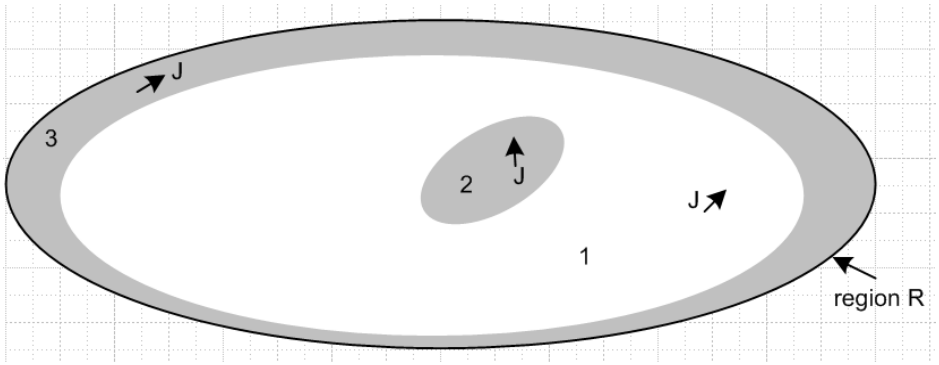
King Gauge Problem Continued PhL 11.4.13

The struggle continues here from the previous doc, and I draw the picture you see below which ended up being the final picture. I wrote down the King condition and wondered which regions it applied to. I then got the idea of using the region 1 King gauge in all three regions and suddenly King Problem #2 seemed to have a solution in terms of my unified wave equation for **A** over entire region R. But at this point, all regions had to have the same μ -- but better than a poke in the eye with a sharp stick! I then assumed different μ's and still got a unified equation! " I am just astounded" he says. I then wrote up the general plan which is now installed (same picture) into Section 1.5 of lines doc.

In the original doc of this title, I pursued this same question at length. Here just since some time has gone by, I will give it another shot starting "afresh", though I am sure I will end up at the same place.

Start with



and make the immediate simplifying assumption that $\mu_1 = \mu_2 = \mu_3 = \mu$. If we consider the *magnetostatic* analog to the Jackson point charge near boundary electrostatic problem, we would maybe have

$$A_z = (\mu/4\pi)(J_z dV/R)$$

as the replacement for the electrostatic problem $\varphi = (1/4\pi\varepsilon)(1/r)$. Then we might argue that $A_z(x)$ does not even see the boundary, and so the lines of **B** would not diffract at the boundary. Perhaps this will somehow help things in what follows. It seems to at least remove one complicating factor.

I lines doc section 1.6 I started out like so: The 3 pre-gauge **A** equations are

$$(\nabla^2 + \beta_0^{(i)2})\,\mathbf{A}^{(i)} = \text{grad}\,[\mu_i\varepsilon_i\, j\omega\varphi^{(i)} + (\text{div}\mathbf{A}^{(i)})\,] - \mu_i\mathbf{J}^{(i)}\,. \qquad (1.3.3)$$

If we decide to have three region-specific King gauges

$$\text{div}\,\mathbf{A}^{(i)} = -\,\mu\varepsilon_i j\omega\,\varphi^{(i)} - \mu\sigma_i\,\varphi^{(i)}$$

we then get in each region

$$(\nabla^2 + \beta^{(i)2})\,\mathbf{A}^{(i)}(\mathbf{x},\omega) = 0$$

since all currents are conduction currents and have been absorbed and this puts me in a bad position.

Before going on, since σ is so violently different, the gauge condition is violently different in each region, and the $(A,\varphi)^{(i)}$ in each region have these violently different gauges, so I see no sense of continuity of these potential functions at the boundaries! That seems like a bad idea if you are trying to unify the regions somehow. So let's try having just one gauge condition for the entire region.

Plan H:

Suppose we assume only the region 1 gauge condition for ALL regions. Then there is only one set of functions (**A**,φ) which give (**E**,**B**) everywhere, I like that. We then have, thinking *only* of region 1:

$\text{div}\,\mathbf{A} = -\mu\varepsilon_1 j\omega\,\varphi - \mu\sigma_1\,\varphi$ // using the common μ in all regions. (global King gauge)

$(\nabla^2 + \beta_1^2)\,\mathbf{A}(\mathbf{x},\omega) = 0$ // since conduction current has been absorbed into β_1

and we are then stuck with our famous homo equation with no source inside region 1, which is correct.

What is the equation for **A** in region 2? This comes from (1.3.3) converted to ω space, pre-gauge:

$(\nabla^2 + \beta_0^{(2)2})\,\mathbf{A} = \text{grad}\,[\mu\varepsilon_2\, j\omega\varphi + (\text{div}\mathbf{A})\,] - \mu\mathbf{J}$

where now **J** is some current in region 2 and I want to try to leave it "as is". Now we have to use the global King gauge to deal with div **A**.

$\text{RHS} = \text{grad}\,[\mu\varepsilon_2\, j\omega\varphi + (-\mu\varepsilon_1 j\omega\,\varphi - \mu\sigma_1\,\varphi)\,] - \mu\mathbf{J}$

$= \mu\varepsilon_2\, j\omega\,\text{grad}\varphi - \mu\varepsilon_1 j\omega\,\text{grad}\varphi - \mu\sigma_1\,\text{grad}\varphi - \mu\mathbf{J}$ ok

$= \mu(j\omega[\varepsilon_2-\varepsilon_1] - \sigma_1)\,\text{grad}\varphi - \mu\mathbf{J}$ ok

Now what do we do? I cannot eliminate φ without using $\mathbf{E} = -\text{grad}\,\varphi - \partial_t\mathbf{A}$ and that then brings in both **A** and **E**. But I don't want to eliminate **E** with $\mathbf{J} = \sigma\mathbf{E}$. But go ahead with $\mathbf{E} = -\text{grad}\,\varphi - \partial_t\mathbf{A}$ **:**

$\text{RHS} = \mu(j\omega[\varepsilon_2-\varepsilon_1] - \sigma_1)\,(-\mathbf{E} - j\omega\mathbf{A}) - \mu\mathbf{J}$ // in region 2 keep in mind

Perhaps since in conductor I could just set **E** = 0 ?? That is, $\sigma_2 = \infty$. Try it

So here I am making an approximation that **E** = 0 inside the region 2 conductor:

$\text{RHS} = \mu(j\omega[\varepsilon_2-\varepsilon_1] - \sigma_1)\,(-j\omega\mathbf{A}) - \mu\mathbf{J}$ ok

Then I end up with

$(\nabla^2 + \beta_0^{(2)2})\,\mathbf{A} = \mu(j\omega[\varepsilon_2-\varepsilon_1] - \sigma_1)\,(-j\omega\mathbf{A}) - \mu\mathbf{J}$ ok // region 2

$(\nabla^2 + \omega^2\mu\varepsilon_2)\,\mathbf{A} = \mu(j\omega[\varepsilon_2-\varepsilon_1] - \sigma_1)\,(-j\omega\mathbf{A}) - \mu\mathbf{J}$ ok

$[\nabla^2 + \omega^2\mu\varepsilon_2 + \mu(j\omega[\varepsilon_2-\varepsilon_1] - \sigma_1)\,(+j\omega)\,]\,\mathbf{A} = -\mu\mathbf{J}$ ok

$[\nabla^2 + \omega^2\mu\varepsilon_2 - \omega^2\mu[\varepsilon_2-\varepsilon_1] - j\omega\mu\sigma_1)\,]\,\mathbf{A} = -\mu\mathbf{J}$ ok

$[\nabla^2 + \omega^2\mu\varepsilon_1 - j\omega\mu\sigma_1)\,]\,\mathbf{A} = -\mu\mathbf{J}$ ok

$[\nabla^2 + \beta^{(1)2}\,]\,\mathbf{A} = -\mu\mathbf{J}$ ok

Wow! Is that really true??? If so, I think we might have made progress here. I conclude that this is the PDE for A in region 2

$[\nabla^2 + \beta^{(1)2}\,]\,\mathbf{A} = -\mu\mathbf{J}$ // for region 2.

where $\beta^{(1)2} = \omega^2\mu\xi_1$. In other words, although we are in region 2, the propagator is that of region 1 !!!

I then have these equations:

$(\nabla^2 + \beta_1^2)\,\mathbf{A}(\mathbf{x},\omega) = 0$ // region 1
$(\nabla^2 + \beta_1^2)\,\mathbf{A}(\mathbf{x},\omega) = -\mu\mathbf{J}$ // region2
$(\nabla^2 + \beta_1^2)\,\mathbf{A}(\mathbf{x},\omega) = -\mu\mathbf{J}$ // region3

This is definitely the first time I have got these equations! I then have this unified equation

$(\nabla^2 + \beta_1^2)\,\mathbf{A}(\mathbf{x},\omega) = -\mu\mathbf{J}$ // region R

where J includes only the currents in region 2 and region 3. Then I get a solution even for FAT conductors! The ingredients to get this result are

- μ = same in all regions
- $\mathbf{E} = 0$ on conductors
- region 1 King gauge applies to all regions

This is a very big breakthrough for me.

Plan H version 2:

Let's just see now what happens if we don't assume all the μ are the same.

$\text{div}\,\mathbf{A} = -\mu_1\varepsilon_1 j\omega\,\varphi - \mu_1\sigma_1\,\varphi$ // (global King gauge)

$(\nabla^2 + \beta_1^2)\,\mathbf{A}(\mathbf{x},\omega) = 0$ // equation for region 1 with conduction current absorbed

$$\beta_1^2 = \omega^2 \mu_1 \xi_1$$

What is the equation for **A** in region 2? This comes from (1.3.3) converted to ω space, pre-gauge:

$$(\nabla^2 + \omega^2 \mu_2 \varepsilon_2)\, \mathbf{A} = \text{grad}\,[\mu_2 \varepsilon_2\, j\omega\varphi + (\text{div}\mathbf{A})\,] - \mu_2 \mathbf{J}$$

Now ignoring the $-\mu_2\mathbf{J}$ term, the RHS is

$$\text{RHS} = \text{grad}\,[\mu_2 \varepsilon_2\, j\omega\varphi + (\text{div}\mathbf{A})\,]$$

$$= \text{grad}\,[\mu_2 \varepsilon_2\, j\omega\varphi + (-\mu_1 \varepsilon_1 j\omega\, \varphi - \mu_1 \sigma_1\, \varphi\,)\,]$$

$$= \text{grad}\,[\mu_2 \varepsilon_2\, j\omega\varphi - \mu_1 \varepsilon_1 j\omega\, \varphi - \mu_1 \sigma_1\, \varphi\,]$$

$$= [\mu_2 \varepsilon_2\, j\omega - \mu_1 \varepsilon_1 j\omega - \mu_1 \sigma_1\,]\, \text{grad}\,\varphi$$

Now make the replacement $\mathbf{E} = -\,\text{grad}\,\varphi - \partial_t \mathbf{A}$ to get

$$= [\mu_2 \varepsilon_2\, j\omega - \mu_1 \varepsilon_1 j\omega - \mu_1 \sigma_1\,]\, (-\mathbf{E} - j\omega\mathbf{A})$$

Then set E = 0 since inside a conductor to get

$$= -\,[\mu_2 \varepsilon_2\, j\omega - \mu_1 \varepsilon_1 j\omega - \mu_1 \sigma_1\,]\, (j\omega\mathbf{A})$$

$$= -\,[\, j\omega(\mu_2 \varepsilon_2 - \mu_1 \varepsilon_1) - \mu_1 \sigma_1\,]\, (j\omega\mathbf{A})$$

$$= -\,[\, -\omega^2(\mu_2 \varepsilon_2 - \mu_1 \varepsilon_1) - j\omega\mu_1 \sigma_1\,]\, \mathbf{A}$$

Then our region 2 equation is

$$(\nabla^2 + \omega^2 \mu_2 \varepsilon_2)\, \mathbf{A} = -\,[\, -\omega^2(\mu_2 \varepsilon_2 - \mu_1 \varepsilon_1) - j\omega\mu_1 \sigma_1\,]\, \mathbf{A} - \mu_2 \mathbf{J}$$

Swing the first term RHS over to get

$$(\nabla^2 + \omega^2 \mu_2 \varepsilon_2 + [\, -\omega^2(\mu_2 \varepsilon_2 - \mu_1 \varepsilon_1) - j\omega\mu_1 \sigma_1\,]\,)\, \mathbf{A} = -\,\mu_2 \mathbf{J}$$

$$(\nabla^2 + \omega^2 \mu_2 \varepsilon_2 - \omega^2 \mu_2 \varepsilon_2 + \omega^2 \mu_1 \varepsilon_1 - j\omega\mu_1 \sigma_1\,]\,)\, A = -\,\mu_2 \mathbf{J}$$

$$(\nabla^2 + \omega^2 \mu_1 \varepsilon_1 - j\omega\mu_1 \sigma_1\,]\,)\, A = -\,\mu_2 \mathbf{J}$$

$$(\nabla^2 + \beta_1^2\,)\, A = -\,\mu_2 \mathbf{J} \qquad \text{// region 2}$$

I am just astounded. Now only two assumptions were made

- **E** = 0 on conductors
- region 1 King gauge applies to all regions

Now I might define

$$\mathbf{J}(\mathbf{x}) = \theta_2(\mathbf{x})\mathbf{J}_2(\mathbf{x}) + \theta_3(\mathbf{x})\mathbf{J}_3(\mathbf{x})$$

Then we have this single equation which covers the entire region R:

$$(\nabla^2 + \beta_1^2)\,\mathbf{A} = -\mu_2\mathbf{J}$$

I keep being amazed by this.

So now I will start a major rewrite of parts of Chapter 1! No more need for that surface current stuff.

Lorenz Gauge Plan H v 2

First do **A**:

Region 1:

$$(\nabla^2 - \mu_1\varepsilon_1\,\partial_t^2)\mathbf{A} = -\mu_1\mathbf{J} \qquad (1.3.5)$$

$$\mathrm{div}\mathbf{A} = -\mu_1\varepsilon_1\,\partial_t\varphi\,. \qquad \text{// Lorenz Gauge} \qquad (1.3.6)$$

Region 2:

Start with the pre-gauge equation

$$(\nabla^2 - \mu_2\varepsilon_2\,\partial_t^2)\,\mathbf{A} = \mathrm{grad}\,[\mu_2\varepsilon_2\,\partial_t\varphi + \mathrm{div}\mathbf{A}\,] - \mu_2\mathbf{J}$$

$$= \mathrm{grad}\,[\mu_2\varepsilon_2\,\partial_t\varphi + (-\mu_1\varepsilon_1\,\partial_t\varphi)\,] - \mu_2\mathbf{J}$$

$$= [\mu_2\varepsilon_2 - \mu_1\varepsilon_1]\partial_t\mathrm{grad}\,\varphi - \mu_2\mathbf{J}$$

Then replace $\mathrm{grad}\varphi = -E - \partial_t A$ and set E = 0 to get

$$= [\mu_2\varepsilon_2 - \mu_1\varepsilon_1]\partial_t(-\mathbf{E} - \partial_t\mathbf{A}) - \mu_2\mathbf{J}$$

$$= -[\mu_2\varepsilon_2 - \mu_1\varepsilon_1]\partial_t^2\mathbf{A} - \mu_2\mathbf{J}$$

then we have

$$(\nabla^2 - \mu_2\varepsilon_2\,\partial_t^2)\,\mathbf{A} = -[\mu_2\varepsilon_2 - \mu_1\varepsilon_1]\partial_t^2\mathbf{A} - \mu_2\mathbf{J}$$

$$(\nabla^2 - \mu_2\varepsilon_2\,\partial_t^2 + [\mu_2\varepsilon_2 - \mu_1\varepsilon_1]\partial_t^2)\,\mathbf{A} = -\mu_2\mathbf{J}$$

$$(\nabla^2 - \mu_1\varepsilon_1\,\partial_t^2)\,\mathbf{A} = -\,\mu_2\mathbf{J}_2$$

So carrying out the program H here we get

$$(\nabla^2 - \mu_1\varepsilon_1\,\partial_t^2)\mathbf{A} = -\,\mu_1\mathbf{J}_1 \qquad \text{region 1}$$
$$(\nabla^2 - \mu_1\varepsilon_1\,\partial_t^2)\mathbf{A} = -\,\mu_2\mathbf{J}_2 \qquad \text{region 2}$$
$$(\nabla^2 - \mu_1\varepsilon_1\,\partial_t^2)\mathbf{A} = -\,\mu_3\mathbf{J}_3 \qquad \text{region 3}$$

$$(\nabla^2 - \mu_1\varepsilon_1\,\partial_t^2)\mathbf{A} = -\,\Sigma_{i=1}^{N}\mu_i\mathbf{J}_i$$

Now do the φ equations.

Region 1:

$$\nabla^2\varphi + \partial_t[\text{div}\,\mathbf{A}] = -\rho/\varepsilon_1$$

Region 2:

$$\nabla^2\varphi + \partial_t[\text{div}\,\mathbf{A}] = -\rho/\varepsilon_2$$

$$\nabla^2\varphi + \partial_t[-\,\mu_1\varepsilon_1\,\partial_t\varphi] = -\rho/\varepsilon_2$$

$$(\nabla^2 - \mu_1\varepsilon_1\,\partial_t^2)\varphi = -\rho/\varepsilon_2$$

thus we get

$$(\nabla^2 - \mu_1\varepsilon_1\,\partial_t^2)\varphi = -\rho/\varepsilon_1 \qquad \text{region 1}$$
$$(\nabla^2 - \mu_1\varepsilon_1\,\partial_t^2)\varphi = -\rho/\varepsilon_2 \qquad \text{region 2}$$
$$(\nabla^2 - \mu_1\varepsilon_1\,\partial_t^2)\varphi = -\rho/\varepsilon_3 \qquad \text{region 3}$$

$$(\nabla^2 - \mu_1\varepsilon_1\,\partial_t^2)\varphi = -\rho/\varepsilon_1 \qquad \text{all of R}$$

Plan H v 2 for the fields E and B

From (1.2.1) and (1.2.2) we can write

$$(\nabla^2 - \mu_1\varepsilon_1\,\partial_t^2)\mathbf{E} = \mu_1\partial_t\mathbf{J}_1 + (1/\varepsilon_1)\,\text{grad}\,\rho_1 = \mu_1\partial_t\mathbf{J}_1 + (1/\varepsilon_1)\,\text{grad}\,\rho_s \qquad \text{// region 1}$$
$$(\nabla^2 - \mu_2\varepsilon_2\,\partial_t^2)\mathbf{E} = \mu_2\partial_t\mathbf{J}_2 + (1/\varepsilon_2)\,\text{grad}\,\rho_2 = \mu_2\partial_t\mathbf{J}_2 \qquad \text{// region 2}$$
$$(\nabla^2 - \mu_3\varepsilon_3\,\partial_t^2)\mathbf{E} = \mu_3\partial_t\mathbf{J}_3 + (1/\varepsilon_3)\,\text{grad}\,\rho_3 = \mu_3\partial_t\mathbf{J}_3 \qquad \text{// region 3}$$

$$(\nabla^2 - \mu_1\varepsilon_1\,\partial_t^2)\mathbf{B} = -\,\mu_1\,\text{curl}\,\mathbf{J}_1 \qquad \text{// region 1}$$
$$(\nabla^2 - \mu_2\varepsilon_2\,\partial_t^2)\mathbf{B} = -\,\mu_2\,\text{curl}\,\mathbf{J}_2 \qquad \text{// region 2}$$

$(\nabla^2 - \mu_3\varepsilon_3 \partial_t^2)\mathbf{B} = -\mu_3 \text{ curl } \mathbf{J_3}$ // region 3

Now maybe I can simplify this somewhat. If

$\mathbf{J_1} = \sigma_1\mathbf{E}$ then the first becomes

$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2)\mathbf{E} = \mu_1\sigma_1\partial_t\mathbf{E} + (1/\varepsilon_1) \text{ grad } \rho_s$ // region 1

$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2 - \mu_1\sigma_1\partial_t)\mathbf{E} = +(1/\varepsilon_1) \text{ grad } \rho_s$ // region 1

And also

curl $\mathbf{J_1} = \varepsilon_1$curl $\mathbf{E} = \varepsilon_1(-\partial_t\mathbf{B})$ so that

$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2)\mathbf{B} = -\mu_1 \text{ curl } \mathbf{J_1}$

$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2)\mathbf{B} = -\mu_1 \varepsilon_1(-\partial_t\mathbf{B})$

$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2 - \mu_1\sigma_1\partial_t)\mathbf{B} = 0$

So we can rewrite the 6 equations as

$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2 - \mu_1\sigma_1\partial_t)\mathbf{E} = +(1/\varepsilon_1) \text{ grad } \rho_s$ // region 1
$(\nabla^2 - \mu_2\varepsilon_2 \partial_t^2)\mathbf{E} = \mu_2\partial_t\mathbf{J_2} + (1/\varepsilon_2) \text{ grad } \rho_2 = \mu_2\partial_t\mathbf{J_2}$ // region 2
$(\nabla^2 - \mu_3\varepsilon_3 \partial_t^2)\mathbf{E} = \mu_3\partial_t\mathbf{J_3} + (1/\varepsilon_3) \text{ grad } \rho_3 = \mu_3\partial_t\mathbf{J_3}$ // region 3

$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2 - \mu_1\sigma_1\partial_t)\mathbf{B} = 0$ // region 1
$(\nabla^2 - \mu_2\varepsilon_2 \partial_t^2)\mathbf{B} = -\mu_2 \text{ curl } \mathbf{J_2}$ // region 2
$(\nabla^2 - \mu_3\varepsilon_3 \partial_t^2)\mathbf{B} = -\mu_3 \text{ curl } \mathbf{J_3}$ // region 3