This is the Title PhL 3.26.05

Algebra done in Appendix G, now replaced by Maple calculations:

$$a^2 \equiv (R^2+\omega^2L^2)^{1/2}\,(G^2+\omega^2C^2)^{1/2} = LC\,(R^2/L^2+\omega^2)^{1/2}(G^2/C^2+\omega^2)^{1/2}$$

$$= \omega^2LC\,(\,1 + R^2/(\omega L)^2)^{1/2}(\,1 + G^2/(\omega C)^2)^{1/2}$$

$$= \omega^2LC\,(\,1 + a/\omega^2)^{1/2}(\,1 + a'/\omega^2)^{1/2} \qquad a = (1/2)(R^2/L^2) \qquad a' = (1/2)$$

$$= \omega^2LC\,[\,1 + a/\omega^2 + b/\omega^4]\,[\,1 + a'/\omega^2 + b'/\omega^4] \qquad b = -(1/8)(R^4/L^4) \qquad b = -(1/8)(G^4/C^4)$$

$$= \omega^2LC\,[\,1 + (a+a')/\omega^2 + (b+b'+aa')/\omega^4] \qquad aa' = (1/4)\;(R^2/L^2)(G^2/C^2)$$

$$= \omega^2LC\,[\,1 + (1/2\omega^2)(R^2/L^2 + G^2/C^2) - (1/8\omega^4)\,[R^4/L^4 + G^4/C^4 - 2(R^2/L^2)(G^2/C^2)\,]$$

$$= \omega^2LC\,[\,1 + (1/2\omega^2)(R^2/L^2 + G^2/C^2) - (1/8\omega^4)\,(R^2/L^2 - G^2/C^2)^2\,]$$

$$a^2 + c = \omega^2LC\,[\,1 + (1/2\omega^2)(R^2/L^2 + G^2/C^2) - (1/8\omega^4)\,(\,R^2/L^2 - G^2/C^2)^2\,] + RG - \omega^2LC$$

$$= LC\,[\,(1/2)(R^2/L^2 + G^2/C^2) - (1/8\omega^2)\,(\,R^2/L^2 - G^2/C^2)^2\,] + RG$$

$$= LC\,[\,(1/2)(R^2/L^2 + G^2/C^2 + 2R/L*G/C) - (1/8\omega^4)\,(\,R^2/L^2 - G^2/C^2)^2\,]$$

$$= LC\,[\,(1/2)(R/L + G/C)^2 - (1/8\omega^2)\,(\,R^2/L^2 - G^2/C^2)^2\,]$$

$$= LC\,(1/2)(R/L + G/C)^2\,\{\,1 - (1/8\omega^2)\,(\,R^2/L^2 - G^2/C^2)^2\;2\,/(R/L + G/C)^2\,\}$$

$$= LC\,(1/2)(R/L + G/C)^2\,\{\,1 - (1/4\omega^2)\,(R/L - G/C)^2\,\}$$

$$\sqrt{a^2+c} = \sqrt{LC/2}\;(R/L + G/C)\,\{\;1 - (1/8\omega^2)\,(R/L - G/C)^2\,\}$$

$$= 1/(\sqrt{2LC}\,)\,(RC + GL)\,\{\;1 - (1/8\omega^2)\,(RC - GL)^2\;(LC)^{-2}\,\}$$

$$\sqrt{\frac{a^2+c}{2}} = 1/(2\sqrt{LC}\,)\,(RC + GL)\,\{\;1 - (1/8\omega^2)\,(RC - GL)^2\;(LC)^{-2}\;\}$$

$$= 1/(2\sqrt{LC}\,)\,(RC + GL) - (1/8\omega^2)\,(RC - GL)^2\;(LC)^{-2} * 1/(2\sqrt{LC}\,)\,(RC + GL)$$

$$= 1/(2\sqrt{LC}\,)\,(RC + GL) - (1/16\omega^2)\,(RC - GL)^2\,(RC + GL)\,(LC)^{-5/2}$$

$$a^2 - c = a^2 + c - 2c = LC\,(1/2)(R/L + G/C)^2\,\{\,1 - (1/4\omega^2)(R/L - G/C)^2\,\} - 2RG + 2\,\omega^2 LC$$

$$= 2\,\omega^2 LC + LC\,(1/2)(R/L + G/C)^2 - 2RG + O(1/\omega^2)$$

$$= 2\,\omega^2 LC + LC\,(1/2)(R/L + G/C)^2 - 2RG$$

$$= 2\,\omega^2 LC + (1/2)(RC-LG)^2/LC$$

$$= 2\,\omega^2 LC\,\{\,1 + (1/2)(RC-LG)^2/LC * 1/(\,2\,\omega^2 LC)\,\}$$

$$= 2\,\omega^2 LC\,\{\,1 + (1/4\omega^2)(RC-LG)^2/L^2C^2\,\}$$

$$\sqrt{\frac{a^2-c}{2}} = \omega\sqrt{LC}\,\{\,1 + (1/8\omega^2)(RC-LG)^2/L^2C^2\,\}$$

$$= \omega\sqrt{LC} + (1/8\omega^2)(RC-LG)^2/L^2C^2 * \omega\sqrt{LC}$$

$$= \omega\sqrt{LC} + (1/8\omega)(RC-LG)^2/(LC)^{3/2}$$

Therefore,

$$\mathrm{Re}(k) = \sqrt{\frac{a^2-c}{2}} = \omega\sqrt{LC} + \frac{(RC-LG)^2}{8\omega(LC)^{3/2}}$$

$$\mathrm{Im}(k) = -\sqrt{\frac{a^2+c}{2}} = -\frac{(RC + GL)}{2\sqrt{LC}} + \frac{(RC - GL)^2\,(RC + GL)}{16\omega^2\,(LC)^{5/2}}$$

Maple verification:

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Therefore

$$\dim(R/L) = \text{ohm/henry} = \text{ohm} * \text{mho}*\sec^{-1} = \sec^{-1}$$

$$\mathrm{Re}(k) = \omega\sqrt{LC} + (1/8\omega)(RC-LG)^2/(LC)^{3/2}$$

$\dim(G/C) = \text{mho/far} = \text{mho}*\text{ohm/sec} = \sec^{-1}$

$$\mathrm{Re}(k) = \sqrt{\frac{a^2-c}{2}} = \omega\sqrt{LC} \qquad \dim(LC) = \sec^2/m^2 \qquad \dim(RC) = \dim(GL) = \sec/m^2$$

$$\text{Im}(k) = -\sqrt{\frac{a^2+c}{2}} = -\ 1/(2\sqrt{LC})\ (RC + GL)\ \{\ 1 -\ (1/8\omega^2)\ (RC - GL)^2\ \ (LC)^{-2}\ \}$$

$$= -\frac{(RC + GL)}{2\sqrt{LC}}\ \left[\ 1 + \frac{(RC - GL)^2}{8\omega^2 L^2 C^2}\ \right]$$

$$= -\ 1/(2\sqrt{LC})\ (RC + GL)\ \{\ 1 -\ (1/8\omega^2)\ (RC - GL)^2\ \ (LC)^{-2}\ \}$$

$$= -\ 1/(2\sqrt{LC})\ (RC + GL)\ + 1/(2\sqrt{LC})\ (RC + GL)\ (1/8\omega^2)\ (RC - GL)^2\ \ (LC)^{-2}$$

$$= -\ 1/(2\sqrt{LC})\ (RC + GL)\ +\ (1/16\omega^2)\ (RC + GL)\ (RC - GL)^2\ \ (LC)^{-5/2}$$

} QED

Note: The expansion $(1+x)^{1/2} = 1+x/2-x^2/8$ used in the proof is valid only if

$R^2/\omega^2L^2 \ll 1$ and $G^2/\omega^2C^2 \ll 1$

which requires that

$\omega \gg (R/L)$ and $\omega \gg (G/C)$

For any values of R,L,C,G we can thus find ω large enough to make our high ω limit valid.

Maple verification

```
a := ((R^2+omega^2*L^2)*(G^2+omega^2*C^2))^(1/4):
c := R*G - omega^2*L*C:
Re_bdp :=  sqrt((a^2-c)/2):
Im_bdp := -sqrt((a^2+c)/2):
limit(Re_bdp/omega,omega = infinity):simplify(%);
```

$$\sqrt{LC}$$

```
limit(Im_bdp,omega = infinity):simplify(%);
```

$$-\frac{1}{2}\frac{RC+GL}{\sqrt{L}\sqrt{C}}$$

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$$a^2 \equiv (R^2+\omega^2L^2)^{1/2}\ (G^2+\omega^2C^2)^{1/2} = RG(1+\omega^2L^2/R^2)^{1/2}(1+\omega^2C^2/G^2)^{1/2}$$

$$\approx\ RG\,(1+\omega^2L^2/2R^2)\,(1+\omega^2C^2/2G^2)\ \approx\ RG\,[1+(1/2)\omega^2(L^2/R^2+C^2/G^2)]$$

$$a^2 + c\ = RG\,[1+(1/2)\omega^2(L^2/R^2+C^2/G^2)] + RG - \omega^2LC$$

$$= [RG +(1/2)\omega^2(GL^2/R+RC^2/G)] + RG - \omega^2LC$$

$$= 2RG +(1/2)\omega^2(GL^2/R+RC^2/G -2LC)$$

$$= 2RG +(1/2)\omega^2(G^2L^2+R^2C^2 - 2LCRG)/RG$$

$$= 2RG +(1/2)\omega^2(RC-GL)^2/RG$$

$$\frac{a^2+c}{2}\ = RG +(\omega/2)^2\,(RC-GL)^2/RG\ = RG\,[\,1 + (\omega/2)^2\,(\frac{RC-GL}{RG})^2\,]$$

$$\sqrt{\frac{a^2+c}{2}}\ = \sqrt{RG}\,[\,1 + (\omega/2)^2\,(\frac{RC-GL}{RG})^2\,]^{1/2} = \sqrt{RG}\,[\,1 + (\omega^2/8)\,(\frac{RC-GL}{RG})^2\,]$$

$a^2 - c$ = same as $a^2 + c$ but the main RG's cancel and the cross term changes sign, so

$$= (1/2)\omega^2(RC+GL)^2/RG$$

Therefore for low ω we find,

$$\text{Re}(k)\ =\ \sqrt{\frac{a^2-c}{2}}\ =\ (\omega/2)\frac{RC+GL}{\sqrt{RG}}$$

$$\text{Im}(k) = -\sqrt{\frac{a^2+c}{2}}\ = -\ \sqrt{RG}\,[\,1 + (\omega^2/8)\,(\frac{RC-GL}{RG})^2\,]$$ QED

Note: The expansion $(1+x)^{1/2} = 1+x/2$ used in the proof is valid only if

$$\omega^2L^2/R^2 \ll 1 \qquad \text{and} \qquad \omega^2C^2/G^2 \ll 1$$

which requires that

$$\omega \ll (R/L) \qquad \text{and} \qquad \omega \ll (G/C)$$

If R = 0 and/or G = 0, our proof fails since at least one of these inequalities cannot be met.

Maple verification, continuing the code shown above:

```
series(Re_bdp,omega,2): factor(%);
```

$$\frac{1}{2}\sqrt{\frac{(LG+RC)^2}{RG}}\,\omega+O(\omega^3)$$

```
series(Im_bdp,omega,3): factor(%);
```

$$-\sqrt{RG}-\frac{1}{8}\frac{\sqrt{RG}\,(-RC+LG)^2}{R^2G^2}\omega^2+O(\omega^4)$$

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```
restart; alias(I\=I,j=sqrt(-1)):
assume(R>0,L>0,C>0);
a := ((R^2+omega^2*L^2)*(G^2+omega^2*C^2))^(1/4):
c := R*G - omega^2*L*C:
G := 0:
Rek :=  sqrt((a^2-c)/2);
```

$$Rek:=\frac{1}{2}\sqrt{2\sqrt{(R^2+\omega^2L^2)\,\omega^2C^2}+2\,\omega^2LC}$$

```
Imk := -sqrt((a^2+c)/2);
```

$$Imk:=-\frac{1}{2}\sqrt{2\sqrt{(R^2+\omega^2L^2)\,\omega^2C^2}-2\,\omega^2LC}$$

```
series(Rek,omega,3);
```

$$\frac{1}{2}\sqrt{2}\,\sqrt{RC}\,\sqrt{\omega}+\frac{1}{4}\frac{\sqrt{2}\,\sqrt{RC}\,L\,\omega^{\left(\frac{3}{2}\right)}}{R}+O\left(\omega^{\left(\frac{5}{2}\right)}\right)$$

```
series(Imk,omega,3);
```

$$-\frac{1}{2}\sqrt{2}\,\sqrt{RC}\,\sqrt{\omega}+\frac{1}{4}\frac{\sqrt{2}\,\sqrt{RC}\,L\,\omega^{\left(\frac{3}{2}\right)}}{R}+O\left(\omega^{\left(\frac{5}{2}\right)}\right)$$