Example of order-interchange of limits problem PhL 9.2.14

1. The original problem

I encountered this when writing version 2 of lines doc Appendix Q. I have this function

```
Rek := sqrt((a^2-c)/2);
```

$$Rek := \frac{1}{2}\sqrt{2\sqrt{(Rdc^2+u^4\,Ldc^2)\left(C^2\,(wd+t\,u^2)^2+u^4\,C^2\right)}-2\,Rdc\,C\,(wd+t\,u^2)+2\,u^4\,Ldc\,C}$$

and I am interested in u→ 0 for a small ω limit, and also a ω_d→ 0 limit for no conductance in a transmission line. See Appendix Q new version.doc.

Let's first look at the limit u → 0:

Rek = (1/2) [2 ($R_{dc}^2C^2\,\omega_d^2)^{1/2}$ - $2R_{dc}C\,\omega_d$]$^{1/2}$ =

= (1/2) [2 $R_{dc}C\,\omega_d$ - $2R_{dc}C\,\omega_d$]$^{1/2}$

= 0

If I then take the limit of this result as ω_d→ 0 , the result is of course still 0.

Now look at the limit ω_d → 0:

Rek = (1/2) [2 {($R_{dc}^2 + u^4L_{dc}^2$)($C^2t^2u^4$+ u^4C^2) }$^{1/2}$ - 2 $R_{dc}C\,t^2u^4$ + $2u^4L_{dc}C$]$^{1/2}$

= (1/2) [2 {($R_{dc}^2 + u^4L_{dc}^2$)$Cu^2(t^2+1)$ }$^{1/2}$ - 2 $R_{dc}C\,t^2u^4$ + $2u^4$ $L_{dc}C$]$^{1/2}$

Now take the limit of this result as u→ 0

Rek = 0

I thought there was going to be a conflict here?

Back up a bit. Ignore the overall square root and leading factor

```
(2*Rek)^2;
```

$$2\sqrt{(Rdc^2+u^4\,Ldc^2)\left(C^2\,(wd+t\,u^2)^2+u^4\,C^2\right)}-2\,Rdc\,C\,(wd+t\,u^2)+2\,u^4\,Ldc\,C$$

Manually simplify this

$2C\sqrt{(R_{dc}^2+u^4L_{dc}^2)(\,(\omega_d + tu^2)^2 + u^4)}$ - $2R_{dc}C\,(\omega_d+ tu^2)$ + $2u^4L_{dc}C$

Now remove the overall common factor of C

$$f(\omega_d,u) \equiv 2\sqrt{(R_{dc}^2+u^4L_{dc}^2)((\omega_d+tu^2)^2+u^4)} - 2R_{dc}(\omega_d+tu^2) + 2u^4L_{dc}$$

What can be said about this function of two complex variables ω_d and u ?

$$f(\omega_d=0,u) \equiv 2\sqrt{(R_{dc}^2+u^4L_{dc}^2)((tu^2)^2+u^4)} - 2R_{dc}(tu^2) + 2u^4L_{dc}$$

$$= 2u^2\sqrt{(R_{dc}^2+u^4L_{dc}^2)((t)^2+1)} - 2R_{dc}(tu^2) + 2u^4L_{dc}$$

$$= u^2\left[2\sqrt{(R_{dc}^2+u^4L_{dc}^2)(t^2+1)} - 2tR_{dc} + 2u^2L_{dc}\right]$$

$$f(\omega_d,u=0) \equiv 2\sqrt{(R_{dc}^2)((\omega_d)^2)} - 2R_{dc}(\omega_d) = 0$$

2. Simpler Case

How about this much simpler function

$$f(\omega_d,u) \equiv \sqrt{(\omega_d+tu^2)^2+u^4}$$

I want to expand this in small u with $\omega_d > 0$. Thus

$$f(\omega_d,u) = \sqrt{\omega_d^2+2\omega_d tu^2+t^2u^4+u^4} = \sqrt{\omega_d^2+2\omega_d tu^2+u^4(t^2+1)}$$

$$= (1/\omega_d)\sqrt{1+2tu^2/\omega_d+u^4(t^2+1)/\omega_d^2}$$

$$\approx (1/\omega_d)\{1+(1/2)[2tu^2/\omega_d+u^4(t^2+1)/\omega_d^2]+(1/2)(-1/2)(1/2)[2tu^2/\omega_d]^2\}$$

$$\approx (1/\omega_d)\{1+(1/2)[2tu^2/\omega_d+u^4(t^2+1)/\omega_d^2]-(1/2)[tu^2/\omega_d]^2\}$$

$$\approx (1/\omega_d)\{1+(1/2)[2tu^2/\omega_d+u^4(t^2+1)/\omega_d^2]-(1/2)t^2u^4/\omega_d^2\}$$

$$\approx (1/\omega_d)\{1+tu^2/\omega_d+(1/2)u^4(t^2+1)/\omega_d^2-(1/2)t^2u^4/\omega_d^2\}$$

$$\approx (1/\omega_d)\{1+tu^2/\omega_d+(1/2)u^4(1)/\omega_d^2+O(u^6)\}$$

If I then take $\omega_d \to 0$, each term blows up!

On the other hand, if I first take $\omega_d \to 0$ I get

$$f(0,u) = \sqrt{(tu^2)^2+u^4} = u^2\sqrt{t^2+1}$$

and this is well-behaved as u→ 0. So what stupidly simple thing is going on here that dumbo fails to comprehend? The series in u must not be a convergent series. Why is that so?

3. Simpler Case Still

```
assume(x>0,z>0);

f := sqrt((x+z^2)^2 +z^4);
```

$$f := \sqrt{x^2 + 2xz^2 + 2z^4}$$

```
series(f,z=0,7);
```

$$x + z^2 + \frac{1}{2}\frac{1}{x}z^4 - \frac{1}{2}\frac{1}{x^2}z^6 + \frac{3}{8}\frac{1}{x^3}z^8 + O(z^{10})$$

This is a power series in complex variable y. What is the radius of convergence of this power series? I have a function here of two complex variables and I expand it in z about the point z = 0.

4. Even Simpler Case Still

```
f := sqrt(x+z);
```

$$f := \sqrt{x+z}$$

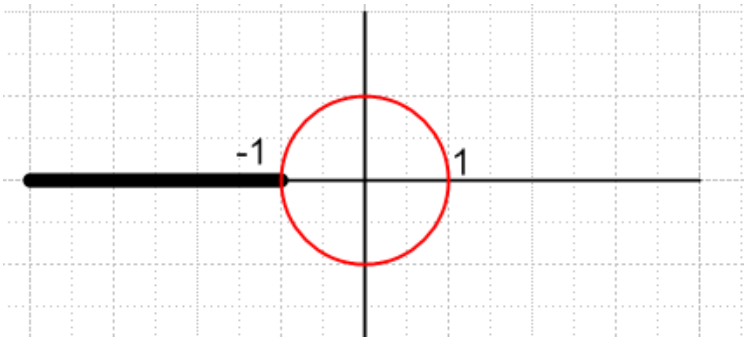
```
series(f,z=0,4);
```

$$\sqrt{x} + \frac{1}{2}\frac{1}{\sqrt{x}}z - \frac{1}{8}\frac{1}{x^{\left(\frac{3}{2}\right)}}z^2 + \frac{1}{16}\frac{1}{x^{\left(\frac{5}{2}\right)}}z^3 + O(z^4)$$

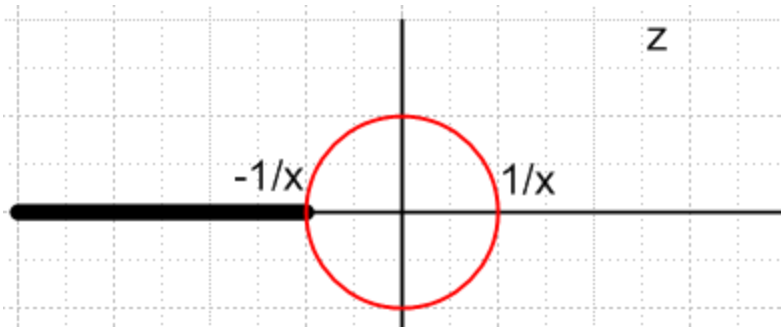
Again, the coefficients all blow up as x → 0. Here is what is going on. Write as

$$f = \sqrt{x}\sqrt{1 + (z/x)}$$

The small y expansion $\sqrt{1+y}$ converges around y = 0 in a disk of radius 1 because that circle hits a branch point,



Similarly, the function $\sqrt{1 + (z/x)}$ expanded about z = 0 has a radius of convergence of 1/x:



As x→ 0, the radius of convergence → 0 as well. In this limit, the power series expansion in z does not converge!

5. Back to the original problem

```
Rek := sqrt((a^2-c)/2);
```

$$Rek := \frac{1}{2}\sqrt{2\sqrt{(Rdc^2+u^4\,Ldc^2)\left(C^2\,(wd+t\,u^2)^2+u^4\,C^2\right)}-2\,Rdc\,C\,(wd+t\,u^2)+2\,u^4\,Ldc\,C}$$

The two complex variables are z = u and x = ω_d . I expand this about u = 0 to get some terms,

```
series(Rek,u=0,5): Rek1 := simplify(collect(%,u));
```

$$Rek1 := \frac{1}{2}\frac{(Rdc+Ldc\,wd)\sqrt{C}}{\sqrt{Rdc}\,\sqrt{wd}}u^2-\frac{1}{4}\frac{(Rdc-Ldc\,wd)\,t\sqrt{C}}{wd^{\left(\frac{3}{2}\right)}\sqrt{Rdc}}u^4+O(u^6)$$

.

You can see that the coefficients are blowing up as $\omega_d \to 0$. There is surely a branch point for the complicated function Rek shown which is some function of ω_d. As $\omega_d \to 0$, this branch point approaches the origin in u-space, causing the u-expansion radius of convergence to shrink. The expansion becomes less and less convergent and finally diverges. In the limit $\omega_d \to 0$, the u expansion is divergent and completely meaningless.

$\omega_d \equiv (\sigma_d/\varepsilon_d) = 10^{-15} / 8.8541877 \times 10^{-12} = 1000/8.85 = 113$ aHz for PE dielectric

$f_d = \omega/2\pi = 18$ Hz.

So I guess the question is this: how "small" is such an ω_d ? For the Belden cable, $\omega_d = 113$ in SI units, maybe that is not so small after all.