varepsilon_d) \qquad \text{(Q.1.3)}$$

This then is our working model for the frequency-dependent real parameter G(ω).

Examples:

A vacuum dielectric has $\sigma_d = 0$ so both $\omega_d = 0$ and $\tan_L = 0$, resulting in G(ω) = 0.

Air as a dielectric at low temperature presumably has $\omega_d = 0$ and $\tan_L << 1$.

Polyethylene (App. Q) has $\sigma_d \sim 10^{-15}$ and $\tan_L \approx .0005$.

Model for L and R

Recall from Chapter 2 these expressions for the high-frequency resistance and internal inductance of a round wire of radius a_1 ,

$$R_1 = \frac{1}{\sigma(2\pi a_1)\delta} \qquad \delta \equiv \sqrt{2/\omega\mu\sigma} \qquad \textit{(2.4.18)}$$

$$L_{1i} = (1/\omega) R_1 = \frac{1}{\omega\sigma(2\pi a_1)\delta} = \frac{1}{\omega\sigma(2\pi a_1)\sqrt{2/\omega\mu\sigma}} = \frac{1}{2\pi a_1}\sqrt{\frac{\mu}{2\sigma\omega}} \quad . \qquad \textit{(2.4.19)}$$

These are for conductor C_1 and similar expressions apply to conductor C_2 for a transmission line of the type considered in Chapter 6. Note that σ and μ are parameters of the conductor, not the dielectric. We can rewrite the above equations in this manner

$$L_{1i} = \frac{1}{2\pi a_1}\sqrt{\frac{\mu}{2\sigma}}\ \omega^{-1/2} \qquad \text{decreases with } \omega$$

$$R_1 = \frac{1}{2\pi a_1}\sqrt{\frac{\mu}{2\sigma}}\,\omega^{+1/2} = \omega L_{1i} \qquad \text{increases with } \omega \qquad \text{(Q.1.4)}$$

We now generalize these results for an arbitrary conductor C_1 as follows

$$L_{1i} = \frac{1}{p_1}\sqrt{\frac{\mu}{2\sigma}}\,\omega^{-1/2} \qquad \text{decreases with } \omega$$

$$R_1 = \omega L_{1i} = \frac{1}{p_1}\sqrt{\frac{\mu}{2\sigma}}\,\omega^{+1/2} \qquad \text{increases with } \omega \qquad \text{(Q.1.5)}$$

where p_1 is the "active" perimeter distance around the cross section of the conductor, as discussed in the text near (4.11.10). For widely spaced thin wires (or a centered coaxial cable), the active perimeter of a round wire is the full perimeter so $p_1 = 2\pi a_1$, but the active perimeter is much less than the total perimeter for a pair of closely spaced conductors as shown in Fig 4.12. We regard this generalization as being "reasonable" if not precise. Summing over the two conductors of our transmission line, we then find for high ω that

$$L_i = \left(\frac{1}{p_1}+\frac{1}{p_2}\right)\sqrt{\frac{\mu}{2\sigma}}\,\omega^{-1/2} \equiv \kappa\,\omega^{-1/2} \qquad \kappa \equiv \left(\frac{1}{p_1}+\frac{1}{p_2}\right)\sqrt{\frac{\mu}{2\sigma}}$$

$$R = \left(\frac{1}{p_1}+\frac{1}{p_2}\right)\sqrt{\frac{\mu}{2\sigma}}\,\omega^{+1/2} \equiv \kappa\,\omega^{+1/2} \qquad \dim(\kappa) = \text{ohm/m} * \sec^{1/2}. \qquad \text{(Q.1.6)}$$

In order to crudely blend these expressions down into the low ω region, we write

$$R(\omega) = R_{dc}\,\theta(\omega<\omega_R) + (\kappa\sqrt{\omega})\,\theta(\omega\ge\omega_R) \qquad \omega_R \equiv (R_{dc}/\kappa)^2$$

$$L_i(\omega) = L_{idc}\,\theta(\omega<\omega_L) + (\kappa/\sqrt{\omega})\,\theta(\omega\ge\omega_L) \qquad \omega_L \equiv (\kappa/L_{i,DC})^2 \qquad \text{(Q.1.7)}$$

where θ(bool) = 1 if bool = true, else 0. Here $R_{dc} = 1/(\sigma A_1) + 1/(\sigma A_2)$ is the DC resistance of the two conductors, while L_{idc} is the DC internal inductance. Examples of computing the latter appear in Appendix C. For a round wire, $L_i = \mu/8\pi$, so for a pair of same, $L_{idc} = \mu/4\pi$ henry/m.

Using parameters shown in Appendix R for Belden 8281 coaxial cable, we may plot R(ω) and $L_i(\omega)$ for our crude model. First, the two expressions are entered into Maple,

```
restart; with(plots): unprotect('Li');
R := Rdc*Heaviside(wR-w) + (kappa*sqrt(w))*Heaviside(w-wR);
```

$$R := Rdc\ \text{Heaviside}(wR - w) + \kappa\sqrt{w}\ \text{Heaviside}(w - wR)$$

```
Li := Lidc*Heaviside(wL-w) + (kappa/sqrt(w))*Heaviside(w-wL);
```

$$Li := Lidc\ \text{Heaviside}(wL - w) + \frac{\kappa\ \text{Heaviside}(w - wL)}{\sqrt{w}}$$

```
kappa := (1/p1 + 1/p2)*sqrt(mu0/(2*sigma)):
```

followed by parameters for the Belden cable obtained in Appendix R,

```
pi := evalf(Pi):
mu0 := 4*pi*1e-7:
sigma := 5.81e7:
a1 :=  (1/2)*0.7874e-3:
a2 := ((1/2)*5020.2 + 11.5) * 1e-6:
p1 := 2*pi*a1: p2 := 2*pi*a2:
R2DC := 3.6091e-3:
R1DC := 32.4819e-3:
Rdc := R2DC + R1DC:
Lidc := .58e-7:
evalf(kappa);
```

$$.00004860306321$$

The two blending frequencies are computed,

```
wR := (Rdc/kappa)^2;
```

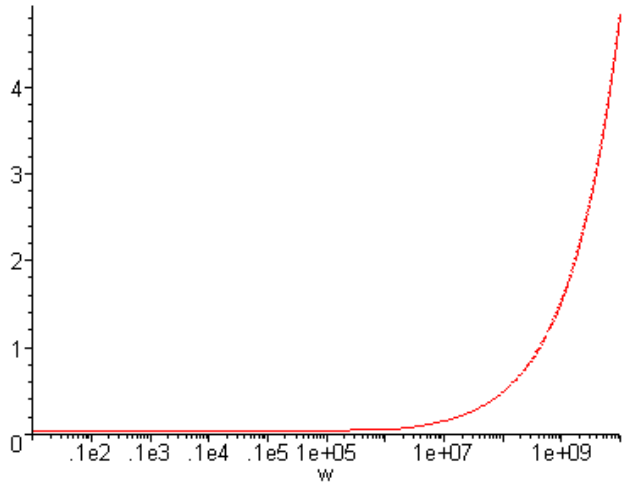
$$wR := 551404.8070$$

```
wL := (kappa/Lidc)^2;
```

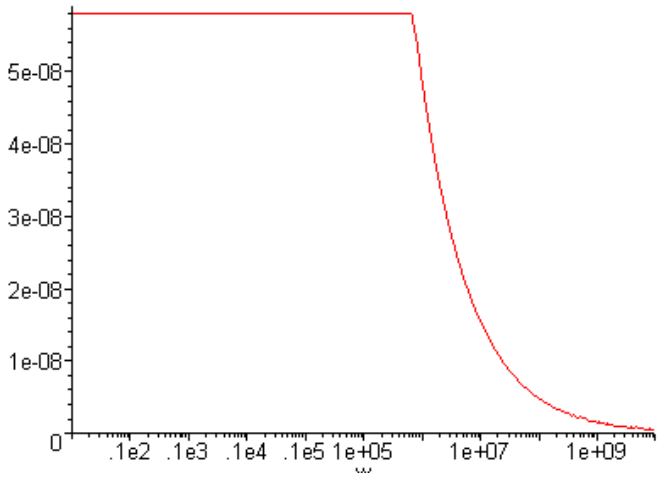
$$wL := 702216.9304$$

and are seen to be in the 1 MHz range. Finally, here are plots for R and L^i versus ω :

```
semilogplot(R,w=1..1e10,numpoints=10000);
```




```
semilogplot(Li,w=1..1e10,numpoints=2000);
```



which duly show the resistance increasing as $\sqrt{\omega}$ and internal inductance decreasing as $1/\sqrt{\omega}$. It would not be difficult to provide a smooth blending function to remove the sharp corners from these plots, but since our only real interest is in very large and very small ω, we leave things as is.

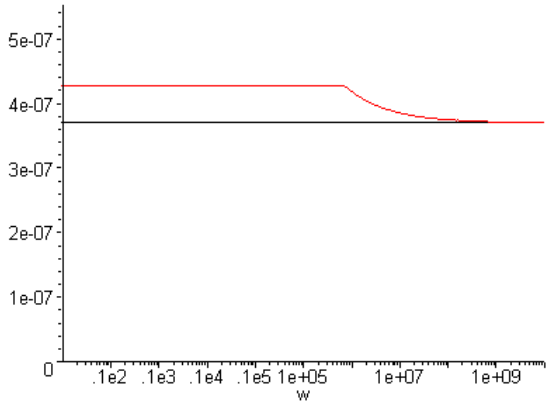
The final step is to add in the external inductance L_e so our model for R(ω) and L(ω) is then

$$R(\omega) = R_{dc}\,\theta(\omega<\omega_R) + (\kappa\sqrt{\omega})\,\theta(\omega\geq\omega_R) \qquad \omega_R \equiv (R_{dc}/\kappa)^2$$

$$L(\omega) = L_e + L_{idc}\,\theta(\omega<\omega_L) + (\kappa/\sqrt{\omega})\,\theta(\omega\geq\omega_L) \qquad \omega_L \equiv (\kappa/L_{i,DC})^2 \ . \qquad \text{(Q.1.8)}$$

For the Belden cable example L_e = 0.37 μH/m = 3.7 x 10^{-7} H/m, giving this plot for L(ω) versus ω:

```
semilogplot([L,Le],w=1..1e10,numpoints=2000,view=[1..1e10,0..1.5*Le],color=[red,black]);
```



L(ω) is shown in red, while L_e is shown in black. This last plot shows that L_e is really the dominant term in L(ω) at all ω. More magnetic energy is stored in the dielectric than inside the conductors.

Simple Transmission Line Parameter Model (Q.1.9)

$$C(\omega) = C = \text{independent of } \omega \qquad \kappa \equiv \left(\frac{1}{p_1} + \frac{1}{p_2}\right)\sqrt{\frac{\mu}{2\sigma}}$$

$$G(\omega) = C\,(\omega_d + \tan_L \omega) \qquad \omega_d \equiv (\sigma_d/\varepsilon_d) \qquad G_{dc} \equiv \omega_d\, C$$

$$R(\omega) = R_{dc}\,\theta(\omega<\omega_R) + (\kappa\sqrt{\omega})\,\theta(\omega\geq\omega_R) \qquad \omega_R \equiv (R_{dc}/\kappa)^2$$

$$L(\omega) = L_e + L_{idc}\,\theta(\omega<\omega_L) + (\kappa/\sqrt{\omega})\,\theta(\omega\geq\omega_L) \qquad \omega_L \equiv (\kappa/L_{idc})^2$$

$$L_eC = \mu_d\varepsilon_d = 1/v_d^2 \qquad \textit{(4.11.34)}$$

The last line shows that L_e has a simple relation to C in terms of the dielectric speed of light v_d. The two p_i are the active perimeters of the conductors, μ and σ are for the conductors, and σ_d, ε_d and μ_d are for the dielectric. R_{dc} and L_{idc} are the DC resistance and internal inductance total for the two conductors.

Q.2 Real and Imaginary part of k(ω)

Here is the separation of k(ω) into its real and imaginary parts:

Fact 1: The real + imaginary decomposition of k is given by (Q.2.1)

$$k = \sqrt{\frac{a^2-c}{2}} - j\sqrt{\frac{a^2+c}{2}} \qquad k \equiv -j\sqrt{(R+j\omega L)(G+j\omega C)}$$

$$jk = \sqrt{\frac{a^2+c}{2}} + j\sqrt{\frac{a^2-c}{2}} \qquad jk = \sqrt{\mathbf{zy}} = \sqrt{(R+j\omega L)(G+j\omega C)}$$

where

$$a \equiv [(R^2+\omega^2L^2)(G^2+\omega^2C^2)]^{1/4} = |k| \qquad \dim(a) = 1/m \qquad a > 0$$

$$c \equiv RG - \omega^2LC \qquad \dim(c) = 1/m^2 \quad c = \text{real}, \ |c| < a^2$$

Proof of Fact 1: Let

$$q \equiv \mathbf{zy} = (R+j\omega L)(G+j\omega C) = (RG-\omega^2LC) + j\omega(LG+RC) = c + j\omega(LG+RC) = |q|\, e^{j\theta}$$

$$\Rightarrow |q|^2 = |(R+j\omega L)(G+j\omega C)|^2 = |(R+j\omega L)|^2|(G+j\omega C)|^2 = (R^2+\omega^2L^2)(G^2+\omega^2C^2) \equiv a^4$$

$$|q| = a^2$$

$$\cos\theta = \mathrm{Re}(q)/|q| = c/a^2$$

Now write

$$s \equiv \sqrt{\mathbf{zy}} = \sqrt{(R+j\omega L)(G+j\omega C)} = \sqrt{q} = |s|\, e^{i\varphi} \qquad |s| = \sqrt{|q|} = a \qquad \varphi = \theta/2$$

$$\mathrm{Re}(s) = |s|\cos\varphi = a\cos(\theta/2) = a\sqrt{(1+\cos\theta)/2} = (a/\sqrt{2})\sqrt{1+c/a^2} = \sqrt{\frac{a^2+c}{2}}$$

$$\mathrm{Im}(s) = |s|\sin\varphi = a\sin(\theta/2) = a\sqrt{(1-\cos\theta)/2} = (a/\sqrt{2})\sqrt{1-c/a^2} = \sqrt{\frac{a^2-c}{2}}$$

$$s = \sqrt{\mathbf{zy}} = \sqrt{\frac{a^2+c}{2}} + j\sqrt{\frac{a^2-c}{2}}$$

$$k = -j\sqrt{\mathbf{zy}} = -js = \sqrt{\frac{a^2-c}{2}} - j\sqrt{\frac{a^2+c}{2}}$$ QED

Maple verification:

```
restart; alias(I=I,j=sqrt(-1)):
assume(R>0,L>0,G>0,C>0,omega>0);
k := (-j)*sqrt((R+j*omega*L)*(G+j*omega*C));
```

$$k := -j\sqrt{(R+j\,\omega\,L)(G+j\,\omega\,C)}$$

```
asq_ := ((R^2+omega^2*L^2)*(G^2+omega^2*C^2))^(1/2);    # a^2
```

$$asq_ := \sqrt{(R^2+\omega^2 L^2)(G^2+\omega^2 C^2)}$$

```
c_ := R*G - omega^2*L*C;
```

$$c_ := RG - \omega^2 LC$$

```
k := factor(evalc(k));
```

$$k := \frac{1}{2}\sqrt{2\sqrt{(R^2+\omega^2 L^2)(G^2+\omega^2 C^2)} - 2RG + 2\omega^2 LC} - \frac{1}{2}j\sqrt{-2\omega^2 LC + 2RG + 2\sqrt{(R^2+\omega^2 L^2)(G^2+\omega^2 C^2)}}$$

```
k :=subs(asq_ = a^2,k);
```

$$k := \frac{1}{2}\sqrt{2a^2 - 2RG + 2\omega^2 LC} - \frac{1}{2}j\sqrt{-2\omega^2 LC + 2RG + 2a^2}$$

```
k := subs(R*G = c + omega^2*L*C,k);
```

$$k := \frac{1}{2}\sqrt{2a^2 - 2c} - \frac{1}{2}j\sqrt{2c + 2a^2}$$

```
factor(expand(asq_^2 - c_^2));    # to show that a^2 > |c|
```

$$\omega^2 (RC + LG)^2$$

Q.3 Real and Imaginary part of k(ω) for large ω

Large ω limit for k(ω)

The model of box (Q.1.9) at high ω reads, setting $L_e = 1/(Cv_d^2)$,

$$G(\omega) = C\,(\omega_d + \tan_L \omega) \qquad \omega_d \equiv (\sigma_d/\varepsilon_d)$$

$$R(\omega) = \kappa\sqrt{\omega}$$

$$L(\omega) = 1/(Cv_d^2) + (\kappa/\sqrt{\omega}) \qquad \kappa \equiv \left(\frac{1}{p_1} + \frac{1}{p_2}\right)\sqrt{\frac{\mu}{2\sigma}} \qquad \text{(Q.3.1)}$$

Defining $t \equiv \tan_L$ and $u \equiv \sqrt{\omega}$, $w_d \equiv \omega_d$, these expressions are

$$G = C(w_d + t\,u^2) \qquad R = \kappa u \qquad L = 1/(Cv_d^2) + \kappa/u \qquad \text{(Q.3.2)}$$

We duly enter these expressions into Maple,

```
G := C*(wd + t*u^2);
```

$$G := C\,(wd + t\,u^2)$$

```
R := kappa*u;
```

$$R := \kappa\,u$$

```
L := 1/(vd^2*C) + kappa/u;
```

$$L := \frac{1}{vd^2\,C} + \frac{\kappa}{u}$$

Next, we enter the intermediate variables a and c from box (Q.2.1),

```
a := ((R^2+u^4*L^2)*(G^2+u^4*C^2))^(1/4);
```

$$a := \left(\left(\kappa^2 u^2 + u^4\left(\frac{1}{vd^2 C} + \frac{\kappa}{u}\right)^2\right)\left(C^2 (wd + t u^2)^2 + u^4 C^2\right)\right)^{\left(\frac{1}{4}\right)}$$

```
c := R*G - u^4*L*C;
```

$$c := \kappa u C (wd + t u^2) - u^4\left(\frac{1}{vd^2 C} + \frac{\kappa}{u}\right) C$$

Large ω limit for Re(k)

The real part of k(ω) is then given from (Q.2.1) as follows:

```
Rek := sqrt((a^2-c)/2);
```

$$Rek := \frac{1}{2}\sqrt{2\sqrt{\left(\kappa^2 u^2 + u^4\left(\frac{1}{vd^2 C} + \frac{\kappa}{u}\right)^2\right)\left(C^2 (wd + t u^2)^2 + u^4 C^2\right)} - 2\kappa u C (wd + t u^2) + 2u^4\left(\frac{1}{vd^2 C} + \frac{\kappa}{u}\right) C}$$

An asymptotic expansion of Re(k) for large ω (large $u \equiv \sqrt{\omega}$) is then found to be,

```
> series(Rek,u=infinity,1): Rek1 :=collect(%,u);
```

$$Rek1 := \frac{1}{2}\sqrt{2\frac{\sqrt{C^2 t^2 + C^2}}{vd^2 C} + 2\frac{1}{vd^2}}\, u^2 + \frac{1}{4}\frac{(2\sqrt{C^2 t^2 + C^2}\,\kappa + 2\kappa C - 2\kappa C t) u}{\sqrt{2\frac{\sqrt{C^2 t^2 + C^2}}{vd^2 C} + 2\frac{1}{vd^2}}} + \frac{1}{2}\sqrt{2\frac{\sqrt{C^2 t^2 + C^2}}{vd^2 C} + 2\frac{1}{vd^2}}$$

$$\left(\frac{\sqrt{C^2 t^2 + C^2}\left(\frac{1}{2}\frac{\left(2\frac{t\,wd}{vd^4} + 2\kappa^2 (C^2 t^2 + C^2)\right) vd^4 C^2}{C^2 t^2 + C^2} - \frac{1}{2} vd^4 C^2 \kappa^2\right)}{vd^2 C\left(2\frac{\sqrt{C^2 t^2 + C^2}}{vd^2 C} + 2\frac{1}{vd^2}\right)} - \frac{1}{8}\frac{(2\sqrt{C^2 t^2 + C^2}\,\kappa + 2\kappa C - 2\kappa C t)^2}{\left(2\frac{\sqrt{C^2 t^2 + C^2}}{vd^2 C} + 2\frac{1}{vd^2}\right)^2}\right)$$

$$+ O\left(\frac{1}{u}\right)$$

which has the form $Au^2 + Bu + C + O(1/u)$. Maple next processes the first three terms by first simplifying them, then expanding them in small parameter t. That is, we now regard $t \equiv \tan_L << 1$. The results are

```
op(1,Rek1); simplify(%): series(%,t=0,3);
```

$$\frac{1}{2}\sqrt{2\frac{\sqrt{C^2t^2+C^2}}{vd^2\,C}+2\frac{1}{vd^2}}\,u^2$$

$$\frac{u^2}{vd}+\frac{1}{8}\frac{u^2}{vd}t^2+O(t^4)$$

```
op(2,Rek1); simplify(%): series(%,t=0,3);
```

$$\frac{1}{4}\frac{(2\sqrt{C^2t^2+C^2}\,\kappa+2\kappa\,C-2\kappa\,C\,t)\,u}{\sqrt{2\frac{\sqrt{C^2t^2+C^2}}{vd^2\,C}+2\frac{1}{vd^2}}}$$

$$\frac{1}{2}vd\,\kappa\,C\,u-\frac{1}{4}vd\,\kappa\,C\,u\,t+\frac{1}{16}vd\,\kappa\,C\,u\,t^2+O(t^3)$$

```
op(3,Rek1); simplify(%): series(%,t=0,3);
```

$$\sqrt{2\frac{\sqrt{C^2t^2+C^2}}{vd^2\,C}+2\frac{1}{vd^2}}\left(\frac{\sqrt{C^2t^2+C^2}\left(\frac{1}{2}\frac{\left(2\frac{t\,wd}{vd^4}+2\kappa^2(C^2t^2+C^2)\right)vd^4\,C^2}{C^2t^2+C^2}-\frac{1}{2}vd^4\,C^2\kappa^2\right)}{vd^2\,C\left(2\frac{\sqrt{C^2t^2+C^2}}{vd^2\,C}+2\frac{1}{vd^2}\right)}-\frac{1}{8}\frac{(2\sqrt{C^2t^2+C^2}\,\kappa+2\kappa\,C-2\kappa\,C\,t)^2}{\left(2\frac{\sqrt{C^2t^2+C^2}}{vd^2\,C}+2\frac{1}{vd^2}\right)^2}\right)$$

$$\frac{1}{8}\frac{2\,wd+vd^4\,C^2\kappa^2}{vd}t+O(t^3)$$

It is important to display the "op" operands to make sure the correct one has been grabbed, since Maple is a little arbitrary in how it orders terms. Collecting the results, replacing u by $\sqrt{\omega}$, and keeping only the leading terms in t = $\tan_L$, the resulting large ω limit for Re(k) is,

$$\mathrm{Re}(k) \approx (1/v_d)\omega + (1/2)v_d\kappa C\sqrt{\omega} + (1/8v_d)(\,2\omega_d + v_d^4C^2\kappa^2)\,\tan_L + O(1/\sqrt{\omega}\,) \qquad \text{(Q.3.3)}$$

<u>Large ω limit for Im(k)</u>

The imaginary part of k(ω) is then given from (Q.2.1) as follows:

```
Imk := - sqrt((a^2+c)/2);
```

$$Imk := -\frac{1}{2}\sqrt{2\sqrt{\left(\kappa^2u^2+u^4\left(\frac{1}{vd^2\,C}+\frac{\kappa}{u}\right)^2\right)\left(C^2(wd+t\,u^2)^2+u^4\,C^2\right)}+2\kappa\,u\,C\,(wd+t\,u^2)-2\,u^4\left(\frac{1}{vd^2\,C}+\frac{\kappa}{u}\right)C}$$

An asymptotic expansion of Im(k) for large ω (large u ≡ $\sqrt{\omega}$) is then found to be,

```
> series(Imk,u=infinity,1): Imk1 :=collect(%,u);
```

$$Imk1 := -\frac{1}{2}\sqrt{2\frac{\sqrt{C^2t^2+C^2}}{vd^2\,C}-2\frac{1}{vd^2}}\,u^2-\frac{1}{4}\frac{(2\sqrt{C^2t^2+C^2}\,\kappa-2\kappa C+2\kappa C t)\,u}{\sqrt{2\frac{\sqrt{C^2t^2+C^2}}{vd^2\,C}-2\frac{1}{vd^2}}}-\frac{1}{2}\sqrt{2\frac{\sqrt{C^2t^2+C^2}}{vd^2\,C}-2\frac{1}{vd^2}}$$
$$\left(\frac{\sqrt{C^2t^2+C^2}\left(\frac{1}{2}\frac{\left(2\frac{t\,wd}{vd^4}+2\kappa^2(C^2t^2+C^2)\right)vd^4C^2}{C^2t^2+C^2}-\frac{1}{2}vd^4C^2\kappa^2\right)}{vd^2\,C\left(2\frac{\sqrt{C^2t^2+C^2}}{vd^2\,C}-2\frac{1}{vd^2}\right)}-\frac{1}{8}\frac{(2\sqrt{C^2t^2+C^2}\,\kappa-2\kappa C+2\kappa C t)^2}{\left(2\frac{\sqrt{C^2t^2+C^2}}{vd^2\,C}-2\frac{1}{vd^2}\right)^2}\right)$$
$$+O\left(\frac{1}{u}\right)$$

which has the form $Au^2 + Bu + C + O(1/u)$. Maple next processes the first three terms by first simplifying them, then expanding them in small parameter $t \equiv \tan_L << 1$. The results are

```
op(1,Imk1); simplify(%): series(%,t=0,4);
```

$$-\frac{1}{2}\sqrt{2\frac{\sqrt{C^2t^2+C^2}}{vd^2\,C}-2\frac{1}{vd^2}}\,u^2$$
$$-\frac{1}{2}\frac{u^2}{vd}t+\frac{1}{16}\frac{u^2}{vd}t^3+O(t^5)$$

```
op(2,Imk1); simplify(%): series(%,t=0,4);
```

$$-\frac{1}{4}\frac{(2\sqrt{C^2t^2+C^2}\,\kappa-2\kappa C+2\kappa C t)\,u}{\sqrt{2\frac{\sqrt{C^2t^2+C^2}}{vd^2\,C}-2\frac{1}{vd^2}}}$$
$$-\frac{1}{2}vd\,\kappa C u-\frac{1}{4}vd\,\kappa C u t-\frac{1}{16}vd\,\kappa C u t^2+O(t^3)$$

```
> op(3,Imk1); simplify(%): series(%,t=0,4);
```

$$-\frac{1}{2}\sqrt{2\frac{\sqrt{C^2t^2+C^2}}{vd^2\,C}-2\frac{1}{vd^2}}$$
$$\left(\frac{\sqrt{C^2t^2+C^2}\left(\frac{1}{2}\frac{\left(2\frac{t\,wd}{vd^4}+2\kappa^2(C^2t^2+C^2)\right)vd^4C^2}{C^2t^2+C^2}-\frac{1}{2}vd^4C^2\kappa^2\right)}{vd^2\,C\left(2\frac{\sqrt{C^2t^2+C^2}}{vd^2\,C}-2\frac{1}{vd^2}\right)}-\frac{1}{8}\frac{(2\sqrt{C^2t^2+C^2}\,\kappa-2\kappa C+2\kappa C t)^2}{\left(2\frac{\sqrt{C^2t^2+C^2}}{vd^2\,C}-2\frac{1}{vd^2}\right)^2}\right)$$
$$\frac{1}{8}\frac{2vd^4C^2\kappa^2-4\,wd}{vd}+O(t^2)$$

Collecting the results, replacing u by $\sqrt{\omega}$, and keeping only the leading terms in t, the resulting large ω limit for Im(k) is,

$$\text{Im}(k) \approx -\omega \tan_L/(2v_d) - (v_d\kappa C/2)\sqrt{\omega} - (1/4v_d)(2\omega_d - v_d^4C^2\kappa^2) + O(1/\sqrt{\omega}) \qquad \text{(Q.3.4)}$$

We now summarize these results as

Fact 2: The large ω asymptotic expansion for k(ω), assuming $\tan_L << 1$, is given by (Q.3.5)

$$\text{Re}(k) \approx \omega/v_d + (1/2)\, v_d\kappa C\sqrt{\omega} + (1/8v_d)(2\omega_d + v_d^4C^2\kappa^2)\tan_L + O(1/\sqrt{\omega})$$
$$\text{Im}(k) \approx -\omega \tan_L/(2v_d) - (v_d\kappa C/2)\sqrt{\omega} - (1/4v_d)(2\omega_d - v_d^4C^2\kappa^2) + O(1/\sqrt{\omega})$$

where $\kappa \equiv \left(\frac{1}{p_1} + \frac{1}{p_2}\right)\sqrt{\frac{\mu}{2\sigma}}$ and $\omega_d \equiv (\sigma_d/\varepsilon_d)$

Keeping only the leading terms for large ω,

$$\text{Re}(k) \approx (\omega/v_d) \qquad v_d = 1/\sqrt{L_eC} \approx 1/\sqrt{LC}$$
$$\text{Im}(k) \approx -\omega \tan_L/(2v_d)$$

Dimension checks!

$\dim(\omega/v_d) = \text{sec}^{-1} * \text{sec/m} = 1/\text{m}$ = correct

$\dim(v_d\kappa C\sqrt{\omega}) = \text{m/sec} * \text{ohm/m} * \text{sec}^{1/2} * \text{far/m} * \text{sec}^{-1/2} = 1/\text{sec} * \text{ohm-far/m} = 1/\text{m}$ OK

$\dim(\omega_d/v_d) = \text{sec}^{-1} * \text{sec/m} = 1/\text{m}$ OK

$\dim(C^2\kappa^2v_d^3) = \text{ohm}^2/\text{m}^2\ \text{sec} * \text{far}^2/\text{m}^2 * \text{m}^3/\text{sec}^3 = (\text{ohm-far})^2\ \text{m}^{-1}\ \text{sec}^{-2} = \text{sec}^2\ \text{m}^{-1}\ \text{sec}^{-2} = 1/\text{m}$

The leading term $\text{Re}(k) = \omega/v_d$ is the traditional result obtained from taking the large ω limit of the expression $k = -j\sqrt{(R+j\omega L)(G+j\omega C)}$ when the four parameters are treated as constants in ω,

$$k = -j\sqrt{(R+j\omega L)(G+j\omega C)} \approx -j\sqrt{(j\omega L)(j\omega C)} = (-j)(j)\omega\sqrt{LC} = \omega\sqrt{LC} \approx \omega\sqrt{L_eC} = \omega/v_d$$

This fact is obvious from the large-ω model stated in (Q.3.1) which we repeat here:

$$G(\omega) = C(\omega_d + \tan_L\omega) \qquad \omega_d \equiv (\sigma_d/\varepsilon_d)$$
$$R(\omega) = \kappa\sqrt{\omega}$$
$$L(\omega) = 1/(Cv_d^2) + (\kappa/\sqrt{\omega}) \qquad \kappa \equiv \left(\frac{1}{p_1} + \frac{1}{p_2}\right)\sqrt{\frac{\mu}{2\sigma}} \qquad \textit{(Q.3.1)}$$

Since $\tan_L << 1$ one has

$$G + j\omega C = C(\omega_d + \tan_L \omega) + j\omega C = C\omega_d + \omega C(\tan_L + j) \approx \omega C j = j\omega C$$
$$R + j\omega L = \kappa\sqrt{\omega} + j\omega\,[1/(C v_d^2) + (\kappa/\sqrt{\omega})\,] \approx j\omega L_e \approx j\omega L$$

$$k = -j\sqrt{(R+j\omega L)(G+j\omega C)} = -j\sqrt{(j\omega L)(j\omega C)} = \omega\sqrt{LC} \approx \omega\sqrt{L_e C} = \omega/v_d$$

Q.4 Real and Imaginary part of k(ω) for small ω

If the DC conductance $G_d = \omega_d C$ is non-vanishing (because $\sigma_d > 0$) we find one set of results, but if the dielectric is a perfect vacuum (perhaps air) with $\sigma_d = 0$, we get a different set of results. The two cases are treated separately below.

Small ω limit for k(ω) for $G_d > 0$

The model of box (Q.1.9) at small ω reads,

$$G(\omega) = C(\omega_d + \tan_L \omega) \qquad \omega_d \equiv (\sigma_d/\varepsilon_d)$$
$$R(\omega) = R_{dc}$$
$$L(\omega) = L_e + L_{idc} \qquad \text{(Q.4.1)}$$

Since L(ω) is a constant in the small ω limit of our model, we represent it in the Maple code as the constant symbol L_{dc}, where $L_{dc} = L_e + L_{idc}$.

Defining $t \equiv \tan_L$ and $u \equiv \sqrt{\omega}$, $w_d \equiv \omega_d$, these expressions are

$$G = C(w_d + t\,u^2) \qquad R = R_{dc} \qquad L = L_{dc} \qquad \text{(Q.4.2)}$$

We duly enter these expressions into Maple,

```
G := C*(wd + t*u^2);
```

$$G := C\,(wd + t\,u^2)$$

```
R := Rdc;
```

$$R := Rdc$$

```
L := 1/(vd^2*C) + Lidc;
```

$$L := \frac{1}{vd^2\,C} + Lidc$$

Next, we enter the intermediate variables a and c from box (Q.2.1),

```
a := ((R^2+u^4*L^2)*(G^2+u^4*C^2))^(1/4);
```

$$a := \left(\left(Rdc^2 + u^4\,Ldc^2\right)\left(C^2\,(wd + t\,u^2)^2 + u^4\,C^2\right)\right)^{\left(\frac{1}{4}\right)}$$

```
c := R*G - u^4*L*C;
```

$$c := Rdc\,C\,(wd + t\,u^2) - u^4\,Ldc\,C$$

Small ω limit for Re(k) for $G_d > 0$

The real part of k(ω) is then given from (Q.2.1) as follows:

```
Rek := sqrt((a^2-c)/2);
```

$$Rek := \frac{1}{2}\sqrt{2\sqrt{\left(Rdc^2 + u^4\,Ldc^2\right)\left(C^2\,(wd + t\,u^2)^2 + u^4\,C^2\right)} - 2\,Rdc\,C\,(wd + t\,u^2) + 2\,u^4\,Ldc\,C}$$

An expansion of Re(k) for small ω (small $u \equiv \sqrt{\omega}$) is then found to be,

```
series(Rek,u=0,3): Rek1 := simplify(collect(%,u));
```

$$Rek1 := \frac{1}{2}\frac{(Ldc\,wd + Rdc)\sqrt{C}}{\sqrt{wd}\,\sqrt{Rdc}}\,u^2 + O(u^4)$$

which has the form $Au^2 + O(u^4)$ and which is independent of $t = \tan_L$. We thus find that, for small ω,

$$\mathrm{Re}(k) \approx (\omega/2)\,(L_{dc}\omega_d + R_{dc})\,\sqrt{C/(\omega_d R_{dc})} + O(\omega^2) \qquad \text{(Q.4.3)}$$

Dimension check:

$$\dim(\mathrm{RHS}) = \sec^{-1} * \mathrm{ohms/m} * [\ \mathrm{far} * \sec * \mathrm{mho}\ ]^{1/2} = \sec^{-1} * \mathrm{ohms/m} * [\sec^2 * \mathrm{mho}^2\ ]^{1/2}$$

$$= \sec^{-1} * \mathrm{ohms/m} * \sec * \mathrm{mho} = 1/\mathrm{m} = \text{correct}$$

Small ω limit for Im(k) for $G_d > 0$

The imaginary part of k(ω) is given from (Q.2.1) as follows:

```
Imk := - sqrt((a^2+c)/2);
```

$$Imk := -\frac{1}{2}\sqrt{2\sqrt{(Rdc^2+u^4\,Ldc^2)\left(C^2\,(wd+t\,u^2)^2+u^4\,C^2\right)}+2\,Rdc\,C\,(wd+t\,u^2)-2\,u^4\,Ldc\,C}$$

An expansion of Im(k) for small ω (small $u \equiv \sqrt{\omega}$) is then found to be,

```
series(Imk,u=0,3): Imk1 :=collect(%,u);
```

$$Imk1 := -\sqrt{Rdc\,C\,wd}-\frac{1}{2}\frac{\sqrt{Rdc\,C\,wd}\,t}{wd}u^2+O(u^4)$$

which has the form $A + Bu^2 + O(u^4)$. We thus find that, for small ω,

$$\mathrm{Im}(k) \approx -\sqrt{R_{dc}C\omega_d}\ [\,1 + (1/2)\tan_L(\omega/\omega_d)] + O(\omega^2) \qquad \text{(Q.4.4)}$$

Fact 3: The small ω limit for k(ω), assuming $G_d > 0$, is given by (Q.4.5)

$$\mathrm{Re}(k) \approx (\omega/2)\,(L_{dc}\omega_d + R_{dc})\sqrt{C/(\omega_d R_{dc})} + O(\omega^2)$$
$$\mathrm{Im}(k) \approx -\sqrt{R_{dc}C\omega_d}\ [\,1 + (\tan_L/2)\,(\omega/\omega_d)] + O(\omega^2) \qquad \omega_d \equiv (\sigma_d/\varepsilon_d)$$

Notice that $\mathrm{Im}(k) \to -\sqrt{R_{dc}C\omega_d} > 0$ as $\omega \to 0$, indicating the presence of loss at DC. This loss is just the ohmic loss in the dielectric due to $\sigma_d > 0$. One can see that the limit $\omega_d \to 0$ of the above expressions is ill defined, which is the reason we treat that case separately.

Fact 3 dimension checks:

$\dim(Cv_d^2R_{dc}) = \text{far/m} * \text{m}^2/\text{sec}^2 * \text{ohms/m} = \text{far-ohm/sec}^2 = \text{sec/sec}^2 = \text{sec}^{-1}$ matches ω_d

$\dim(R_{dc}C\omega_d) = \text{ohm/m*far/m*sec}^{-1} = \text{ohm-far sec}^{-1}/\text{m}^2 = 1/\text{m}^2$

$\dim(v_d^2\sqrt{R_{dc}C\omega_d}) = \text{m}^2/\text{sec}^2 * 1/\text{m} = \text{m/sec}^2$

$\dim(\omega\ \omega_d/(\ v_d^2\sqrt{R_{dc}C\omega_d}\ )) = \text{sec}^{-2} * \text{sec}^2/\text{m} = 1/\text{m} = \text{correct!}$

Small ω limit for k(ω) for $G_d = 0$

Here we rerun the Maple code shown above setting wd = 0 at the start. For Re(k) we find,

```
Rek := sqrt((a^2-c)/2);
```

$$Rek := \frac{1}{2}\sqrt{2\sqrt{(Rdc^2 + u^4\,Ldc^2)(C^2 t^2 u^4 + u^4 C^2)} - 2\,Rdc\,C\,t\,u^2 + 2\,u^4\,Ldc\,C}$$

```
series(Rek,u=0,3): Rek1 := simplify(collect(%,u));
```

$$Rek1 := \frac{1}{2}\sqrt{2}\,\sqrt{Rdc}\,\sqrt{C}\,\sqrt{\sqrt{t^2+1} - t}\,u + O(u^3)$$

which has the form Au + O(u³). Expanding the coefficient A for small t = tan$_L$ then gives

```
op(1,Rek1); simplify(%): series(%,t=0,2);
```

$$\frac{1}{2}\sqrt{2}\,\sqrt{Rdc}\,\sqrt{C}\,\sqrt{\sqrt{t^2+1} - t}$$

$$\frac{1}{2}\sqrt{2}\,\sqrt{Rdc}\,\sqrt{C} - \frac{1}{4}\sqrt{2}\,\sqrt{Rdc}\,\sqrt{C}\,t + O(t^2)$$

from which we find that with t << 1,

$$\mathrm{Re}(k) \approx \sqrt{R_{dc}C/2}\ \sqrt{\omega}\,(1 - \tan_L/2) + O(\omega^{3/2}) \qquad \text{(Q.4.6)}$$

For Im(k) the corresponding results are

```
Imk := - sqrt((a^2+c)/2);
```

$$Imk := -\frac{1}{2}\sqrt{2\sqrt{(Rdc^2 + u^4\,Ldc^2)(C^2 t^2 u^4 + u^4 C^2)} + 2\,Rdc\,C\,t\,u^2 - 2\,u^4\,Ldc\,C}$$

```
series(Imk,u=0,3): Imk1 :=collect(%,u);
```

$$Imk1 := -\frac{1}{2}\sqrt{2\,Rdc\sqrt{C^2 t^2 + C^2} + 2\,Rdc\,C\,t}\,u + O(u^3)$$

which again has the form Au + O(u³). Expanding the coefficient A for small t = tan$_L$ then gives

```
op(1,Imk1); simplify(%): series(%,t=0,2);
```

$$-\frac{1}{2}\sqrt{2\,Rdc\sqrt{C^2 t^2 + C^2} + 2\,Rdc\,C\,t}$$

$$-\frac{1}{2}\sqrt{2}\,\sqrt{Rdc}\,\sqrt{C} - \frac{1}{4}\sqrt{2}\,\sqrt{Rdc}\,\sqrt{C}\,t + O(t^2)$$

so that

$$\mathrm{Im}(k) \approx -\sqrt{R_{dc}C/2}\ \sqrt{\omega}\,(1 + \tan_L/2) + O(\omega^{3/2})$$

The results are then.

Fact 4: The small ω limit for k(ω), assuming $G_d = 0$, is given by (Q.4.7)

$$\mathrm{Re}(k) \approx +\sqrt{R_{dc}C/2}\ \sqrt{\omega}\,(1 - \tan_L/2) + O(\omega^{3/2})$$

$$\mathrm{Im}(k) \approx \; -\sqrt{R_{dc}C/2}\;\sqrt{\omega}\,(\,1+\tan_L/2) \;+O(\omega^{3/2})$$

In this case, Im(k) → 0 as ω→0 so there is no DC loss in the dielectric, as one would expect with $\sigma_d = 0$.

Q.5 Real and Imaginary part of $Z_0(\omega)$

Here is the separation of $Z_0(\omega)$ into its real and imaginary parts:

Fact 5: The real + imaginary decomposition of Z_0 is given by (Q.5.1)

$$Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = \sqrt{\frac{z}{y}} = (b/\sqrt{2})\,[\,\sqrt{1+(d/a^2)} \;- j\sigma\sqrt{1-(d/a^2)}\;]$$

where $b = \left(\frac{R^2+\omega^2L^2}{G^2+\omega^2C^2}\right)^{1/4} = |Z_0|$ $\quad a = [(R^2+\omega^2L^2)(G^2+\omega^2C^2)]^{1/4}$ as in (Q.2.1)

$\sigma = \mathrm{sign}(RC-LG)$ $\quad d = RG + \omega^2LC$

Proof of Fact 5: The proof is similar to that of Fact 1. Let

$$q = \frac{z}{y} = \frac{R+j\omega L}{G+j\omega C} = |q|\,e^{j\theta} \qquad -\pi < \theta < \pi \text{ (but see few lines below) .}$$

Then

$$|q|^2 = \left|\frac{R+j\omega L}{G+j\omega C}\right|^2 = \frac{R^2+\omega^2L^2}{G^2+\omega^2C^2} \equiv b^4 \qquad \Rightarrow \qquad |q| = \sqrt{\frac{R^2+\omega^2L^2}{G^2+\omega^2C^2}} = b^2 .$$

Next,

$$q = \frac{R+j\omega L}{G+j\omega C} = \frac{R+j\omega L}{G+j\omega C}\,\frac{G-j\omega C}{G-j\omega C} = \frac{(R+j\omega L)(G-j\omega C)}{G^2+\omega^2C^2} = \frac{RG+j\omega(LG-RC)+\omega^2LC}{G^2+\omega^2C^2}$$

so

$$\mathrm{Re}(q) = \frac{RG+\omega^2LC}{G^2+\omega^2C^2} > 0 \qquad \Rightarrow \qquad -\pi/2 < \theta < \pi/2$$

$$\mathrm{Im}(q) = \frac{\omega(LG-RC)}{G^2+\omega^2C^2} \qquad \mathrm{sign}[\mathrm{Im}(q)] = \mathrm{sign}(LG-RC) = \mathrm{sign}(\theta) \equiv \Sigma \;.$$

Then

$$\cos\theta = \mathrm{Re}(q)/|q| = \frac{(RG+\omega^2LC)}{G^2+\omega^2C^2}\sqrt{\frac{G^2+\omega^2C^2}{R^2+\omega^2L^2}} = \frac{RG+\omega^2LC}{\sqrt{(G^2+\omega^2C^2)(R^2+\omega^2L^2)}}$$

$\equiv d/a^2$ where $a = [(R^2+\omega^2L^2)(G^2+\omega^2C^2)]^{1/4}$ and $d \equiv RG + \omega^2LC$.

Now write

$$s \equiv \sqrt{\frac{z}{y}} = \sqrt{q} = |s|\, e^{i\varphi} \qquad |s| = \sqrt{|q|} = b = |Z_0| \qquad \varphi = \theta/2 \qquad -\pi/4 < \varphi < \pi/4$$

$$\mathrm{Re}(s) = |s|\cos\varphi = b\cos(\theta/2) = b\sqrt{(1+\cos\theta)/2} = (b/\sqrt{2})\sqrt{1+d/a^2}$$
$$\mathrm{Im}(s) = |s|\sin\varphi = b\sin(\theta/2) = \Sigma\, b\sqrt{(1-\cos\theta)/2} = \Sigma\,(b/\sqrt{2})\sqrt{1-d/a^2}$$

giving the result

$$s = (b/\sqrt{2})\,[\sqrt{1+d/a^2} + j\Sigma\sqrt{1-d/a^2}\,]$$

where $b = \left(\frac{R^2+\omega^2L^2}{G^2+\omega^2C^2}\right)^{1/4} = |Z_0|$ $\qquad a = [(R^2+\omega^2L^2)(G^2+\omega^2C^2)]^{1/4}$

$\Sigma = \mathrm{sign}(LG-RC)$ $\qquad d \equiv RG + \omega^2LC$.

Letting $\sigma = -\Sigma = \mathrm{sign}(RC-LG)$ one gets the final form,

$$s = (b/\sqrt{2})\,[\sqrt{1+d/a^2} - j\sigma\sqrt{1-d/a^2}\,]$$

where $b = \left(\frac{R^2+\omega^2L^2}{G^2+\omega^2C^2}\right)^{1/4} = |Z_0|$ $\qquad a = [(R^2+\omega^2L^2)(G^2+\omega^2C^2)]^{1/4}$

$\sigma = \mathrm{sign}(RC-LG)$ $\qquad d \equiv RG + \omega^2LC$.

Our Maple verification of this result is a bit ugly so we omit the code.

An alternate geometric derivation giving the same results begins as follows:

$$Z_0 = \sqrt{\frac{R+zL}{G+zC}} = \sqrt{\frac{L}{C}}\sqrt{\frac{z+(R/L)}{z+(G/C)}} = \sqrt{\frac{L}{C}}\sqrt{\frac{z+r}{z+g}} \quad \text{where } r = (R/L) \text{ and } g = (G/C) .$$

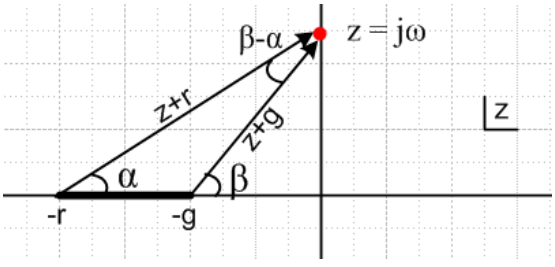


Fig Q.1

The drawing shows the complex z plane for the function $Z_0(z) = \sqrt{\frac{L}{C}}\sqrt{\frac{z+r}{z+g}}$ in the particular case that r > g, where the z-plane has a branch cut from -r to -g. The z values of interest are only those on the positive imaginary axis where $z = j\omega$.

Reader Exercise: Finish this derivation and obtain the results shown in (Q.5).

Hint: $\cos(\beta-\alpha) = (rg+\omega^2)/(AB)$ and $\sin(\beta-\alpha) = \omega(r-g)/(AB)$ where $A = \sqrt{\omega^2+r^2}$ and $B = \sqrt{\omega^2+g^2}$.

Q.6 Real and Imaginary part of $Z_0(\omega)$ for large ω

For large ω, our models for G,R,L were found to be

$$G = C(w_d + t\,u^2) \qquad R = \kappa u \qquad L = 1/(Cv_d^2) + \kappa/u \qquad (Q.3.2)$$

These expressions are entered into Maple as before. Next, we enter the intermediate variables a, b, and d from box (Q.5.1)

```
a := ((R^2+u^4*L^2)*(G^2+u^4*C^2))^(1/4);
```

$$a := \left(\left(\kappa^2 u^2 + u^4\left(\frac{1}{vd^2\,C}+\frac{\kappa}{u}\right)^2\right)\left(C^2\,(wd+t\,u^2)^2 + u^4\,C^2\right)\right)^{\left(\frac{1}{4}\right)}$$

```
b := ((R^2+u^4*L^2)/(G^2+u^4*C^2))^(1/4);
```

$$b := \left(\frac{\kappa^2 u^2 + u^4\left(\frac{1}{vd^2\,C}+\frac{\kappa}{u}\right)^2}{C^2\,(wd+t\,u^2)^2 + u^4\,C^2}\right)^{\left(\frac{1}{4}\right)}$$

```
d := R*G + u^4*L*C;
```

$$d := \kappa u\,C\,(wd+t\,u^2) + u^4\left(\frac{1}{vd^2\,C}+\frac{\kappa}{u}\right)C$$

Large ω limit for $Re(Z_0)$

The real part of $Z_0(\omega)$ is then given from (Q.5.1) as follows:

```
ReZ := (b/sqrt(2))*sqrt(1+(d/a^2));
```

$$ReZ := \frac{1}{2}\left(\frac{\kappa^2 u^2 + u^4\left(\frac{1}{vd^2\,C}+\frac{\kappa}{u}\right)^2}{C^2\,(wd+t\,u^2)^2 + u^4\,C^2}\right)^{\left(\frac{1}{4}\right)}\sqrt{2}\sqrt{1+\frac{\kappa u\,C\,(wd+t\,u^2)+u^4\left(\frac{1}{vd^2\,C}+\frac{\kappa}{u}\right)C}{\sqrt{\left(\kappa^2 u^2 + u^4\left(\frac{1}{vd^2\,C}+\frac{\kappa}{u}\right)^2\right)\left(C^2\,(wd+t\,u^2)^2 + u^4\,C^2\right)}}}$$

An asymptotic expansion of $Re(Z_0)$ for large ω (large $u \equiv \sqrt{\omega}$) is then found to be,

```
> series(ReZ,u=infinity,4): ReZ1 :=collect(%,u);
```

$$ReZ1 := \frac{1}{2}\left(\frac{1}{vd^4\,C^2\,(C^2\,t^2+C^2)}\right)^{\left(\frac{1}{4}\right)}\sqrt{2}\sqrt{1+\frac{C}{\sqrt{C^2\,t^2+C^2}}} + \frac{\frac{1}{4}\frac{\left(\frac{1}{vd^4\,C^2\,(C^2\,t^2+C^2)}\right)^{\left(\frac{1}{4}\right)}\sqrt{2}\left(-\frac{vd^2\,C^2\,\kappa}{\sqrt{C^2\,t^2+C^2}}+\frac{(\kappa\,C\,t+\kappa\,C)\,vd^2\,C}{\sqrt{C^2\,t^2+C^2}}\right)}{\sqrt{1+\frac{C}{\sqrt{C^2\,t^2+C^2}}}}+\frac{1}{4}\left(\frac{1}{vd^4\,C^2\,(C^2\,t^2+C^2)}\right)^{\left(\frac{1}{4}\right)}\kappa\,vd^2\,C\sqrt{2}\sqrt{1+\frac{C}{\sqrt{C^2\,t^2+C^2}}}}{u} + O\left(\frac{1}{u^2}\right)$$

which has the form A + B/u + O(1/u²) . The two leading terms are then expanded for small t to get

```
op(1,ReZ1); simplify(%): series(%,t=0,3);
```

$$\frac{1}{2}\left(\frac{1}{vd^4\,C^2\,(C^2\,t^2+C^2)}\right)^{\left(\frac{1}{4}\right)}\sqrt{2}\sqrt{1+\frac{C}{\sqrt{C^2\,t^2+C^2}}}$$

$$\frac{1}{vd\,C}-\frac{3}{8}\frac{1}{vd\,C}t^2+O(t^4)$$

```
op(2,ReZ1); simplify(%): series(%,t=0,3);
```

$$\frac{\frac{1}{4}\frac{\left(\frac{1}{vd^4\,C^2\,(C^2\,t^2+C^2)}\right)^{\left(\frac{1}{4}\right)}\sqrt{2}\left(-\frac{vd^2\,C^2\,\kappa}{\sqrt{C^2\,t^2+C^2}}+\frac{(\kappa\,C\,t+\kappa\,C)\,vd^2\,C}{\sqrt{C^2\,t^2+C^2}}\right)}{\sqrt{1+\frac{C}{\sqrt{C^2\,t^2+C^2}}}}+\frac{1}{4}\left(\frac{1}{vd^4\,C^2\,(C^2\,t^2+C^2)}\right)^{\left(\frac{1}{4}\right)}\kappa\,vd^2\,C\sqrt{2}\sqrt{1+\frac{C}{\sqrt{C^2\,t^2+C^2}}}}{u}$$

$$\frac{1}{2}\frac{vd\,\kappa}{u}+\frac{1}{4}\frac{vd\,\kappa}{u}t-\frac{3}{16}\frac{vd\,\kappa}{u}t^2+O(t^3)$$

Collecting the results, replacing u by $\sqrt{\omega}$, and keeping only the leading terms in t = $\tan_L$, the resulting large ω limit for Re(Z_0) is,

$$\mathrm{Re}(Z_0) \approx 1/(v_d C) + (v_d \kappa/2)\, 1/\sqrt{\omega} + O(1/\omega)$$

<u>Large ω limit for Im(Z_0)</u>

The imaginary part of $Z_0(\omega)$ is given from (Q.5.1) as follows:

```
ImZ := - sigma*(b/sqrt(2))*sqrt(1-(d/a^2));
```

$$ImZ := -\frac{1}{2}\sigma\left(\frac{\kappa^2\,u^2+u^4\left(\frac{1}{vd^2\,C}+\frac{\kappa}{u}\right)^2}{C^2\,(wd+t\,u^2)^2+u^4\,C^2}\right)^{\left(\frac{1}{4}\right)}\sqrt{2}\sqrt{1-\frac{\kappa\,u\,C\,(wd+t\,u^2)+u^4\left(\frac{1}{vd^2\,C}+\frac{\kappa}{u}\right)C}{\sqrt{\left(\kappa^2\,u^2+u^4\left(\frac{1}{vd^2\,C}+\frac{\kappa}{u}\right)^2\right)\left(C^2\,(wd+t\,u^2)^2+u^4\,C^2\right)}}}$$

An asymptotic expansion of Im($Z_0(\omega)$) for large ω (large $u \equiv \sqrt{\omega}$) is then found to be,

```
> series(ImZ,u=infinity,4): ImZ1 :=collect(%,u);
```

$$ImZ1 := -\frac{1}{2}\sigma\left(\frac{1}{vd^4 C^2 (C^2 t^2 + C^2)}\right)^{\left(\frac{1}{4}\right)} \sqrt{2}\sqrt{1-\frac{C}{\sqrt{C^2 t^2 + C^2}}} + \frac{-\frac{1}{4}\frac{\sigma\left(\frac{1}{vd^4 C^2 (C^2 t^2 + C^2)}\right)^{\left(\frac{1}{4}\right)} \sqrt{2}\left(\frac{vd^2 C^2 \kappa}{\sqrt{C^2 t^2 + C^2}} - \frac{(\kappa C t + \kappa C)\, vd^2 C}{\sqrt{C^2 t^2 + C^2}}\right)}{\sqrt{1-\frac{C}{\sqrt{C^2 t^2 + C^2}}}} - \frac{1}{4}\sigma\left(\frac{1}{vd^4 C^2 (C^2 t^2 + C^2)}\right)^{\left(\frac{1}{4}\right)} vd^2 C \kappa \sqrt{2}\sqrt{1-\frac{C}{\sqrt{C^2 t^2 + C^2}}}}{u} + O\left(\frac{1}{u^2}\right)$$

which has the form A + B/u + $O(1/u^2)$. Maple next processes the first two terms by first simplifying them, then expanding them in small parameter $t \equiv \tan_L << 1$. The results are

```
op(1,ImZ1); simplify(%): series(%,t=0,4);
```

$$-\frac{1}{2}\sigma\left(\frac{1}{vd^4 C^2 (C^2 t^2 + C^2)}\right)^{\left(\frac{1}{4}\right)} \sqrt{2}\sqrt{1-\frac{C}{\sqrt{C^2 t^2 + C^2}}}$$

$$-\frac{1}{2}\frac{\sigma}{vd\,C}t + \frac{5}{16}\frac{\sigma}{vd\,C}t^3 + O(t^5)$$

```
op(2,ImZ1); simplify(%): series(%,t=0,4);
```

$$\frac{-\frac{1}{4}\frac{\sigma\left(\frac{1}{vd^4 C^2 (C^2 t^2 + C^2)}\right)^{\left(\frac{1}{4}\right)} \sqrt{2}\left(\frac{vd^2 C^2 \kappa}{\sqrt{C^2 t^2 + C^2}} - \frac{(\kappa C t + \kappa C)\, vd^2 C}{\sqrt{C^2 t^2 + C^2}}\right)}{\sqrt{1-\frac{C}{\sqrt{C^2 t^2 + C^2}}}} - \frac{1}{4}\sigma\left(\frac{1}{vd^4 C^2 (C^2 t^2 + C^2)}\right)^{\left(\frac{1}{4}\right)} vd^2 C \kappa \sqrt{2}\sqrt{1-\frac{C}{\sqrt{C^2 t^2 + C^2}}}}{u}$$

$$\frac{1}{2}\frac{\sigma\, vd\, \kappa}{u} - \frac{1}{4}\frac{\sigma\, vd\, \kappa}{u}t - \frac{3}{16}\frac{\sigma\, vd\, \kappa}{u}t^2 + O(t^3)$$

Collecting the results, replacing u by $\sqrt{\omega}$, and keeping only the leading terms in t, the resulting large ω limit for Im(Z_0) is,

$$\mathrm{Im}(Z_0) \approx -\sigma \tan_L/(2v_d C) + \sigma(v_d\kappa/2)/\sqrt{\omega} + O(1/\omega)$$

Finally, the sign σ is the sign of expression RC-LG,

```
e := expand(R*C-L*G,u);
```

$$e := \kappa\, u\, C - \frac{wd}{vd^2} - \frac{t\, u^2}{vd^2} - \frac{C\,\kappa\, wd}{u} - C\,\kappa\, u\, t$$

```
series(e,u=infinity,1): a1 :=  collect(%,u);
```

$$a1 := -\frac{t\, u^2}{vd^2} + (\kappa\, C - C\,\kappa\, t)\, u - \frac{wd}{vd^2} + O\left(\frac{1}{u}\right)$$

so for large ω and small $t = \tan_L$ we have

$$\sigma = \text{sign}\ [\ -(\tan_L / v_d^2)\ \omega + \kappa C\sqrt{\omega}\ - \omega_d / v_d^2\ ]$$

which is then $\sigma = -1$ for very large ω.

Dimension check:

$$\dim(\omega / v_d^2) = \sec^{-1} * \sec^2/m^2 = \sec/m^2$$

$$\dim(\kappa C\sqrt{\omega}\ ) = \text{ohm/m} * \sec^{1/2} * \text{far/m} * \sec^{-1/2} = \text{far-ohm/m}^2 = \sec/m^2$$

We now summarize these results as

Fact 6: The large ω asymptotic expansion for $Z_0(\omega)$, assuming $\tan_L << 1$, is given by (Q.3.5)

$$\text{Re}(Z_0) \approx 1/(v_d C) + (v_d \kappa/2)\ 1/\sqrt{\omega} + O(1/\omega)$$
$$\text{Im}(Z_0) \approx -\sigma \tan_L/(2 v_d C) + \sigma(v_d \kappa/2)/\sqrt{\omega} + O(1/\omega)$$

where $\kappa \equiv \left(\frac{1}{p_1} + \frac{1}{p_2}\right)\sqrt{\frac{\mu}{2\sigma}}$ $\qquad \sigma = \text{sign}\ [\ -(\tan_L / v_d^2)\ \omega + \kappa C\sqrt{\omega}\ - \omega_d / v_d^2\ ]$

Keeping only the leading terms for large ω,

$$\text{Re}(Z_0) \approx 1/(v_d C) = \sqrt{L_e/C} \approx \sqrt{L/C} \qquad v_d = 1/\sqrt{L_e C} \approx 1/\sqrt{LC}$$
$$\text{Im}(Z_0) \approx +\tan_L/(2 v_d C) = (1/2)\tan_L \sqrt{L_e/C} \qquad \sigma = -1$$

The leading term $\text{Re}(Z_0) \approx \sqrt{L/C}$ is the traditional result obtained from taking the large ω limit of the expression $Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$ when the four parameters are treated as constants in ω. This fact is obvious from the large-ω model stated in (Q.3.1) which we repeat here:

$$G(\omega) = C\ (\omega_d + \tan_L \omega\ ) \qquad \omega_d \equiv (\sigma_d/\varepsilon_d)$$
$$R(\omega) = \kappa\sqrt{\omega}$$
$$L(\omega) = 1/(C v_d^2) + (\kappa/\sqrt{\omega}\ ) \qquad \kappa \equiv \left(\frac{1}{p_1} + \frac{1}{p_2}\right)\sqrt{\frac{\mu}{2\sigma}} \qquad \textit{(Q.3.1)}$$

Since $\tan_L << 1$ one has

$$G + j\omega C = C(\omega_d + \tan_L \omega) + j\omega C = C\omega_d + \omega C(\tan_L + j) \approx \omega C j = j\omega C$$
$$R + j\omega L = \kappa\sqrt{\omega} + j\omega\,[1/(Cv_d^2) + (\kappa/\sqrt{\omega})\,] \approx j\omega L_e \approx j\omega L$$

Q.7 Real and Imaginary part of $Z_0(\omega)$ for small ω

For small ω the parameter model is that stated in (Q.4.2.)

$$G = C(w_d + t\,u^2) \qquad R = R_{dc} \qquad L = L_e + L_{idc} \qquad \textit{(Q.4.2)}$$

These expressions are entered into Maple followed by expressions for the intermediate parameters of (Q.5.1)

```
a := ((R^2+u^4*L^2)*(G^2+u^4*C^2))^(1/4);
```

$$a := \left(\left(Rdc^2 + u^4\left(\frac{1}{vd^2\,C} + Lidc\right)^2\right)\left(C^2\,(wd + t\,u^2)^2 + u^4\,C^2\right)\right)^{\left(\frac{1}{4}\right)}$$

```
b := ((R^2+u^4*L^2)/(G^2+u^4*C^2))^(1/4);
```

$$b := \left(\frac{Rdc^2 + u^4\left(\frac{1}{vd^2\,C} + Lidc\right)^2}{C^2\,(wd + t\,u^2)^2 + u^4\,C^2}\right)^{\left(\frac{1}{4}\right)}$$

```
d := R*G + u^4*L*C;
```

$$d := Rdc\,C\,(wd + t\,u^2) + u^4\left(\frac{1}{vd^2\,C} + Lidc\right)C$$

The real part of $Z_0(\omega)$ is then given from (Q.5.1) as follows:

An expansion of $\mathrm{Re}(Z_0)$ for small ω (small $u \equiv \sqrt{\omega}$) is then found to be,