New approach to the functions k and Z_0 PhL 7.4.14

Let jω = z, a complex variable. Then we have

$$Z_0 = \sqrt{\frac{R + zL}{G + zC}} = \sqrt{\frac{L}{C}}\sqrt{\frac{z + (R/L)}{z + (G/C)}} = \sqrt{\frac{L}{C}}\sqrt{\frac{z + r}{z + g}} \quad \text{where } r = (R/L) \text{ and } g = (G/C)$$

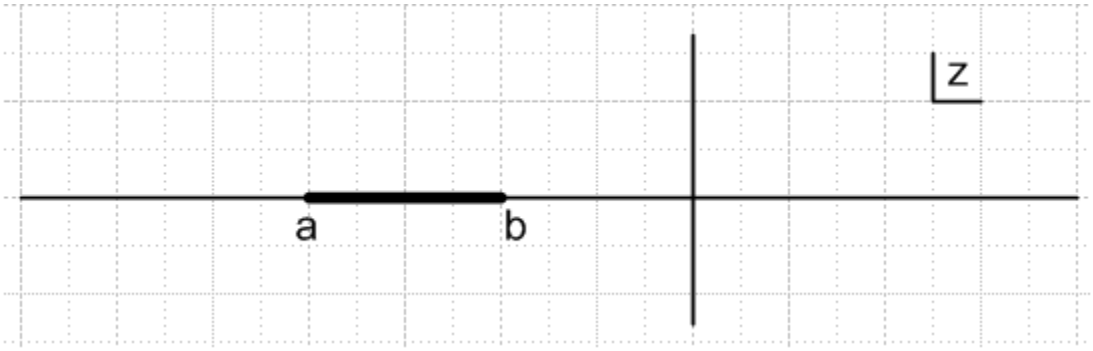
Forget the overall constant and just consider then

$$f(z) = \sqrt{\frac{z + r}{z + g}} = \sqrt{\frac{z - (-r)}{z - (-g)}}$$

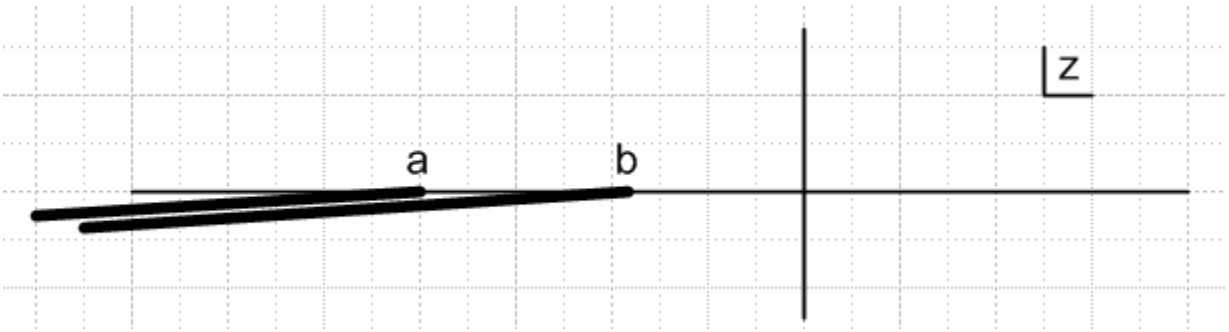
This has branch points at z = -r and z = -g. Let

a = min(-r,-g)
b = max(-r,-g)

Here is the situation, where I draw the cuts so that f(z) is real and positive on the positive z axis and also on the distance negative axis.



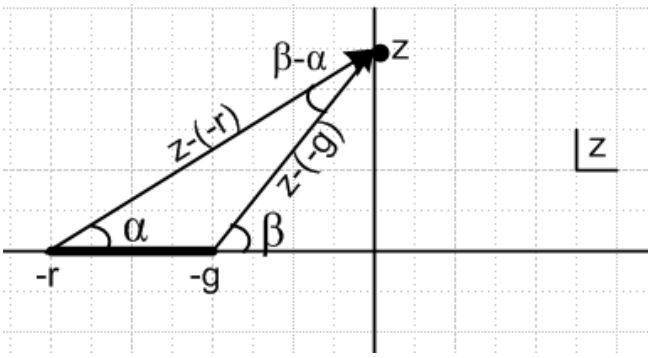
This is one of several ways one could draw these cuts. Another is this



But these are really the same since there won't be a discontinuity on the far left.

We are interested in z lying on the imaginary axis where ω is real! Thus, only the upper half plane of z is of interest to us!!! Also, the angles have these restricted ranges

$0 < \alpha < \pi/2$
$0 < \beta < \pi/2$



The angle β-α starts at 0 when z = 0, then as you go up the z axis, this angle increases to some max angle, then it decreases to 0 again. I think the max angle is less than π/2 and perhaps smaller.

Fact: β-α ranges from 0 to π/2 at most.

Now we can say (for now I assume that r > g for the moment)

$$\frac{z+r}{z+g} = |z+r|\, e^{j\alpha} \;/\; |z+g|\, e^{j\beta} = \left|\frac{z+r}{z+g}\right|\, e^{j(\alpha-\beta)}$$

$$f(z) = \left|\frac{z+r}{z+g}\right|^{1/2} e^{j(\alpha-\beta)/2}$$

$$= \left|\frac{z+r}{z+g}\right|^{1/2} \left(\cos[(\alpha-\beta)/2] + j \sin[(\alpha-\beta)/2] \right)$$

$$= \left|\frac{z+r}{z+g}\right|^{1/2} \left(\cos[(\beta-\alpha)/2] - j \sin[(\beta-\alpha)/2] \right)$$

$$= \left|\frac{z+r}{z+g}\right|^{1/2} \left(\cos(\theta/2) - j \sin(\theta/2) \right) \qquad \theta \equiv \beta-\alpha$$

Range of θ: First, I can see that this is true

α ranges	0 to π/2		
β ranges	0 to π/2		
β-α ranges	0 to π/2	θ	
(β-α)/2 ranges	0 to π/4	θ/2	
θ ranges	0 to π/2		
θ/2 ranges	0 to π/4		sin(θ/2) >0
			cos(θ/2) >0

I can see that

$\sin\alpha = \omega / |z+r|$
$\sin\beta = \omega / |z+g|$

$\cos\alpha = r / |z+r|$
$\cos\beta = g / |z+g|$

Now maybe define

$A = |z+r|$
$B = |z+g|$

Then we have

$\sin\alpha = \omega / A$
$\sin\beta = \omega / B$

$\cos\alpha = r / A$
$\cos\beta = g / B$

$\cos(\alpha-\beta) = \cos\alpha\cos\beta+\sin\alpha\sin\beta = (r/A)(g/B) + \omega^2/(AB) = (rg + \omega^2)/(AB) = \cos\theta$
$\sin(\alpha-\beta) = \sin\alpha\cos\beta-\sin\beta\cos\alpha = (\omega/A)(g/B) - (\omega/B)(r/A) = \omega(g-r)/(AB) = -\sin\theta$

Define $\theta = \beta-\alpha$ so then

$\cos\theta = (rg + \omega^2)/(AB)$
$\sin\theta = \omega(r-g)/(AB)$

Then

$f(z) = \sqrt{A/B}\,[\,\cos(\theta/2) - j\sin(\theta/2)\,]$

$\cos(\theta/2) = \sqrt{1+\cos\theta}\,/\sqrt{2} = \sqrt{1+(rg + \omega^2)/(AB)}\,/\sqrt{2}$
$\sin(\theta/2) = \sqrt{1-\cos\theta}\,/\sqrt{2} = \sqrt{1-(rg + \omega^2)/(AB)}\,/\sqrt{2}$

So reinstall the leading factor to get

$$Z_0 = (1/\sqrt{2}\,)\sqrt{\frac{L}{C}}\;\sqrt{A/B}\;\left[\;\sqrt{1+(rg + \omega^2)/(AB)}\;-j\sqrt{1-(rg + \omega^2)/(AB)}\;\right]$$

$A = \sqrt{\omega^2+r^2}$ $B = \sqrt{\omega^2+g^2}$

How does this compare to my previous calculation? Results there were

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{z}{y}} = (a/\sqrt{2})\,[\sqrt{1+\beta/\alpha} + j\sigma\sqrt{1-\beta/\alpha}\,]$$

where $a = \left(\frac{R^2+\omega^2L^2}{G^2+\omega^2C^2}\right)^{1/4} = |Z_0|$ $\quad \alpha = \sqrt{(G^2+\omega^2C^2)(R^2+\omega^2L^2)}$

$\sigma = \text{sign}(LG-RC)$ $\quad \beta \equiv RG + \omega^2LC$

Let's operate on the original boxed results a big

$$\text{sign}(r-g) = \text{sign}((R/L)-(G/C)) = \text{sign}\left(\frac{R}{L} - \frac{G}{C}\right) = \text{sign}\,[(RC-LG)/LC] = \text{sign}(RC-LG) = -\sigma$$

Since I assumed r > g, I must have sign(r-g) = 1 and thus σ = -1.

Next,

$$a = \left(\frac{R^2+\omega^2L^2}{G^2+\omega^2C^2}\right)^{1/4} = \sqrt{L/C}\,\left(\frac{r^2+\omega^2}{g^2+\omega^2}\right)^{1/4} = \sqrt{L/C}\,\sqrt{A/B}$$

$$\alpha = \sqrt{(G^2+\omega^2C^2)(R^2+\omega^2L^2)} = LC\sqrt{(g^2+\omega^2)(r^2+\omega^2)} = LCAB$$

$$\beta \equiv (RG + \omega^2LC) = LC(\omega^2 + (R/L)(G/C)) = LC(\omega^2 + rg)$$

$$\beta/\alpha = LC(\omega^2 + rg)/\,[LC\sqrt{(g^2+\omega^2)(r^2+\omega^2)}\,] = (\omega^2 + rg)\,/\,\sqrt{(g^2+\omega^2)(r^2+\omega^2)}$$

$$1 + \beta/\alpha = 1 + (\omega^2 + rg)\,/\,\sqrt{(g^2+\omega^2)(r^2+\omega^2)} = 1 + (\omega^2 + rg)/(AB)$$

So here is a translation of my *previous* result:

$$Z_0 = (1/\sqrt{2})\left(\frac{R^2+\omega^2L^2}{G^2+\omega^2C^2}\right)^{1/4}[\sqrt{1+\beta/\alpha} - j\sqrt{1-\beta/\alpha}\,]$$

and the radicals agree, so I get the same result the new way as the old way.

The results agree including sign of imaginary part.

So far I have assumed r > g. What happens the other way? The angle β-α will change sign. Then sin(θ/2) changes to negative, cos(θ/2) stays the same sign. The result will then have the opposite sign imaginary part.