This was installed on 9/24/14.

Appendix S: Details of the Chapter 4 Averaging Procedure

Here we redo the averaging process outlined in Section 4.11 (b) in more detail, *and* we maintain the transverse derivatives of the vector potential which were neglected in that Section. Since the process is explained there, here we just show what happens to the various equations. Corresponding equation numbers from Chapter 4 are shown in italics. The function arguments suppressed in Chapter 4 for points $\mathbf{x}_1$ and $\mathbf{x}_2$ are shown in red as $(\mathbf{x}_1,\mathbf{x}_2)$. New terms arising from the previously neglected transverse vector potential derivatives are shown in blue.

The potential difference between two conductors in the transmission line limit is given by

$$\begin{aligned} V(\mathbf{x}_1,\mathbf{x}_2) &\equiv \varphi_{12}(\mathbf{x}_1) - \varphi_{12}(\mathbf{x}_2) \\ &= \frac{1}{4\pi\xi_d}\, q(z) \int_{-\infty}^{\infty} dz' \left\{ \int_{C_1} dx_1'\, dy_1'\, \alpha_1(x_1',y_1') \frac{1}{R_{11}} - \int_{C_2} dx_2'\, dy_2'\, \alpha_2(x_2',y_2') \frac{1}{R_{12}} \right\} \\ &- \frac{1}{4\pi\xi_d}\, q(z) \int_{-\infty}^{\infty} dz' \left\{ \int_{C_1} dx_1'\, dy_1'\, \alpha_1(x_1',y_1') \frac{1}{R_{21}} - \int_{C_2} dx_2'\, dy_2'\, \alpha_2(x_2',y_2') \frac{1}{R_{22}} \right\} \end{aligned} \qquad \textit{(4.4.1)} \quad (S.1)$$

and later

$$V(\mathbf{x}_1,\mathbf{x}_2) = q(z) \frac{1}{4\pi\xi_d} \left\{ \int_{C_1} dx_1'\, dy_1'\, \alpha_1(x_1',y_1') \ln(s_{21}^2/s_{11}^2) - \int_{C_2} dx_2'\, dy_2'\, \alpha_2(x_2',y_2') \ln(s_{22}^2/s_{12}^2) \right\}$$

$$s_{21}^2 = (x_2-x_1')^2 + (y_2-y_1')^2 \qquad s_{22}^2 = (x_2-x_2')^2 + (y_2-y_2')^2 \qquad \textit{(4.4.6)}$$

$$s_{11}^2 = (x_1-x_1')^2 + (y_1-y_1')^2 \qquad s_{12}^2 = (x_1-x_2')^2 + (y_1-y_2')^2 \ . \qquad (S.2)$$

Then

$$\frac{1}{C'(\mathbf{x}_1,\mathbf{x}_2)} = \frac{V(\mathbf{x}_1,\mathbf{x}_2)}{q(z)} \qquad (S.3)$$

$$= \frac{1}{4\pi\xi_d} \left\{ \int_{C_1} dx_1'\, dy_1'\, \alpha_1(x_1',y_1') \ln(s_{21}^2/s_{11}^2) - \int_{C_2} dx_2'\, dy_2'\, \alpha_2(x_2',y_2') \ln(s_{22}^2/s_{12}^2) \right\}$$

$$= \frac{1}{4\pi\xi_d} K(\mathbf{x}_1,\mathbf{x}_2) \text{ and } \quad V(\mathbf{x}_1,\mathbf{x}_2) = q(z) \frac{1}{4\pi\xi_d} K(\mathbf{x}_1,\mathbf{x}_2) \qquad \textit{(4.4.7)}$$

where

$$K(\mathbf{x}_1,\mathbf{x}_2) = \frac{1}{4\pi\xi_d} \left\{ \int_{C_1} dx_1'\, dy_1'\, \alpha_1(x_1',y_1') \ln(s_{21}^2/s_{11}^2) - \int_{C_2} dx_2'\, dy_2'\, \alpha_2(x_2',y_2') \ln(s_{22}^2/s_{12}^2) \right\} \qquad (S.4)$$

Continuing along,

$$L_e(\mathbf{x}_1,\mathbf{x}_2) = (\mu_d/4\pi) K(\mathbf{x}_1,\mathbf{x}_2) \ . \qquad \textit{(4.4.11)} \qquad (S.5)$$

$$Z_0(\mathbf{x}_1,\mathbf{x}_2) = (1/4\pi)\, K(\mathbf{x}_1,\mathbf{x}_2) \sqrt{\mu_d/\varepsilon_d} \qquad \textit{(4.4.14)} \qquad (S.6)$$

We them move from the V section to the W section which is Section 4.10 :

$$W(\mathbf{x}_1,\mathbf{x}_2) = i(z)\,\frac{\mu_d}{4\pi}\,\{\int_{C_1} dx_1'\,dy_1'\,b_1(x_1',y_1')\,\ln(s_{21}^2/s_{11}^2) - \int_{C_2} dx_2'\,dy_2'\,b_2(x_2',y_2')\,\ln(s_{22}^2/s_{12}^2)\}$$

(4.10.4) (S.7)

$$W(\mathbf{x}_1,\mathbf{x}_2) = L_e(\mathbf{x}_1,\mathbf{x}_2)\; i(z)\;. \qquad \textit{(4.10.7)} \qquad \text{(S.8)}$$

$$L_e(\mathbf{x}_1,\mathbf{x}_2) = \frac{W(\mathbf{x}_1,\mathbf{x}_2)}{i(z)} = \frac{\mu_d}{4\pi}\,K_L(\mathbf{x}_1,\mathbf{x}_2) \qquad \textit{(4.10.8)} \qquad \text{(S.9)}$$

where K_L is a dimensionless function we can compare to K,

$$K_L(\mathbf{x}_1,\mathbf{x}_2) \equiv \int_{C_1} dx_1'\,dy_1'\,b_1(x_1',y_1')\,\ln(s_{21}^2/s_{11}^2) - \int_{C_2} dx_2'\,dy_2'\,b_2(x_2',y_2')\,\ln(s_{22}^2/s_{12}^2)\;. \textit{(4.10.9)}$$

(S.10)

$$K(\mathbf{x}_1,\mathbf{x}_2) \equiv \int_{C_1} dx_1'\,dy_1'\,\alpha_1(x_1',y_1')\,\ln(s_{21}^2/s_{11}^2) - \int_{C_2} dx_2'\,dy_2'\,\alpha_2(x_2',y_2')\,\ln(s_{22}^2/s_{12}^2)\;. \textit{(4.4.8)}$$

Moving along to Section 4.12 (a) we find (evaluated at some point **x**)

$$\mathbf{E} = -\,\text{grad}\,\varphi - \partial_t\mathbf{A} \qquad \textit{(1.3.1)}$$

$$\text{div}\,\mathbf{A} = -\,\mu_d\varepsilon_d\,\partial_t\varphi - \mu\sigma\varphi\;. \qquad \text{// the King gauge} \qquad \textit{(1.3.18)} \qquad \text{(S.11)}$$

Both the above equations are exact and can be rewritten as:

$$E_z(\mathbf{x}) = -\,\partial_z\varphi(\mathbf{x}) - j\omega A_z(\mathbf{x})$$

$$\partial_z A_z(\mathbf{x}) + (\partial_x A_x + \partial_y A_y) = -\,j\,(\beta_d^2/\omega)\,\varphi(\mathbf{x})\;. \qquad \textit{(4.12.3)} \qquad \text{(S.12)}$$

where the blue term $(\partial_x A_x + \partial_y A_y)$ was assumed to vanish in Chapter 4, but now we maintain it in what follows, continuing through the development of Section 4.12. Taking into account both conductors gives

$$E_z(\mathbf{x}) = -\,\partial_z\varphi_{12}(\mathbf{x}) - j\omega A_{z12}(\mathbf{x}) \qquad \textit{(4.12.4a)}$$

$$\partial_z A_{z12}(\mathbf{x}) + (\partial_x A_{x12}(\mathbf{x}) + \partial_y A_{y12}(\mathbf{x})) = -\,j\,(\beta_d^2/\omega)\varphi_{12}(\mathbf{x})\;. \qquad \textit{(4.12.4b)} \qquad \text{(S.13)}$$

Then evaluate at $\mathbf{x}_1$ and $\mathbf{x}_2$ and subtract to get

$$E_z(\mathbf{x}_1) - E_z(\mathbf{x}_2) = -\partial_z[\varphi_{12}(\mathbf{x}_1) - \varphi_{12}(\mathbf{x}_2)] - j\omega[A_{z12}(\mathbf{x}_1) - A_{z12}(\mathbf{x}_2)] \qquad \textit{(4.12.5a)}$$

$$\partial_z[A_{z12}(\mathbf{x}_1) - A_{z12}(\mathbf{x}_2)] + (\partial_x A_{x12}(\mathbf{x}_1) - \partial_x A_{x12}(\mathbf{x}_2)) + (\partial_y A_{y12}(\mathbf{x}_1) - \partial_y A_{y12}(\mathbf{x}_2))$$

$$= -\,j\,(\beta^2/\omega)[\varphi_{12}(\mathbf{x}_1) - \varphi_{12}(\mathbf{x}_2)]\;. \qquad \textit{(4.12.5b)} \qquad \text{(S.14)}$$

We then define

$$V(x_1,x_2) \equiv \varphi_{12}(x_1) - \varphi_{12}(x_2) \qquad (4.4.1)$$
$$W(x_1,x_2) \equiv A_{z12}(x_1) - A_{z12}(x_2) \qquad (4.10.1) \qquad \text{(S.15)}$$

and rewrite the previous equation pair as

$$E_z(x_1) - E_z(x_2) = -\partial_z V(x_1,x_2) - j\omega W(x_1,x_2) \qquad (4.12.6a) \qquad \text{(S.16)}$$
$$\partial_z W(x_1,x_2) = (\partial_x A_{x12}(x_1) - \partial_x A_{x12}(x_2)) + (\partial_y A_{y12}(x_1) - \partial_y A_{y12}(x_2)) - j\,(\beta_d^2/\omega)\, V(x_1,x_2) \qquad (4.12.6b)$$

Then define function T to represent the Transverse derivatives.

$$T(x_1,x_2) \equiv (\partial_x A_{x12}(x_1) - \partial_x A_{x12}(x_2)) + (\partial_y A_{y12}(x_1) - \partial_y A_{y12}(x_2)) \quad . \quad \text{// dim(T) = tesla} \qquad \text{(S.17)}$$

We can then "double average" V,W and T as demonstrated in (4.12.13) to obtain from (S.16),

$$\partial_z W(z) = T(z) - j\,(\beta_d^2/\omega)\, V(z) \qquad \text{(S.18)}$$

where

$$W(z) \equiv \langle W(x_1,x_2)\rangle_{c1,c2} \qquad (4.12.7)$$
$$V(z) \equiv \langle V(x_1,x_2)\rangle_{c1,c2} \qquad (4.12.7)$$
$$T(z) \equiv \langle T(x_1,x_2)\rangle_{c1,c2} \quad . \qquad \text{(S.19)}$$

Equation (4.12.6a) from (S.16) is then double-averaged to give

$$[E_{z1}(z) - E_{z2}(z)] = -\partial_z V(z) - j\omega\, W(z) \qquad (4.12.6a) \qquad \text{(S.20)}$$

where for example,

$$E_{z1}(z) \equiv \langle E_{z1}(x_1)\rangle_{c1,c2} = (1/P_1) \int_{c1} ds_1\, (1/P_2) \int_{c2} ds_2\, E_{z1}(x_1) = (1/P_1) \int_{c1} ds_1\, E_{z1}(x_1)$$
$$= \langle E_{z1}(x_1)\rangle_{c1} \quad . \qquad \text{(S.21)}$$

Then $E_{z1}(x_1) = Z_{s1}(x_1)\, i_1(z)$ gets double-averaged in the same way to define Z_{s1}. At this point we have

$$[Z_{s1} + Z_{s2}]\, i(z) = -\partial_z V(z) - j\omega\, W(z)$$
$$\partial_z W(z) = -j\,(\beta_d^2/\omega)\, V(z) + T(z) \qquad (4.12.11) \qquad \text{(S.22)}$$

where T(z) was not present in (4.12.11). From (S.9) we write the L_e equation then double-average it,

$$L_e(x_1,x_2) = \frac{W(x_1,x_2)}{i(z)} \qquad \rightarrow \qquad L_e = \frac{W(z)}{i(z)} \quad . \qquad (4.10.8) \qquad \text{(S.23)}$$

Since then $W(z) = L_e\, i(z)$, the equation pair (S.22) maybe be rewritten,

$$[Z_{s1} + Z_{s2}]\, i(z) = -\partial_z V(z) - j\omega W(z)$$
$$L_e \partial_z i(z) = -j\,(\beta_d^2/\omega)\, V(z) + T(z) \qquad \text{(S.24)}$$

or

$$\partial_z V(z) = -[\, Z_{s1} + Z_{s2} + j\omega L_e]\, i(z)$$
$$\partial_z\, i(z) = -[\, j\beta_d^2/(\omega L_e)]\, V(z) + T(z)/L_e \qquad \textit{(4.12.14)} \qquad \text{(S.25)}$$

These are the classical **transmission line equations**, usually written as

$$\frac{dV(z)}{dz} = -\mathbf{z}\, i(z) \qquad \frac{di(z)}{dz} = -\mathbf{y}\, V(z) + T(z)/L_e \qquad \textit{(4.12.15)} \qquad \text{(S.26)}$$

where

$$\mathbf{z} = Z_{s1} + Z_{s2} + j\omega L_e = R + j\omega L \qquad \text{// } \mathbf{z} \text{ and R are ohms/m} \qquad \text{(S.27)}$$
$$\mathbf{y} = j\beta_d^2/(\omega L_e) = G + j\omega C = j\omega C' . \qquad \text{// } \mathbf{y} \text{ and G are mhos/m} \qquad \textit{(4.12.16)} \qquad \text{(S.28)}$$

Applying ∂_z to the transmission line equations (S.26) then re-using them results in the following second order transmission line equations:

$$\frac{d^2V(z)}{dz^2} - \mathbf{zy}\, V(z) = (-\mathbf{z}/L_e)T(z) \qquad \frac{d^2i(z)}{dz^2} - \mathbf{zy}\, i(z) = (1/L_e)\,\partial_z T(z) \ . \qquad \textit{(4.12.17)} \qquad \text{(S.29)}$$

which are now inhomogeneous differential equations due to $T(z) \neq 0$.

In the **z** equation (S.27) both R and L can be represented as double averages,

$$R = <\mathrm{Re}(\mathbf{z})>_{c1,c2} = <\mathrm{Re}[Z_{s1}(\mathbf{x_1}) + Z_{s2}(\mathbf{x_2}) + j\omega L_e(\mathbf{x_1},\mathbf{x_2})]>_{c1,c2}$$
$$j\omega L = <\mathrm{Im}(\mathbf{z})>_{c1,c2} = <\mathrm{Im}[Z_{s1}(\mathbf{x_1}) + Z_{s2}(\mathbf{x_2}) + j\omega L_e(\mathbf{x_1},\mathbf{x_2})]>_{c1,c2} \ . \qquad \text{(S.30)}$$

However, in the **y** equation (S.28) this cannot be done because the number $L_e \equiv <L_e(\mathbf{x_1},\mathbf{x_2})>_{c1,c2}$ is in the denominator instead of the numerator. Thus we must regard the numbers G,C and C' in (S.28) as being defined by the quantity $j\beta_d^2/(\omega L_e)$. However, we can invert both sides of the **y** equation to get

$$1/\mathbf{y} = \omega L_e/(j\beta_d^2) = (1/C')/(j\omega) \qquad \text{(S.31)}$$

which can be interpreted as

$$\omega/(j\beta_d^2) * <L_e(\mathbf{x_1},\mathbf{x_2})>_{c1,c2} = (1/j\omega) * \left<\frac{1}{C'(\mathbf{x_1},\mathbf{x_2})}\right>_{c1,c2} \qquad \text{(S.32)}$$

or

$$\omega L_e/(j\beta_d^2) = \left<\frac{1}{C'}\right>/(j\omega) \qquad \text{(S.33)}$$

or

$$L_e \left<\frac{1}{C'}\right>^{-1} = (\beta_d^2/\omega^2) = \mu_d \xi_d \ . \qquad \text{// using (1.5.1a)} \qquad \textit{(4.12.19)} \qquad \text{(S.34)}$$

Comparing (S.33) with (S.31) shows that the number C' appearing in (S.31) and (S.28) can be interpreted in this manner,

$$C' = \left\langle \frac{1}{C'} \right\rangle^{-1} . \qquad \text{(S.35)}$$

Going back a bit, we had

$$\frac{1}{C'(\mathbf{x_1},\mathbf{x_2})} = V(\mathbf{x_1},\mathbf{x_2})/q(z) = \frac{1}{4\pi\xi_d} K(\mathbf{x_1},\mathbf{x_2}) \qquad \textit{(4.4.7)} \qquad \textit{(S.3)}$$

$$L_e(\mathbf{x_1},\mathbf{x_2}) = \frac{W(\mathbf{x_1},\mathbf{x_2})}{i(z)} = \frac{\mu_d}{4\pi} K_L(\mathbf{x_1},\mathbf{x_2}) \qquad \textit{(4.10.8)} \qquad \textit{(S.9)}$$

which can be double averaged to get

$$\left\langle \frac{1}{C'} \right\rangle = V(z)/q(z) = \frac{1}{4\pi\xi_d} K \qquad \text{(S.36)}$$

$$L_e = \frac{W(z)}{i(z)} = \frac{\mu_d}{4\pi} K_L . \qquad \text{(S.37)}$$

Inserting these last expressions into (S.34) gives,

$$L_e \left\langle \frac{1}{C'} \right\rangle^{-1} = \mu_d \xi_d$$

or

$$\left[\frac{\mu_d}{4\pi} K_L \right] \left[4\pi\xi_d / K \right] = \mu_d \xi_d$$

or

$$K_L = K . \qquad \textit{(4.12.20)} \qquad \text{(S.38)}$$

This equality has thus survived the double averaging procedure.