The King Boundary Conditions PhL 10.20.13

I eventually derived all four of the King boundary conditions quoted below, including the ξ one, but at the time of writing this doc, things were very confused. As part of this effort, I scoured sections of Jackson looking for hints, as detailed below. Then while browsing all my and Jim's E&M books I just stumbled onto Jim's old Panofsky book and found what appeared to be their method of dealing with my King Problem #2 which was no **J** in the **A** wave equation (and where I just made up "applied currents") . They have this poorly explained **j'** sitting there (quoted below). This gave me some reinforcement, and eventually led to my resolution of the King Problem #2!

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1. Introduction

King shows these boundary conditions,

$$\epsilon_0(\hat{n}_1,\mathbf{E}_1) + \xi_2(\hat{n}_2,\mathbf{E}_2) = -\bar{n}_1 \qquad (32a)$$

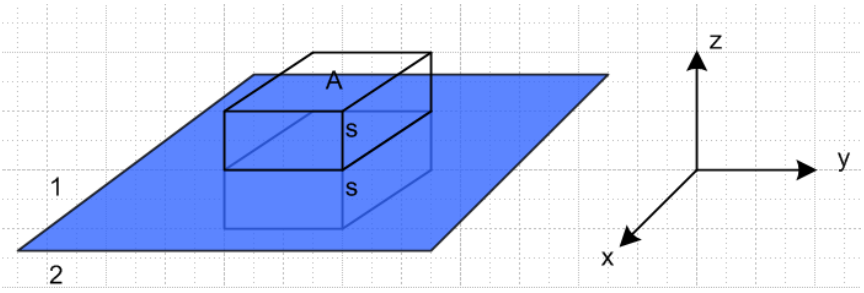
$$[\hat{n}_1,\mathbf{E}_1] + [\hat{n}_2,\mathbf{E}_2] = 0 \qquad (32b)$$

$$\nu_0[\hat{n}_1,\mathbf{B}_1] + \nu_2[\hat{n}_2,\mathbf{B}_2] = -\overline{\mathbf{n}_m\mathbf{v}_1} \qquad (32c)$$

$$(\hat{n}_1,\mathbf{B}_1) + (\hat{n}_2,\mathbf{B}_2) = 0 \qquad (32d)$$

My job is to figure out how he obtains these results. In lines after (1.1.12) I have a derivation of my version of the first (32a) above, and it contains no ξ. Here is my derivation:

Wee put a tiny "Gaussian box" straddling the two media. Area A and height 2s are both very small so the fields are the same at all points in the box.



For the electric displacement **D** we consider

$$\text{div } \mathbf{D} = \rho \qquad \Leftrightarrow \qquad \int_V \rho \, dV = \int_S \mathbf{D} \bullet d\mathbf{A} \qquad (1.1.13)$$

where volume V is the box shown. The surface integral is

$$\int_S \mathbf{D} \bullet d\mathbf{A} = D_z^{(1)} A - D_z^{(2)} A + \text{contributions from the sides of the box}$$

The contributions from the pairs of box sides cancel since the fields are assumed constant over the box on either side of the boundary. Assuming the only charge density is a surface charge density n on the boundary between the two media, the volume integral is nA and then the conclusion, generalized to the perpendicular field component, is

$$[D_\perp^{(1)} - D_\perp^{(2)}] = n \qquad \text{or} \qquad [\varepsilon_1 E_\perp^{(1)} - \varepsilon_2 E_\perp^{(2)}] = n \qquad (1.1.22)$$

The only way I can get his result is if I use $\mathbf{D} = \xi \mathbf{E}$. This would be a major change in thinking and requires going back to relearn the notion of D. There is just no other way! [Wrong! See King meets Stak. Yes the result (1.1.22) is correct as stated. But it is *also* true that $\xi_1 E_\perp^{(1)} = \xi_2 E_\perp^{(2)}$, and THIS is the equation you see in King above as (32a).]

2. Review of Jackson Sections which Seem Relevant

Due diligence!

Jackson 1Ed pp 103-108 talks about averaging fields in the dielectric world. His page 107 drawing shows the origin of the extra "ρ" which appears in (4.34) when you take div of the averaged field. So he ends up with this fact that div E = 4πρ - 4π div P which we can write as div (E + 4πP) = 4πρ [cgs units all] and then you have div D = 4πρ. He does not talk about conductivity of the dielectric in this discussion, it might change the way things work. He only brings in σ on page 222 in his plane wave discussion.

Comment: Since div D = 4πρ, we conclude that D is "driven by" only free charge. In contrast, since we also have div E = 4πρ - 4π div P, we see that E is "driven by" both free charge AND polarization charge!

Jackson 3Ed on this same subject starts on p 151 where he uses the word "ponderable" without any comment. Ordering is changed in this edition, so lets read this section right now.

Chapter 4. He says the fancy averaging will be done later in Chapter 6 and here in Chapter 4 he is just going to give a summary. Fine. He has the same little picture with the same unclear comment about it. If we apply the divergence theorem to div P we get

$$\text{div } \mathbf{P} = \rho \qquad \Leftrightarrow \qquad \int_V \rho \, dV = \int_S \mathbf{P} \bullet d\mathbf{A}$$

This says that the total charge in a volume equals the surface integral of **P,** nothing flows here. So if P is non-uniform, you might get a non-zero left side, and that is what his picture is showing, where the circle is the volume V,

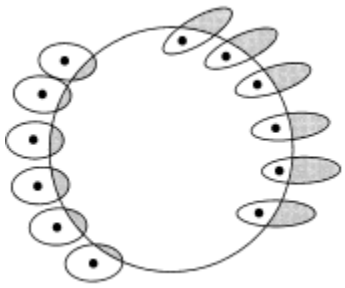


Figure 4.2 Origin of polarization-charge density. Because of spatial variation of polarization, more molecular charge may leave a given small volume than enters it. Only molecules near the boundary are shown.

The word "enters" seems wrong to me. It just seems to be the total charge that lies inside the volume at some instant in time. Nothing is entering or leaving, we are at an instant in time. So OK. He then goes on to obtain

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} \tag{4.36}$$

The constant χ_e is called the *electric susceptibility* of the medium. The displacement **D** is therefore proportional to **E**,

$$\mathbf{D} = \epsilon \mathbf{E} \tag{4.37}$$

where

$$\epsilon = \epsilon_0(1 + \chi_e) \tag{4.38}$$

is the electric permittivity; $\epsilon/\epsilon_0 = 1 + \chi_e$ is called the *dielectric constant* or relative electric permittivity.

If the dielectric is not only isotropic, but also uniform, then ϵ is independent of position. The divergence equations (4.35) can then be written

$$\nabla \cdot \mathbf{E} = \rho/\epsilon \tag{4.39}$$

All problems *in that medium* are reduced to those of preceding chapters, except that the electric fields produced by given charges are reduced by a factor ϵ_0/ϵ. The reduction can be understood in terms of a polarization of the atoms that produce fields in opposition to that of the given charge. One immediate consequence is that the capacitance of a capacitor is increased by a factor of ϵ/ϵ_0 if the empty space between the electrodes is filled with a dielectric with dielectric constant ϵ/ϵ_0 (true only to the extent that fringing fields can be neglected).

OK, this is the standard stuff. Note that a conductive medium has no free charge, but is perhaps different from his discussion above.

Chapter 6. Let's now look at the macroscopic discussion of Chapter 6. He starts off with the four Maxwell's with D and H appearing, just as I have them in lines. Then come **A** and φ. And then the wave equations,

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\rho/\epsilon_0 \qquad (6.15)$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J} \qquad (6.16)$$

Jackson has a footnote now on Lorenz right in his book, good. Not much mea culpa. Meanwhile, the above equations must be in free space. Notice that a discussion of E,B and A,φ says nothing about "media" and D and H.

He then starts into gauge transformations. The Lorenz condition is (6.14). On page 241 he refers to my Λ transformation as a "restricted" gauge transformation because I guess (6.14) is preserved. So within the Lorenz condition world, there is a whole **class** of potential pairs (A,φ). [*********]

Jackson works for a while in the Coulomb gauge div A = 0 and in that gauge he shows that

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J}_t \qquad (6.30)$$

where A is driven only by "transverse" current defined in a certain manner. I skip this since I am not going to be working in this gauge, but it is a good fact to know.

Now we come to Section 6.4 where he does the wave equation Green's function stuff. On page 245 he shows the usual particular solution. He comments on adding a homo solution. Then he is off on the incident wave at t = -∞ and all that scattering stuff. His green's gives the retarded potential stuff.

Next is **Section 6.5**. Here he deals with wave equations for E and B and gives their retarded integral solutions. This stuff is developed and Jefimenco is given a shout-out. He then ends up comparing a Feynman and a Heaviside form for the solution in this retarded world, and claims they are the same.

Next is **Section 6.6** on the macroscopic business. Jackson gives here a much more elaborate development of the averaging process which I have not read, but we end up below (6.92) where he starts to talk about current density **J**! I am all ears. Earlier he did averaging for ρ, here he only outlines the averaging process for J which he says is much messier. At one point he writes

$$\mathbf{J}(\mathbf{x}, t) = \left\langle \sum_{j\ (\text{free})} q_j\mathbf{v}_j\delta(\mathbf{x} - \mathbf{x}_j) + \sum_{n\ (\text{molecules})} q_n\mathbf{v}_n\delta(\mathbf{x} - \mathbf{x}_n) \right\rangle \qquad (6.97)$$

so he allows for current within a molecule. Sadly, there is still no discussion of conductivity of the medium, even in this messy section which involves magnetization. The free current sum shown above could be the sum of plus and minus ions so that the total charge is 0 but the current is not.

Section 6.7 is then off on Poynting and energy and all that stuff.

Section 6.8 does talk about "dispersive media", maybe the first time in the book. But dispersive here must means that ε = ε(ω).

Section 6.9 he claims is based on a book by Fano, Chu and Adler, and it is the $e^{j\omega t}$ stuff, but Jackson uses $e^{-i\omega t}$ as time dependence. Called a harmonic field. Here is a good comment

The input impedance $Z = R - iX$ (electrical engineers please read as $Z = R + jX$!) follows from (6.136) with its definition, $V_i = ZI_i$. Its real and imaginary parts are

He draws some very general EE situation where there is a medium of σ with boundary S which may go out to infinity and thereby involve radiation. The medium has conductivity σ. But nothing useful for me here I think.

Section 6.10 is off on symmetries.

Section 6.11 on magnetic monopoles. Fancy section worth reading some day.

Section 6.12 continues the above, Dirac quantization of charge idea. Did not read.

Section 6.13 is on Hertz vectors, I never heard of such things.

or *Hertz vectors*, introduced by Hertz (1889) and Righi (1901). As the name suggests, these potentials put the electric and magnetic polarization densities to the fore. We consider linear, isotropic media with sources of external polarization densities, $\mathbf{P}_{\text{ext}}$ and $\mathbf{M}_{\text{ext}}$, but no separate macroscopic charge or current. The media are described by electric and magnetic susceptibilities, ϵ and μ. [The realistic situation of frequency-dependent quantities can be abstracted by choosing a unique sinusoidal time dependence and then using Fourier superposition.]

The macroscopic fields are written

$$\mathbf{D} = \epsilon\mathbf{E} + \mathbf{P}_{\text{ext}}; \qquad \mathbf{B} = \mu\mathbf{H} + \mu_0\mathbf{M}_{\text{ext}} \tag{6.164}$$

What is he saying here??? What does "external" mean? This section does not seem to have a bearing on me, but he says Born and Wolf use it.

So that is the end of Chapter 6, and nothing really on a conducting medium. But let's look at Chapter 7 where I think conductivity will appear.

Chapter 7. In **Section 7.1** medium is non-conducting. We writes Maxwell's for now sources but with D and H, so it is not vacuum, it is a dielectric that does not conduct. Here is how he deals with the harmonic issue:

$$\begin{aligned}\mathbf{E}(\mathbf{x}, t) &= \mathscr{E}e^{ik\mathbf{n}\cdot\mathbf{x}-i\omega t}\\ \mathbf{B}(x, t) &= \mathscr{B}e^{ik\mathbf{n}\cdot\mathbf{x}-i\omega t}\end{aligned} \tag{7.8}$$

and so we have fancy fonts entering the scene! He then writes down a whole page of properties of the scripted fields.

Section 7.2 does polarization and Stokes parameters (did not appear in 1Ed)

Section 7.3 is on reflection etc at a dielectric boundary. We are in a wave context here and the whole chapter 7 in fact is on Waves.

Section 7.4 on total internal reflection and some new name I don't know.

Section 7.5. For the first time, we are going to thing of ε as being complex! The energy loss here is damping in molecules, it is not due to general conductivity. But then he looks at the DC world which I think involves free current. On page 312 we see **Ohm's Law** for the first time, J = σE. Then here we go: but first I will back up and quote a few earlier results.

electrons per molecule with binding frequency ω_j and damping constant γ_j, then the dielectric constant, $\epsilon/\epsilon_0 = 1 + \chi_e$, is given by

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{Ne^2}{\epsilon_0 m} \sum_j f_j(\omega_j^2 - \omega^2 - i\omega\gamma_j)^{-1} \tag{7.51}$$

where the *oscillator strengths* f_j satisfy the sum rule,

$$\sum_j f_j = Z \tag{7.52}$$

With suitable quantum-mechanical definitions of f_j, γ_j, and ω_j, (7.51) is an accurate description of the atomic contribution to the dielectric constant.

This is his "simple model" for ε(ω). Then he has

If some fraction f_0 of the electrons per molecule are "free" in the sense of having $\omega_0 = 0$, the dielectric constant is singular at $\omega = 0$. If the contribution of the free electrons is exhibited separately, (7.51) times ϵ_0 becomes

$$\epsilon(\omega) = \epsilon_b(\omega) + i\frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)} \tag{7.56}$$

where $\epsilon_b(\omega)$ is the contribution of all the other dipoles. The singular behavior can be understood if we examine the Maxwell–Ampère equation

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{d\mathbf{D}}{dt}$$

and assume that the medium obeys Ohm's law, $\mathbf{J} = \sigma\mathbf{E}$ and has a "normal" dielectric constant ϵ_b. With harmonic time dependence the equation becomes

$$\nabla \times \mathbf{H} = -i\omega\left(\epsilon_b + i\frac{\sigma}{\omega}\right)\mathbf{E} \tag{7.57}$$

Nothing has happened except J = σE. Here is maybe a critical statement

If, on the other hand, we did not insert Ohm's law explicitly but attributed instead all the properties of the medium to the dielectric constant, we would identify the quantity in brackets on the right-hand side of (7.57) with $\epsilon(\omega)$. Comparison with (7.56) yields an expression for the conductivity:

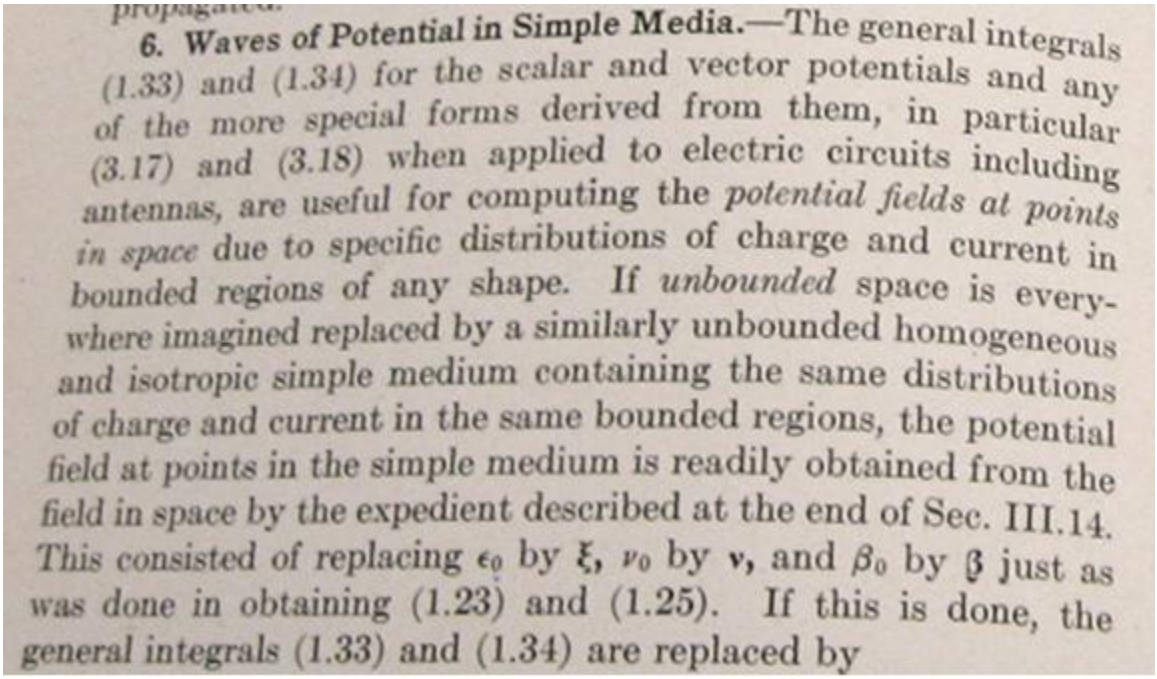
$$\sigma = \frac{f_0 Ne^2}{m(\gamma_0 - i\omega)} \tag{7.58}$$

So here he is talking about replacing ε with ξ and his simple ε(ω) model in fact makes a prediction for σ and in fact σ is complex! Ouch, but that is what the simple model says. Fine. He claims this is the Drude model of 1900. He goes on

This is essentially the model of Drude (1900) for the electrical conductivity, with $f_0 N$ being the number of free electrons per unit volume in the medium. The damping constant γ_0/f_0 can be determined empirically from experimental data on the conductivity. For copper, $N \simeq 8 \times 10^{28}$ atoms/m^3 and at normal temperatures the low-frequency conductivity is $\sigma \simeq 5.9 \times 10^7\ (\Omega \cdot \mathrm{m})^{-1}$. This gives $\gamma_0/f_0 \simeq 4 \times 10^{13}\ \mathrm{s}^{-1}$. Assuming that $f_0 \sim 1$, this shows that up to frequencies well beyond the microwave region ($\omega \lesssim 10^{11}\ \mathrm{s}^{-1}$) conductivities of metals are essentially real (i.e., current in phase with the field) and independent of frequency. At higher frequencies (in the infrared and beyond) the conductivity is complex and varies with frequency in a way described qualitatively by the simple result (7.58). The problem of electrical conductivity is really a quantum-mechanical one in which the Pauli principle plays an important role. The free electrons are actually valence electrons of the isolated atoms that become quasi-free and move relatively unimpeded through the lattice (provided their energies lie in certain intervals or bands) when the atoms are brought together to form a solid. The damping effects come from collisions involving appreciable momentum transfer between the electrons and lattice vibrations, lattice imperfections, and impurities.*

So he is saying that for reasonable frequencies ω, γ_0 is huge (up to $\omega = 10^{11} = 100$ GHz which is "well beyond the microwave region), so σ shown above is then real.

Pause: Jackson has thus stated the notion of thinking of $\varepsilon \rightarrow \xi$ as a way of incorporating conductivity into your model. I am very glad to see Jackson broach this notion. It is what King says as well,



propagated.

6. Waves of Potential in Simple Media.—The general integrals (1.33) and (1.34) for the scalar and vector potentials and any of the more special forms derived from them, in particular (3.17) and (3.18) when applied to electric circuits including antennas, are useful for computing the *potential fields at points in space* due to specific distributions of charge and current in bounded regions of any shape. If *unbounded* space is everywhere imagined replaced by a similarly unbounded homogeneous and isotropic simple medium containing the same distributions of charge and current in the same bounded regions, the potential field at points in the simple medium is readily obtained from the field in space by the expedient described at the end of Sec. III.14. This consisted of replacing ϵ_0 by ξ, ν_0 by ν, and β_0 by β just as was done in obtaining (1.23) and (1.25). If this is done, the general integrals (1.33) and (1.34) are replaced by

However, Jackson is only saying $\varepsilon \rightarrow \xi$ in the context of the curl H equation, and this is really nothing new to me. We are just combining the conduction and displacement current. No one is saying $\mathbf{D} = \xi \mathbf{E}$.

I think I am finally onto something here.

Jackson goes on to talk about the plasma high frequency limit.

He then does a detailed study of ε(ω) for water!

Section 7.6 is on ionosphere propagation and "whistlers". No mention of faster than c.

Section 7.7 is magnetohydrodynamics. He allows here how you can have current with no ρ in a conducting medium with Ohm's law. A new feature here is this: [****** v x B in lines!!!]

in (5.159) generalized for a fluid in motion to $\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, in accord with the discussion of Section 5.15. The generalization of (5.160), but for the magnetic induction, is

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{1}{\mu\sigma} \nabla^2 \mathbf{B} \qquad (7.68)$$

Here the idea is that you don't have a simple Ohm's law because charges curve due to B fields. Why were we allowed to ignore this in our transmission line discussion? Probably the second term is small, but where does someone argue that? I could compute it. Here we are talking about a fluid and v is the fluid velocity of the charged particles. Alfvén waves appear.

So somewhere I have to comment on why the v x B term for charges in my dielectric don't matter. They would be electrons. Probably v is small due to collisions. [v and B are parallel?]

Section 7.8 is group velocity.

Section 7.9 is a spreading pulse.

Section 7.10 is Kramers-Kronig

Section 7.11 is about the arrival of a signal. The **Somerfeld Precursor** is mentioned. The **Brillouin Precursor** (second) is also mentioned. Again no mention of faster than c as in our class.

And so ends Chapter 7!

Chapter 8. On waveguides.

Section 8.1. Boundary conditions at a conductor surface.

$$\mathbf{n} \cdot \mathbf{D} = \Sigma \qquad (8.1)$$

Here Σ is surface charge and this is just the Gauss box idea so that $D_{\mathbf{n}} = \Sigma$ in our SI units. Then

$$\mathbf{n} \cdot (\mathbf{B} - \mathbf{B}_c) = 0$$
$$\mathbf{n} \times (\mathbf{E} - \mathbf{E}_c) = 0 \qquad (8.3)$$

He is not allowing the dielectric to have a conductivity.

I have perused the rest of Chapter 8, nothing there for my current problem.

3. Panofsky and Phillips

I just happened to see this while reviewing Jim's books. It is on line at scribd but not for free. Tried http://www.mediafire.com/download/5nkxpmjdwpd/panofsky.rar but these rar never pan out. But

this time it worked, rar handled by filzip, and I have it! I have the 1955 original edition hard cover, but the download is the 1962 second edition. I will now work with the newer edition.

It is page 240 (210 of the 1955) and they are writing wave equations for the potentials. They quote the Lorentz condition in (14-2) as being the King gauge condition which includes conductivity! They come up with this pair of wave equations for the potentials

We obtain the symmetrical set of equations:

$$\nabla^2\mathbf{A} - \mu\epsilon\frac{\partial^2\mathbf{A}}{\partial t^2} - \mu\sigma\frac{\partial\mathbf{A}}{\partial t} = -\mu\mathbf{j}', \qquad (14\text{–}4)$$

$$\nabla^2\phi - \mu\epsilon\frac{\partial^2\phi}{\partial t^2} - \mu\sigma\frac{\partial\phi}{\partial t} = -\rho/\epsilon. \qquad (14\text{–}5)$$

This deals with one of my issues. I never could get a current on the right of the first equation, but they have somehow done it. I can now backtrack to see how they did it. This is the first time ever I have seen the above wave equation for A with j on the right. Even King never stated it.

The Lorentz condition is not quite the arbitrary subsidiary relation it appears at first sight. We shall see at once that it has the advantage of introducing complete symmetry between the scalar and vector potentials, i.e., it makes both potentials satisfy the same wave equation as that obeyed by the fields. Later we shall find that the Lorentz condition also assures a relativistic covariant relation between the scalar and vector potentials. To determine the equations satisfied by the potentials we need only introduce the defining equations (7–40) and (14–1) and the Lorentz condition into the first and fourth of Maxwell's equations, Eqs. (9–7).

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[but I thought adding σ broke the relativity thing? ********]

I will now look up the supporting equations

$$\mathbf{B} = \nabla \times \mathbf{A}. \qquad (7\text{–}40)$$

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}, \qquad (14\text{–}1)$$

Maxwell's field equations become

$$\begin{aligned}
&(1) && \nabla \cdot \mathbf{D} = \rho_{\text{true}}, \\
&(2) && \nabla \cdot \mathbf{B} = 0, \\
&(3) && \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \\
&(4) && \nabla \times \mathbf{H} = \mathbf{j}_{\text{true}} + \frac{\partial \mathbf{D}}{\partial t}.
\end{aligned} \tag{9–7}$$

The field equations (9–7) appear formally simpler than Eqs. (9–6), but they are actually more complicated physically. The solution of these equations is possible only if "constitutive" equations are available connecting $\mathbf{D}$ to $\mathbf{E}$, $\mathbf{j}$ to $\mathbf{E}$, and $\mathbf{H}$ to $\mathbf{B}$, such as $\mathbf{D} = \kappa\epsilon_0\mathbf{E}$, $\mathbf{j}_{\text{true}} = \sigma\mathbf{E}$, $\mathbf{H} = \mathbf{B}/\mu$, for a linear medium, or whatever forms apply for a nonlinear medium.

But none of these equations helps I don't think. What do they mean by j' with the prime? Here is their answer:

Here $\mathbf{j}'$ represents a current given by $\mathbf{j}' = \sigma\mathbf{E}'$, that part of the current density that is produced by the external electromotive forces. It does not contain any part of the current induced by the electric fields in conducting media and represented by the last term on the left side of Eq. (14–4).

This is the "applied" business again! Ouch! At least I now have someone talking about this concept. It was imbued in my original lines doc, but I never had a reference for it. Maybe I will go look at my lines old notes again right now.