Appendix Q: Properties of the function $\beta_d'(\omega)$

According to Section 5.3 (a), when losses are included in the transmission line model, one must make the following replacement to the exponent in the general traveling wave factor $e^{j(\omega t-\beta_d z)}$:

$$\beta_d = (\omega/v_d) \rightarrow \beta_d' = -j\sqrt{\mathbf{zy}} = -j\sqrt{(R+j\omega L)(G+j\omega C)} \ . \qquad (Q.1)$$

Fact 1: The real and imaginary decomposition of β_d' is given by

$$\beta_d' = \sqrt{\frac{a^2-c}{2}} - j\sqrt{\frac{a^2+c}{2}} \qquad (Q.2)$$

where

$$a \equiv (R^2+\omega^2L^2)^{1/4}(G^2+\omega^2C^2)^{1/4} \qquad \dim(a) = 1/m \qquad a > 0 \qquad (Q.3)$$

$$c \equiv RG - \omega^2LC \qquad \dim(c) = 1/m^2 \qquad c = \text{real},\ |c| < a^2 \qquad (Q.4)$$

Proof: Let

$$q \equiv \mathbf{zy} = (R+j\omega L)(G+j\omega C) = (RG-\omega^2LC) + j\omega(LG+RC) = c + j\omega(LG+RC) = |q|\, e^{j\theta}$$

Then

$$|q|^2 = |(R+j\omega L)(G+j\omega C)|^2 = |(R+j\omega L)|^2|(G+j\omega C)|^2 = (R^2+\omega^2L^2)(G^2+\omega^2C^2) = a^4$$

$$|q| = a^2$$

$$\cos\theta = c/|q| = c/a^2$$

Now write

$$s \equiv \sqrt{\mathbf{zy}} = \sqrt{(R+j\omega L)(G+j\omega C)} = \sqrt{q} = |s|\, e^{i\varphi} \qquad |s| = \sqrt{|q|} = a \qquad \varphi = \theta/2$$

$$\mathrm{Re}(s) = |s|\cos\varphi = a\cos(\theta/2) = a\sqrt{(1+\cos\theta)/2} = (a/\sqrt{2})\sqrt{1+c/a^2} = \sqrt{\frac{a^2+c}{2}}$$

$$\mathrm{Im}(s) = |s|\sin\varphi = a\sin(\theta/2) = a\sqrt{(1-\cos\theta)/2} = (a/\sqrt{2})\sqrt{1-c/a^2} = \sqrt{\frac{a^2-c}{2}}$$

$$s = \sqrt{\mathbf{zy}} = \sqrt{\frac{a^2+c}{2}} + j\sqrt{\frac{a^2-c}{2}}$$

$$\beta_d' = -j\sqrt{\mathbf{zy}} = -js = \sqrt{\frac{a^2-c}{2}} - j\sqrt{\frac{a^2+c}{2}} \qquad \text{QED}$$

Maple verification, where β_d' = bdp and a2_ = a^2 :

```
restart; alias(I\=I,j=sqrt(-1)):
assume(R>0,L>0,G>0,C>0,omega>0);
bdp := (-j)*sqrt((R+j*omega*L)*(G+j*omega*C));
```

$$bdp := -j\sqrt{(R+j\,\omega\,L)\,(G+j\,\omega\,C)}$$

```
a2_ := ((R^2+omega^2*L^2)*(G^2+omega^2*C^2))^(1/2);
```

$$a2_ := \sqrt{(R^2+\omega^2\,L^2)\,(G^2+\omega^2\,C^2)}$$

```
c_ := R*G - omega^2*L*C;
```

$$c_ := R\,G-\omega^2\,L\,C$$

```
bdp := factor(evalc(bdp));
```

$$bdp := \frac{1}{2}\sqrt{2\sqrt{(R^2+\omega^2\,L^2)\,(G^2+\omega^2\,C^2)}-2\,R\,G+2\,\omega^2\,L\,C}-\frac{1}{2}j\sqrt{2\sqrt{(R^2+\omega^2\,L^2)\,(G^2+\omega^2\,C^2)}+2\,R\,G-2\,\omega^2\,L\,C}$$

```
bdp :=subs(a2_ = a^2,bdp);
```

$$bdp := \frac{1}{2}\sqrt{2\,a^2-2\,R\,G+2\,\omega^2\,L\,C}-\frac{1}{2}j\sqrt{2\,a^2+2\,R\,G-2\,\omega^2\,L\,C}$$

```
bdp := subs(R*G = c + omega^2*L*C,bdp);
```

$$bdp := \frac{1}{2}\sqrt{2\,a^2-2\,c}-\frac{1}{2}j\sqrt{2\,a^2+2\,c}$$

```
factor(expand(a2_^2 - c_^2));   # to show that a^2 > |c|
```

$$\omega^2\,(R\,C+L\,G)^2$$

Fact 2: In the high frequency limit,

$$\mathrm{Re}(\beta_d') = \omega\sqrt{LC}$$

$$\mathrm{Im}(\beta_d') = -\frac{RC+GL}{2\sqrt{LC}} \qquad \text{(Q.5)}$$

Proof: At high ω write

$$a^2 \equiv (R^2+\omega^2L^2)^{1/2}\,(G^2+\omega^2C^2)^{1/2} = LC\,(R^2/L^2+\omega^2)^{1/2}(G^2/C^2+\omega^2)^{1/2}$$

$$= \omega^2LC\,(\,1 + R^2/[\omega^2L^2])^{1/2}(\,1 + G^2/[\omega^2C^2])^{1/2}$$

$$\approx \omega^2LC\,(\,1 + R^2/[2\omega^2L^2] +)\,(\,1 + G^2/[2\omega^2C^2]) \qquad \text{// } (1+x)^{1/2} = 1+x/2 \text{ twice}$$

$$\approx\ \omega^2LC\,[1\ + (1/2\omega^2)(R^2/L^2+G^2/C^2)\,]$$

$$a^2 + c = \omega^2LC\ \ [1\ + (1/2\omega^2)(R^2/L^2+G^2/C^2)\,] + RG - \omega^2LC$$

$$= \omega^2LC\ \ [(1/2\omega^2)(R^2/L^2+G^2/C^2)\,] + RG$$

$$= (1/2)LC(R^2/L^2+G^2/C^2) + RG$$

$$= (1/2)\,[LC(R^2/L^2+G^2/C^2) + 2RG] = (1/2LC)\,[L^2C^2(R^2/L^2+G^2/C^2) + 2RGLC]$$

$= (1/2LC)\,[R^2C^2+G^2L^2 + 2RGLC] \;\; = (1/2)\,[RC+GL]^2/LC$

$a^2 - c = \omega^2 LC\;[1 \; + (1/2\omega^2)(R^2/L^2+G^2/C^2)\,] - RG + \omega^2 LC \;\approx 2\omega^2\; LC$

Therefore

$$\mathrm{Re}(\beta'_d) \;=\; \sqrt{\frac{a^2-c}{2}} \;= \omega\sqrt{LC} \qquad\qquad \mathrm{Im}(\beta'_d) = -\sqrt{\frac{a^2+c}{2}} \;= -\;\frac{RC+GL}{2\sqrt{LC}}$$

QED

Note: The expansion $(1+x)^{1/2} = 1+x/2$ used in the proof is valid only if

$R^2/\omega^2L^2 \;<< 1$ and $G^2/\omega^2C^2 \;<<\; 1$

which requires that

$\omega \;>> (R/L)$ and $\omega \;>> (G/C)$

For any values of R,L,C,G we can thus find ω large enough to make our high ω limit valid.

Maple verification

```
a := ((R^2+omega^2*L^2)*(G^2+omega^2*C^2))^(1/4):
c := R*G - omega^2*L*C:
Re_bdp :=  sqrt((a^2-c)/2):
Im_bdp := -sqrt((a^2+c)/2):
limit(Re_bdp/omega,omega = infinity):simplify(%);
```

$$\sqrt{LC}$$

```
limit(Im_bdp,omega = infinity):simplify(%);
```

$$-\frac{1}{2}\frac{RC+GL}{\sqrt{L}\sqrt{C}}$$

Fact 3: In the low frequency limit,

$$\mathrm{Re}(\beta'_d) \;=\; (\omega/2)\frac{RC+GL}{\sqrt{RG}}$$

$$\mathrm{Im}(\beta'_d) \;= -\;\sqrt{RG}\,[\,1 + (\omega^2/8)\,(\frac{RC-GL}{RG})^2\;] \qquad\qquad (Q.6)$$

Proof:

$a^2 \equiv (R^2+\omega^2L^2)^{1/2}\,(G^2+\omega^2C^2)^{1/2} \;= RG(1+\omega^2L^2/R^2)^{1/2}(1+\omega^2C^2/G^2)^{1/2}$

$\approx\; RG\,(1+\omega^2L^2/2R^2)\,(1+\omega^2C^2/2G^2) \;\approx\; RG\,[1+(1/2)\omega^2(L^2/R^2+C^2/G^2)]$

$a^2 + c \;= RG\,[1+(1/2)\omega^2(L^2/R^2+C^2/G^2)] + RG - \omega^2LC$

$= [RG +(1/2)\omega^2(GL^2/R+RC^2/G)] + RG - \omega^2LC$

$= 2RG + (1/2)\omega^2(GL^2/R + RC^2/G - 2LC)$

$= 2RG + (1/2)\omega^2(G^2L^2 + R^2C^2 - 2LCRG)/RG$

$= 2RG + (1/2)\omega^2(RC-GL)^2/RG$

$$\frac{a^2+c}{2} = RG + (\omega/2)^2 (RC-GL)^2/RG = RG \left[1 + (\omega/2)^2 \left(\frac{RC-GL}{RG}\right)^2 \right]$$

$$\sqrt{\frac{a^2+c}{2}} = \sqrt{RG} \left[1 + (\omega/2)^2 \left(\frac{RC-GL}{RG}\right)^2 \right]^{1/2} = \sqrt{RG} \left[1 + (\omega^2/8) \left(\frac{RC-GL}{RG}\right)^2 \right]$$

$a^2 - c$ = same as $a^2 + c$ but the main RG's cancel and the cross term changes sign, so

$= (1/2)\omega^2(RC+GL)^2/RG$

Therefore for low ω we find,

$$\mathrm{Re}(\beta'_d) = \sqrt{\frac{a^2-c}{2}} = (\omega/2)\frac{RC+GL}{\sqrt{RG}}$$

$$\mathrm{Im}(\beta'_d) = -\sqrt{\frac{a^2+c}{2}} = -\sqrt{RG} \left[1 + (\omega^2/8) \left(\frac{RC-GL}{RG}\right)^2 \right]$$ QED

Note: The expansion $(1+x)^{1/2} = 1+x/2$ used in the proof is valid only if

$\omega^2L^2/R^2 << 1$ and $\omega^2C^2/G^2 << 1$

which requires that

$\omega << (R/L)$ and $\omega << (G/C)$

If R = 0 and/or G = 0, our proof fails since at least one of these inequalities cannot be met.

Maple verification, continuing the code shown above:

```
series(Re_bdp,omega,2): factor(%);
```

$$\frac{1}{2}\sqrt{\frac{(LG+RC)^2}{RG}}\,\omega + O(\omega^3)$$

```
series(Im_bdp,omega,3): factor(%);
```

$$-\sqrt{RG} - \frac{1}{8}\frac{\sqrt{RG}\,(-RC+LG)^2}{R^2G^2}\omega^2 + O(\omega^4)$$

Fact 4: In the low frequency limit with G = 0 ,

$$\mathrm{Re}(\beta'_d) = \sqrt{\omega RC/2}$$
$$\mathrm{Im}(\beta'_d) = -\sqrt{\omega RC/2}$$

Proof:

$$a^2 \equiv (R^2+\omega^2L^2)^{1/2}\,(\omega^2C^2)^{1/2} = \omega RC\,(1+\omega^2L^2/R^2)^{1/2}$$

$$\approx \omega RC(1+\omega^2L^2/2R^2)$$

$$a^2 + c = \omega RC(1+\omega^2L^2/2R^2) - \omega^2LC \approx \omega RC$$

$$a^2 - c = \omega RC(1+\omega^2L^2/2R^2) + \omega^2LC \approx \omega RC$$

Therefore

$$\mathrm{Re}(\beta'_d) = \sqrt{\frac{a^2-c}{2}} = \sqrt{\omega RC/2}$$
$$\mathrm{Im}(\beta'_d) = -\sqrt{\frac{a^2+c}{2}} = -\sqrt{\omega RC/2}$$ QED

Summary of results:

Summary: β_d' and its limits (Q.7)

$\beta_d' \equiv -j\sqrt{\mathbf{zy}} = -j\sqrt{(R+j\omega L)(G+j\omega C)}$ // definition of the wavenumber $\beta_d'(\omega)$

$$\beta_d' = \sqrt{\frac{a^2-c}{2}} - j\sqrt{\frac{a^2+c}{2}} \qquad \sqrt{\mathbf{zy}} = \sqrt{\frac{a^2+c}{2}} + j\sqrt{\frac{a^2-c}{2}}$$

$a \equiv (R^2+\omega^2L^2)^{1/4}\,(G^2+\omega^2C^2)^{1/4}$ $\quad \mathrm{Re}(\beta'_d) = \mathrm{Im}(\sqrt{\mathbf{zy}})$

$c \equiv RG - \omega^2LC$ $\quad \mathrm{Im}(\beta'_d) = -\mathrm{Re}(\sqrt{\mathbf{zy}})$

For large ω: $\mathrm{Re}(\beta'_d) = \omega\sqrt{LC}$

$$\mathrm{Im}(\beta'_d) = -\frac{RC+GL}{2\sqrt{LC}} \qquad \omega >> (R/L) \text{ and } \omega >> (G/C)$$

For small ω: $\mathrm{Re}(\beta'_d) = (\omega/2)\frac{RC+GL}{\sqrt{RG}}$

$$\mathrm{Im}(\beta'_d) = -\sqrt{RG}\,[\,1 + (\omega^2/8)\,(\frac{RC-GL}{RG})^2\,] \qquad \omega << (R/L) \text{ and } \omega << (G/C)$$

For small ω: $\mathrm{Re}(\beta'_d) = \sqrt{\omega RC/2}$

$\mathrm{Im}(\beta'_d) = -\sqrt{\omega RC/2}$ $\quad$ $G = 0$ and $\omega << (R/L)$