Better expression for R and Li PhL 9.6.14

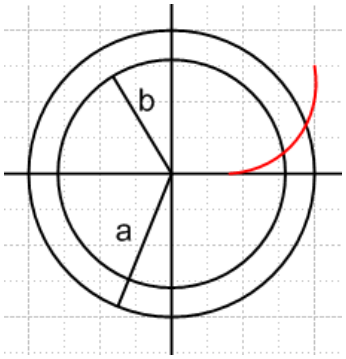
This entire effort is ill conceived. I was thinking that "R" was the effective bulk R of the round conductor, so here I integrate J_z over the cross section, but really R is the real part of the surface impedance, and I already have an expression for that for the symmetric round wire in Chapter 2! And the inductance is the imaginary part L_i. Oy. It is the SURFACE of the conductors that is involved with transmission lines.

My current Appendix Q model with simple Heaviside functions seems too crude. In that model, C = C and G = expression valid for all ω. The problem is with L_i and R. I think I can come up with a better and smoother model for these two parameters based on physics.

At high ω, the current flows in a sheaf and I have expressions for L_i and R for such a sheaf. In lines doc this is treated in App C and App K and App R for Belden. See App R section (f).

I will now " try a few things" in hopes of coming up with something better.

Let's start at relatively high ω so we have this picture



where we see the annulus and a crude plot of J_z versus radius. Our model here from (D.10.11) for J_z in a given partial wave m is,

$$J_z(r,m) = -(j/2)\,\sigma\,\eta_m\,I\,R_{dc}\,(a\beta)\,\sqrt{\frac{a}{r}}\,e^{(1+j)(r-a)/\delta}$$

Also from (D.10.15),

$$E_z(r,\theta) = -(j\omega/\sigma)\sqrt{\frac{a}{r}}\,e^{(1+j)(r-a)/\delta}\,n(\theta)\,(\beta/k)$$

$$J_z(r,\theta) = -j\omega\sqrt{\frac{a}{r}}\,e^{(1+j)(r-a)/\delta}\,n(\theta)\,(\beta/k)$$

In a separate file (App D filed) I just verified that J_z integrates to I as long as $a/\delta >> 1$. If this last is not true, the erf appears in the integral.

Of course I claim to have this more general answer, at least in partial waves,

$J_z(r,m) = \sigma\,(1/4)\,\eta_m\, I\, R_{dc}\ (a\beta')\, f_m(r)$ Note that $J_z(r,-m) = J_z(r,m)$ for Ch 6 case

$J_z(r,\theta) = \Sigma_{m=-\infty}^{\infty}\ J_z(r,m)\, e^{jm\theta}$

$= J_z(r,0) + 2\,\sigma\,(1/4)\, I\, R_{dc}\ (a\beta')\ \Sigma_{m=1}^{\infty}\,\eta_m\, f_m(r)\cos(m\theta)$

But maybe I can just assume a coaxial cable so we have a symmetric in θ situation. Then J_z is what I computed in Ch 2. Here that is just

$J_z(r,\theta) = J_z(r,0) = \sigma\,(1/4)\, I\, R_{dc}\ (a\beta')\, f_0(r) = \sigma\,(1/4)\, I\, R_{dc}\ (a\beta')\, f_0(r)$

$= \sigma\,(1/4)\, I\, R_{dc}\ (a\beta')\, 2\, J_0(x)/J_1(x_a) = \sigma\,(1/2)\, I\, R_{dc}\ x_a\, J_0(x)/J_1(x_a)$

$= (1/2)\, I\,(1/\pi a)\ (\beta)\, J_0(x)/J_1(x_a)\quad \beta \neq \beta' \quad = \quad (I/2\pi a)\ (\beta)\, J_0(x)/J_1(x_a)$

which agrees with (2.2.30).

So let's perhaps simplify the problem and just work on coaxial cable center conductor. We then have

$J_z(r,\theta) = (I/2\pi a)\ (\beta)\, J_0(\beta' r)/J_1(\beta' a)$ $\beta'^2 = \beta^2 - k^2$

and perhaps I could add a factor $n(\theta) / \langle n(\theta)\rangle = n(\theta)/ N_0$ to *simulate* the θ dependence.

So here then is the question: given the above J_z function, what can I say about R and L_i for the center conductor? Since k^2 appears here, we have a somewhat circular situation since I am trying to create a model from which I can compute k^2!! I think I will have to make the ansatz that $\beta' \approx \beta$. I recently studied this question in " Appendix M rewrite done Aug 19, 2014 original.doc" and it seems reasonable there based on my simple 2-phase model for k. So let's just assume that

$J_z(r) = (I/2\pi a)\ (\beta)\, J_0(\beta r)/J_1(\beta a)$ ansatz

From this $J_z(r)$ function, I should be able to compute both R and L_i at all ω !

$$L_i = \mu_i \int_{in}\, dS\, (H_i/I)^2 \qquad \text{(C.3.7)}$$

The magnetic field H_i at radius r is determined by

$$2\pi r\, H(r) = I_{enc} = \int_0^r r\, dr \int_{-\pi}^{\pi} d\theta\, J_z(r) = 2\pi \int_0^r r\, dr\, J_z(r)$$

$$= 2\pi \frac{\beta I}{2\pi a J_1(\beta a)} \int_0^r r\, dr\, J_0(\beta r) = \frac{\beta I}{a J_1(\beta a)} \int_0^r r\, dr\, J_0(\beta r)$$

Now let

$x = \beta r \qquad dx = \beta dr \qquad xdx = \beta^2 r dr$

so that

$$2\pi r\, H(r) = \frac{\beta I}{a J_1(\beta a)} \int_0^{\beta r} x dx/\beta^2\, J_0(x) = \frac{I}{a\beta J_1(\beta a)} \int_0^{\beta r} x J_0(x)\, dx$$

This I know is a doable integral. GR7 p 630 says

2. $\int x Z_0(x)\, dx = x Z_1(x)$

so the integral is just $[xJ_1(x)]|^{\beta r}_0 = (\beta r)J_1(\beta r)$, so we find that

$$2\pi r\, H(r) = \frac{I}{a\beta J_1(\beta a)} \quad (\beta r)J_1(\beta r) = I\,(r/a)\, J_1(\beta r)/\, J_1(\beta a)$$

which is very simple. When r = a you get I as expected. So:

$H(r) = I\,(1/2\pi a)\, J_1(\beta r)/\, J_1(\beta a)$ amp/m

Then I can compute the internal inductance of the center conductor this way

$2L_i = \mu \int_{in} dS\, |H/I|^2$ // correction made for complex fields, see below

$= \mu\, 2\pi \int_0^a r\, dr\, |(1/2\pi a)\, J_1(\beta r)/\, J_1(\beta a)|^2$

Do a check right here: For small ω, $J_1(z) = z/2$ so

$2L_i = \mu\, 2\pi\, (1/2\pi a)^2 \int_0^a r\, dr\, (r/a)^2 = \mu\, 2\pi\, (1/4\pi^2 a^2)\, a^{-2}\, (1/4)a^4 = \mu/8\pi$

But this seems wrong by factor of 2. So get this cleaned up before continuing.

$= \mu\, 2\pi\, (1/2\pi a)^2\, |1/\, J_1(\beta a)|^2 \int_0^a r\, dr\, |J_1(\beta r)|^2$

$= \mu\, 2\pi\, (1/4\pi^2 a^2)\, |1/\, J_1(\beta a)|^2 \int_0^a r\, dr\, |J_1(\beta r)|^2$

$= \mu\, (1/2\pi a^2)\, |1/\, J_1(x_a)|^2 \int_0^a r\, dr\, |J_1(\beta r)|^2$

$= \mu\, (1/2\pi a^2)\, |1/\, J_1(x_a)|^2\ K$

The integral here is

$$K = \int_0^a r\,dr\,|J_1(\beta r)|^2 = \int_0^{\beta a} x dx/\beta^2 * |J_1(x)|^2 = (1/\beta^2) \int_0^{\beta a} x dx\ |J_1(x)|^2$$

Check: For small ω, have $J_1(\beta r) \approx (\beta r/2)$ and then $K = \int_0^a r\,dr\,(|\beta| r/2)^2 = |\beta^2|/4 * \int_0^a r^3\,dr$

$$= |\beta^2|/4 * (1/4)\, a^4 = |\beta^2|\, a^4 .$$

Then we get

$$2L_i = \mu\,(1/2\pi a^2)\,(2/|\beta|a)^2\ |\beta^2|\, a^4 = \mu\,(1/2\pi)\,(2)^2 = 2\mu/\pi \quad \Rightarrow \quad L_i = \mu/\pi = \text{wrong}$$

Something is wrong, but I don't pursue this since topic not relevant.

STOP. With first = you can see that K is a real number, but this fact seems lost in the next form. Let's just leave it as is and write

$$K = \int_0^a r\,dr\,|J_1(\beta r)|^2 = \int_0^a r\,dr\,J_1(\beta r)\,J_1^*(\beta r)$$

Side Note: If β is very small, we get $K \approx a^2/2$ and then

$$2L_i = \mu\,(1/2\pi a^2)(2/|x_a|)\ a^2/2 = \mu\,(1/2\pi a^2)(2/a|\beta|)\ a^2/2 = \mu\,(1/4\pi)(2/a|\beta|) = \text{wrong!!}$$

Now I think we can claim from the series form of J

10.2.2 $$J_\nu(z) = (\tfrac{1}{2}z)^\nu \sum_{k=0}^{\infty} (-1)^k \frac{(\frac{1}{4}z^2)^k}{k!\,\Gamma(\nu+k+1)}.$$

that $J_\nu^*(z) = J_\nu(z^*)$. I thought this is "real analytic" but I see from the web that is a not the right meaning of real analytic, there seems to be no name for this thing. If the power series has real coefficients, then you conclude that $J_\nu^*(z) = J_\nu(z^*)$, so fine. We then have

$$K = \int_0^a r\,dr\,J_1(\beta r)\,J_1(\beta^* r)$$

Luckily I can evaluate this integral using

5.54

1.[10] $$\int x\,Z_p(\alpha x)\,\mathfrak{Z}_p(\beta x)\,dx = \frac{\alpha x\,Z_{p+1}(\alpha x)\,\mathfrak{Z}_p(\beta x) - \beta x\,Z_p(\alpha x)\,\mathfrak{Z}_{p+1}(\beta x)}{\alpha^2-\beta^2} = \frac{\beta x\,Z_p(\alpha x)\,\mathfrak{Z}_{p-1}(\beta x) - \alpha x\,Z_{p-1}(\alpha x)\,\mathfrak{Z}_p(\beta x)}{\alpha^2-\beta^2}$$

WA 134(8)

where I will take the first form. Then

$$K = \int_0^a x\,dx\, J_1(\beta^* x)\, J_1(\beta x)$$

$$= \frac{1}{\beta^{*2} - \beta^2}\ [\ \beta^* x\, J_1(\beta^* x) J_0(\beta x) - \beta x\, J_0(\beta^* x) J_1(\beta x)\ ]\ |^a_0$$

The zero end evaluation gives nothing since $J_1 = 0$ there, so we then have

$$K = \frac{1}{\beta^{*2} - \beta^2}\ [\ \beta^* a\, J_1(\beta^* a) J_0(\beta a) - \beta a\, J_0(\beta^* a) J_1(\beta a)\ ]$$

$$= \frac{1}{\beta^{*2} - \beta^2}\ [\ x_a^* J_1(x_a^*) J_0(x_a) - x_a\, J_0(x_a^*) J_1(x_a)\ ]$$

Now we know that $(z^2)^* = (z^*)^2$. Reverse order in both top and bottom to get

$$K = \frac{1}{\beta^2 - [\beta^2]^*}\ [x_a\, J_0(x_a^*) J_1(x_a) - \text{c.c.}\ ] = \frac{1}{2j\,\mathrm{Im}(\beta^2)}\ 2j\ \mathrm{Im}[x_a\, J_0(x_a^*) J_1(x_a)]$$

$$= \frac{1}{\mathrm{Im}(\beta^2)}\ \mathrm{Im}[x_a\, J_0(x_a^*) J_1(x_a)] = \text{real!}$$

Just verifying that K is real.

Side Note: If β is very small, we get $K = \frac{1}{\mathrm{Im}(\beta^2)}\ \mathrm{Im}[(x_a^2)/2] = \frac{1}{\mathrm{Im}(\beta^2)}\ \mathrm{Im}[\beta^2 a^2/2] = a^2/2$ so OK.

But I want to write this a different way. Note that

$$\beta^2 = -j\omega\mu\sigma \qquad \Rightarrow \qquad \mathrm{Im}\beta^2 = -\omega\mu\sigma$$

Go back so that

$$K = \frac{1}{\beta^2 - [\beta^2]^*}\ [x_a\, J_0(x_a^*) J_1(x_a) - \text{c.c.}\ ] \qquad \dim K = m^2$$

$$= \frac{1}{2j\,\mathrm{Im}(\beta^2)}\ [x_a\, J_0(x_a^*) J_1(x_a) - \text{c.c.}\ ]$$

$$= \frac{1}{2j(-\omega\mu\sigma)}\ [x_a\, J_0(x_a^*) J_1(x_a) - x_a^*\, J_0(x_a) J_1(x_a^*)\ ]$$

$$= \frac{1}{2j(-\omega\mu\sigma)}\ [x_a\, J_0^*(x_a) J_1(x_a) - x_a^*\, J_0(x_a) J_1^*(x_a)]$$

So we then have

$$2L_i = \mu\,(1/2\pi a^2)\,|1/\,J_1(x_a)|^2\;\; K$$

$$= (\mu/2\pi a^2)\,\frac{1}{J_1(x_a)\,J_1^*(x_a)}\;\frac{1}{2j(-\omega\mu\sigma)}\;[x_a\,J_0^*(x_a)J_1(x_a)\;\; - x_a^*\,J_0(x_a)\,J_1^*(x_a)]$$

$$= \;(\mu/2\pi a^2)\,\frac{1}{2j(-\omega\mu\sigma)}\;\;[x_a\,J_0^*(x_a)/J_1^*(x_a)\;\; - x_a^*\,J_0(x_a)/J_1(x_a)]$$

$$= \;(1/4\pi a^2)\,\frac{1}{j(-\omega\sigma)}\,[\;\frac{x_a\,J_0^*(x_a)}{J_1^*(x_a)}\;\; - \frac{x_a^*\,J_0(x_a)}{J_1(x_a)}\;\;]$$

$$= \;(j/4\pi a^2\sigma\omega)\,[\;\frac{x_a\,J_0^*(x_a)}{J_1^*(x_a)}\;\; - \frac{x_a^*\,J_0(x_a)}{J_1(x_a)}\;\;]$$

and finally here is my result:

$$L_i = \frac{j}{8\pi a^2\sigma\omega}\;\;\;[\;\frac{x_a\,J_0^*(x_a)}{J_1^*(x_a)}\;\; - \frac{x_a^*\,J_0(x_a)}{J_1(x_a)}\;\;]$$

But OK, now combine the two terms to get 2jIm(..) to get

$$L_i = \frac{j}{8\pi a^2\sigma\omega}\;\;\;2j\;\mathrm{Im}\,[\;\frac{x_a\,J_0^*(x_a)}{J_1^*(x_a)}\;]\;\;\;= -\;\frac{1}{4\pi a^2\sigma\omega}\;\mathrm{Im}\,[\;\frac{x_a\,J_0^*(x_a)}{J_1^*(x_a)}\;]\qquad\qquad //\text{ real at least}$$

Now evaluate this at small β meaning small x_a to get

$$J_0 = 1\qquad\qquad\qquad J_1 = x_a/2$$

$$\mathrm{Im}\,[\;\frac{x_a\,J_0^*(x_a)}{J_1^*(x_a)}\;]\;= \mathrm{Im}\,[\;\frac{x_a}{x_a^*/2}\;]\;= 2\;\;\mathrm{Im}\;\;[\;\frac{x_a}{x_a^*}\;]\;\;=\;\;2\;\mathrm{Im}\,[\frac{\beta}{\beta^*}\;]$$

Now write $\beta \equiv e^{j3\pi/4}\sqrt{\omega\mu\sigma}$ from (2.2.12) to get

$$= 2\;\mathrm{Im}\,[\frac{e^{j3\pi/4}\sqrt{\omega\mu\sigma}}{e^{-j3\pi/4}\sqrt{\omega\mu\sigma}}\;]\;= 2\;\mathrm{Im}[e^{+j3\pi/2}] = 2\;\mathrm{Im}[-j]\;\;= -2$$

So our low ω limit is then

$$L_i = -\;\;\frac{1}{4\pi a^2\sigma\omega}\,[\;-2] = \text{wrong!!!}$$

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So we then have

$$L_i = \mu\, 2\pi\, (1/2\pi a)^2\, |1/\, J_1(\beta a)|^2 * (1/\beta^2) \int_0^{\beta a} x\,dx\, |J_1(x)|^2$$

$$= \frac{2\pi\mu}{4\pi^2 a^2\, \beta^2 |J_1(\beta a)|^2} \int_0^{\beta a} x\,dx\, |J_1(x)|^2 = \frac{\mu}{2\pi x_a^2 |J_1(x_a)|^2} \int_0^{\beta a} x\,dx\, |J_1(x)|^2$$

Maple cannot do this integral. But GR7 can using

2. $$\int x\,[Z_p(\alpha x)]^2\,dx = \frac{x^2}{2}\left\{[Z_p(\alpha x)]^2 - Z_{p-1}(\alpha x)\, Z_{p+1}(\alpha x)\right\}$$ WA 135(11)

which says

$$\int_0^{\beta a} x\,dx\, [J_1(x)]^2 = [\, (x^2/2)\, \{\, [J_1(x)]^2 - J_0(x)J_2(x)\, \}|^{\beta a}_0$$

$$= (\beta^2 a^2/2)\, \{\, [J_1(\beta a)]^2 - J_0(\beta a)J_2(\beta a)\, \}$$

$$= (x_a^2/2)\, \{\, [J_1(x_a)]^2 - J_0(x_a)J_2(x_a)\, \}$$

and so

$$L_i = \frac{\mu}{2\pi x_a^2 (J_1(x_a))^2}\ (x_a^2/2)\, \{\, [J_1(x_a)]^2 - J_0(x_a)J_2(x_a)\, \}$$

$$= \frac{\mu}{4\pi (J_1(x_a))^2}\, \{\, [J_1(x_a)]^2 - J_0(x_a)J_2(x_a)\, \}\}$$

$$= \frac{\mu}{4\pi}\left(1 - \frac{J_0(x_a)J_2(x_a)}{[J_1(x_a)]^2}\right) \qquad x_a = \beta a$$

Check some limits here. At low ω, x_a is small and we use $J_n(x) = (x/2)^n / n!$ so

$$J_0 = 1 \qquad J_1 = x_a/2 \qquad J_2 = (x_a/2)^2/2$$

so then

$$L_i = \frac{\mu}{4\pi}\left(1 - \frac{(x_a/2)^2/2}{[\, x_a/2]^2}\right) = \frac{\mu}{4\pi}\left(1 - \frac{1/2}{1}\right) = \mu/8\pi \quad \text{hurray!!}$$

Fine, but L_i is complex, so I think I need to fix something in Appendix C interpretation. There I write

$$u(r) = (1/2)\,\mu\, H(r)^2. \qquad \text{(C.3.1)}$$

So what does it mean that the energy density has an r-dependent phase? I suspect this should say

$$u(r) = (1/2)\,\mu\, |H(r)|^2. \qquad \text{(C.3.1)}$$

My appendix is only for DC applications. Jackson says $u_{em} = (\mathbf{E}\bullet\mathbf{D} + \mathbf{B}\bullet\mathbf{H})/2$. But if we have monochrome fields, this usual thing happens

$$\mathbf{H} = \mathbf{H_1}\cos(\omega t)$$

$$2\langle u\rangle/\mu = \langle \mathbf{H}\bullet\mathbf{H}\rangle = \int dt\, [\mathbf{H_1}\cos(\omega t)]^2 = (1/2)\,\mathbf{H_1}\bullet\mathbf{H_1}$$

And if

$$\mathbf{H} = \mathbf{H_1}e^{j\omega t}$$

then

$$\mathbf{H}\bullet\mathbf{H}^* = \mathbf{H_1}\bullet\mathbf{H_1} = 4\langle u\rangle/\mu$$

so the rule is this

$$\langle u\rangle = (1/4)\,\mu\,\mathbf{H}\bullet\mathbf{H}^*$$

so you add a * to one of the fields, and you add an extra overall 1/2. To quote from Jackson

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} \qquad \text{(6.109)}$$

We now define the complex Poynting vector

$$\mathbf{S} = \tfrac{1}{2}(\mathbf{E}\times\mathbf{H}^*) \qquad \text{(6.132)}$$

so that is an example of this extra factor of 1/2. So I will now REDO the above calculation