Fact 5: The real + imaginary decomposition of Z_0 is given by

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{\mathbf{z}}{\mathbf{y}}} = (a/\sqrt{2}\,)\,[\,\sqrt{1+\beta/\alpha}\; - j\sigma\sqrt{1-\beta/\alpha}\;]$$

where $\quad a = \left(\frac{R^2+\omega^2L^2}{G^2+\omega^2C^2}\right)^{1/4} = |Z_0| \qquad \alpha = \sqrt{(G^2+\omega^2C^2)(R^2+\omega^2L^2)}$

$\sigma = \text{sign}(RC-LG) \qquad \beta \equiv RG + \omega^2LC$

Proof of Fact 5: The proof is similar to the proof of Fact 1. Let

$$q = \frac{\mathbf{z}}{\mathbf{y}} = \frac{R+j\omega L}{G+j\omega C} = |q|\, e^{j\theta} \qquad -\pi < \theta < \pi \text{ (but see few lines below)}$$

Then

$$|q|^2 = \left|\frac{R+j\omega L}{G+j\omega C}\right|^2 = \frac{R^2+\omega^2L^2}{G^2+\omega^2C^2} \equiv a^4 \qquad \Rightarrow \qquad |q| = \sqrt{\frac{R^2+\omega^2L^2}{G^2+\omega^2C^2}} = a^2$$

Next,

$$q = \frac{R+j\omega L}{G+j\omega C} = \frac{R+j\omega L}{G+j\omega C}\;\frac{G-j\omega C}{G-j\omega C} = \frac{(R+j\omega L)(G-j\omega C)}{G^2+\omega^2C^2} = \frac{RG+j\omega(LG-RC)+\omega^2LC}{G^2+\omega^2C^2}$$

so

$$\text{Re}(q) = \frac{RG + \omega^2LC}{G^2+\omega^2C^2} > 0 \qquad \Rightarrow \qquad -\pi/2 < \theta < \pi/2$$

$$\text{Im}(q) = \frac{\omega(LG-RC)}{G^2+\omega^2C^2} \qquad \text{sign}[\text{Im}(q)] = \text{sign}(LG-RC) = \text{sign}(\theta) \equiv \Sigma \;.$$

Then

$$\cos\theta = \text{Re}(q)/|q| = \frac{(RG + \omega^2LC)}{G^2+\omega^2C^2}\sqrt{\frac{G^2+\omega^2C^2}{R^2+\omega^2L^2}} = \frac{RG + \omega^2LC}{\sqrt{(G^2+\omega^2C^2)(R^2+\omega^2L^2)}}$$

$$\equiv \beta/\alpha \qquad \text{where } \alpha \equiv \sqrt{(G^2+\omega^2C^2)(R^2+\omega^2L^2)} \text{ and } \beta \equiv RG + \omega^2LC$$

Now write

$$s \equiv \sqrt{\frac{\mathbf{z}}{\mathbf{y}}} = \sqrt{q} = |s|\, e^{i\varphi} \qquad |s| = \sqrt{|q|} = a = |Z_0| \qquad \varphi = \theta/2 \qquad -\pi/4 < \varphi < \pi/4$$

$$\text{Re}(s) = |s|\cos\varphi = a\cos(\theta/2) = a\sqrt{(1+\cos\theta)/2} = (a/\sqrt{2}\,)\sqrt{1+\beta/\alpha}$$
$$\text{Im}(s) = |s|\sin\varphi = a\sin(\theta/2) = \Sigma\, a\sqrt{(1-\cos\theta)/2} = \Sigma\,(a/\sqrt{2}\,)\sqrt{1-\beta/\alpha}$$

giving the result

$$s = (a/\sqrt{2})\,[\sqrt{1+\beta/\alpha} + j\Sigma\sqrt{1-\beta/\alpha}\,]$$

where $a = \left(\frac{R^2+\omega^2L^2}{G^2+\omega^2C^2}\right)^{1/4} = |Z_0|$ $\qquad \alpha = \sqrt{(G^2+\omega^2C^2)(R^2+\omega^2L^2)}$

$\Sigma = \text{sign}(LG-RC)$ $\qquad \beta \equiv RG + \omega^2LC$

Letting $\sigma = -\Sigma = \text{sign}(RC-LG)$ one gets the final form,

$$s = (a/\sqrt{2})\,[\sqrt{1+\beta/\alpha} - j\sigma\sqrt{1-\beta/\alpha}\,]$$

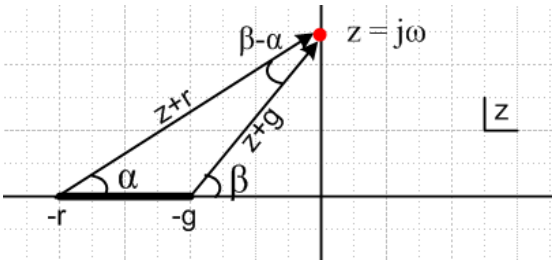
where $a = \left(\frac{R^2+\omega^2L^2}{G^2+\omega^2C^2}\right)^{1/4} = |Z_0|$ $\qquad \alpha = \sqrt{(G^2+\omega^2C^2)(R^2+\omega^2L^2)}$

$\sigma = \text{sign}(RC-LG)$ $\qquad \beta \equiv RG + \omega^2LC$

Our Maple verification of this result is a bit ugly so we omit the code.

An alternate geometric derivation giving the same results begins as follows:

$$Z_0 = \sqrt{\frac{R+zL}{G+zC}} = \sqrt{\frac{L}{C}}\sqrt{\frac{z+(R/L)}{z+(G/C)}} = \sqrt{\frac{L}{C}}\sqrt{\frac{z+r}{z+g}}$$ where $r = (R/L)$ and $g = (G/C)$.



The drawing shows the complex z plane for the function $Z_0(z) = \sqrt{\frac{L}{C}}\sqrt{\frac{z+r}{z+g}}$ in the particular case that r > g, in which case the z-plane has a branch cut from -r to -g. The z values of interest are only those on the positive imaginary axis where $z = j\omega$.

Reader Exercise: Finish this derivation and obtain the same results shown in **

Hint: $\cos(\beta-\alpha) = (rg+\omega^2)/(AB)$ and $\sin(\beta-\alpha) = \omega(r-g)/(AB)$ where $A = \sqrt{\omega^2+r^2}$ and $B = \sqrt{\omega^2+g^2}$.

Fact 6: In the high frequency limit,

$$\text{Re}(Z_0) \approx \sqrt{L/C} + (1/8\omega^2)\,(RC+3GL)(RC-GL) / (C^{5/2}L^{3/2})$$

$\mathrm{Im}(Z_0) \approx -(1/2\omega)(RC-GL)/(L^{1/2}C^{3/2})$ $\qquad\qquad$ $\omega >> R/L$ and $\omega >> G/C$

Proof of Fact 6:

We first enter in Maple the basic parameters, omitting the σ in $\mathrm{Im}(Z_0)$ just for the moment,

```
restart; alias(I=I,j=sqrt(-1)):
assume(R>0,L>0,G>0,C>0,omega>0):
ReZ0 := (a/sqrt(2))*sqrt(1+(beta/alpha)):
ImZ0 :=-signum(R*C-L*G)*(a/sqrt(2))*sqrt(1-(beta/alpha));
```

$$ImZ0 := -\frac{1}{2}\,\mathrm{signum}(R\,C - L\,G)\,a\,\sqrt{2}\,\sqrt{1-\frac{\beta}{\alpha}}$$

```
a := ((R^2+omega^2*L^2)/(G^2+omega^2*C^2))^(1/4);
```

$$a := \left(\frac{R^2+\omega^2 L^2}{G^2+\omega^2 C^2}\right)^{\left(\frac{1}{4}\right)}$$

```
alpha := ((R^2+omega^2*L^2)*(G^2+omega^2*C^2))^(1/2);
```

$$\alpha := \sqrt{(R^2+\omega^2 L^2)(G^2+\omega^2 C^2)}$$

```
beta := R*G + omega^2*L*C;
```

$$\beta := R\,G + \omega^2 L\,C$$

We then request an expansion of $\mathrm{Re}(Z_0)$ for large ω:

```
ReZ0:= series(ReZ0,omega=infinity,3);
```

$$ReZ0 := \left(\frac{L^2}{C^2}\right)^{\left(\frac{1}{4}\right)} + \frac{\left(\frac{L^2}{C^2}\right)^{\left(\frac{1}{4}\right)}\left(-\frac{1}{8}\frac{L^2G^2+R^2C^2}{L^2C^2}+\frac{1}{4}\frac{RG}{LC}\right)+\frac{1}{4}\frac{\left(\frac{L^2}{C^2}\right)^{\left(\frac{1}{4}\right)}\left(R^2-\frac{L^2G^2}{C^2}\right)}{L^2}}{\omega^2} + O\left(\frac{1}{\omega^4}\right)$$

The first term is the expected $\sqrt{L/C}$. The $1/\omega^2$ term can be simplified as follows:

```
term2 := op(2,ReZ0);
```

$$term2 := \frac{\left(\frac{L^2}{C^2}\right)^{\left(\frac{1}{4}\right)} \left(-\frac{1}{8}\frac{L^2 G^2 + R^2 C^2}{L^2 C^2} + \frac{1}{4}\frac{R G}{L C}\right) + \frac{1}{4}\frac{\left(\frac{L^2}{C^2}\right)^{\left(\frac{1}{4}\right)} \left(R^2 - \frac{L^2 G^2}{C^2}\right)}{L^2}}{\omega^2}$$

```
term2 := factor(simplify(term2));
```

$$term2 := -\frac{1}{8}\frac{(3GL + RC)(GL - RC)}{L^{\left(\frac{3}{2}\right)} C^{\left(\frac{5}{2}\right)} \omega^2}$$

and then the result is

$$\mathrm{Re}(Z_0) = \sqrt{L/C} + (1/8\omega^2)\,(RC+3GL)(RC-GL) / (C^{5/2}L^{3/2}) + \mathrm{order}(1/\omega^4)$$

For the imaginary part we do a similar set of steps:

```
ImZ0 := series(ImZ0,omega=infinity,3);
```

$$ImZ0 := -\frac{1}{4}\frac{\mathrm{signum}(RC - LG)\left(\frac{L^2}{C^2}\right)^{\left(\frac{1}{4}\right)} \sqrt{2}\sqrt{2\frac{L^2 G^2 + R^2 C^2}{L^2 C^2} - 4\frac{RG}{LC}}}{\omega} + O\left(\frac{1}{\omega^3}\right)$$

```
term1 := op(1,ImZ0);
```

$$term1 := -\frac{1}{4}\frac{\mathrm{signum}(RC - LG)\left(\frac{L^2}{C^2}\right)^{\left(\frac{1}{4}\right)} \sqrt{2}\sqrt{2\frac{L^2 G^2 + R^2 C^2}{L^2 C^2} - 4\frac{RG}{LC}}}{\omega}$$

```
term1 := factor(simplify(term1));
```

$$term1 := -\frac{1}{2}\frac{RC - LG}{C^{\left(\frac{3}{2}\right)} \sqrt{L}\,\omega}$$

with the result

$$\mathrm{Im}(Z_0) = -\,(1/2\omega)(RC-GL) / (L^{1/2}C^{3/2})$$

Fact 7: In the low frequency limit with G > 0,

$$\mathrm{Re}(Z_0) \approx \sqrt{R/G} - (\omega^2/8)\,(3RC+GL)(RC-GL) / (C^{5/2}L^{3/2})$$

$\mathrm{Im}(Z_0) \approx -(\omega/2)(RC-GL)/(R^{1/2}G^{3/2})$ $\omega << R/L$ and $\omega << G/C$

Proof of Fact 7:

The Maple expressions are entered as in Fact 6, and then:

```
series(ReZ0,omega=0,3):
simplify(%):
factor(%);
```

$$\frac{\sqrt{R}}{\sqrt{G}} - \frac{1}{8}\frac{(3RC+LG)(RC-LG)}{R^{\left(\frac{3}{2}\right)}\,G^{\left(\frac{5}{2}\right)}}\omega^2 + O(\omega^4)$$

```
series(ImZ0,omega=0,3):
simplify(%);
```

$$-\frac{1}{2}\frac{RC-LG}{\sqrt{R}\,G^{\left(\frac{3}{2}\right)}}\omega + O(\omega^3)$$

so that

$\mathrm{Re}(Z_0) = \sqrt{R/G} - (\omega^2/8)(3RC+GL)(RC-GL)/(C^{5/2}L^{3/2}) + \mathrm{order}(\omega^4)$

$\mathrm{Im}(Z_0) = -(\omega/2)(RC-GL)/(R^{1/2}G^{3/2}) + \mathrm{order}(\omega^3)$

Fact 8: In the low frequency limit with G = 0, both components diverge as:

$\mathrm{Re}(Z_0) \approx (1/\sqrt{2})\sqrt{R/C}\ 1/\sqrt{\omega}$

$\mathrm{Im}(Z_0) \approx -(1/\sqrt{2})\sqrt{R/C}\ 1/\sqrt{\omega}$

Proof of Fact 8:

We first enter the general forms, setting G = 0,

```
assume(R>0,L>0,C>0,omega>0):
ReZ0 := (a/sqrt(2))*sqrt(1+(beta/alpha)):
ImZ0 :=-(a/sqrt(2))*sqrt(1-(beta/alpha));
```

$$ImZ0 := -\frac{1}{2} a \sqrt{2} \sqrt{1-\frac{\beta}{\alpha}}$$

```
a := ((R^2+omega^2*L^2)/(omega^2*C^2))^(1/4);
```

$$a := \left(\frac{R^2+\omega^2 L^2}{\omega^2 C^2}\right)^{\left(\frac{1}{4}\right)}$$

```
alpha := ((R^2+omega^2*L^2)*(omega^2*C^2))^(1/2);
```

$$\alpha := \sqrt{(R^2+\omega^2 L^2)\,\omega^2 C^2}$$

```
beta := omega^2*L*C;
```

$$\beta := \omega^2 L C$$

The expansions are then,

```
series(ReZ0,omega=0,2);
```

$$\frac{1}{2}\frac{\left(\frac{R^2}{C^2}\right)^{\left(\frac{1}{4}\right)}\sqrt{2}}{\sqrt{\omega}} + \frac{1}{4}\frac{\left(\frac{R^2}{C^2}\right)^{\left(\frac{1}{4}\right)}\sqrt{2}\,L\sqrt{\omega}}{R} + \frac{1}{16}\frac{\left(\frac{R^2}{C^2}\right)^{\left(\frac{1}{4}\right)}L^2\sqrt{2}\,\omega^{\left(\frac{3}{2}\right)}}{R^2} + O\left(\omega^{\left(\frac{5}{2}\right)}\right)$$

```
series(ImZ0,omega=0,3);
```

$$-\frac{1}{2}\frac{\left(\frac{R^2}{C^2}\right)^{\left(\frac{1}{4}\right)}\sqrt{2}}{\sqrt{\omega}} + \frac{1}{4}\frac{\left(\frac{R^2}{C^2}\right)^{\left(\frac{1}{4}\right)}\sqrt{2}\,L\sqrt{\omega}}{R} - \frac{1}{16}\frac{\left(\frac{R^2}{C^2}\right)^{\left(\frac{1}{4}\right)}L^2\sqrt{2}\,\omega^{\left(\frac{3}{2}\right)}}{R^2} + O\left(\omega^{\left(\frac{5}{2}\right)}\right)$$

with the final results,

$$\mathrm{Re}(Z_0) = (1/\sqrt{2})\sqrt{R/C}\,\frac{1}{\sqrt{\omega}}$$

$$\mathrm{Im}(Z_0) = -(1/\sqrt{2})\sqrt{R/C}\,\frac{1}{\sqrt{\omega}}\ .$$

These results are obvious without using Maple,

$$Z_0 = \sqrt{\frac{R}{j\omega C}} = \sqrt{\frac{R}{\omega C}}\,(j)^{-1/2} = \sqrt{\frac{R}{\omega C}}\,e^{-j\pi/4} = \sqrt{\frac{R}{\omega C}}\,[\,1-j]/\sqrt{2}\,,$$

but the expansions provide more terms if needed.