11.23.13

This doc is where I developed my new version of Section 1.3 (c) which no longer mentions surface currents J_m. Probably the section has evolved further since it was installed into lines doc. *This* doc is just for archive purposes, could be thrown out some day.

(c) The Potential Wave Equations in the King and Lorenz Gauges with Conductors

We refer to a certain gauge condition below as "the King gauge" because King (see Refs.) made extensive use of this condition in his books and papers at least as early as 1945. Perhaps this gauge has some official name, but we are not aware of it.

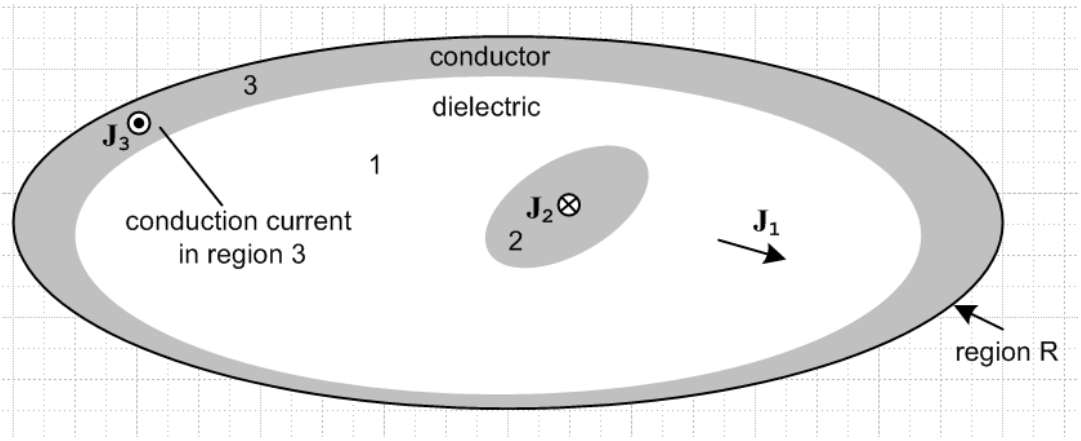
We start with this King gauge and treat **A** and then φ. Then we do the Lorenz gauge case for **A** and φ, and finally we look at the wave equations for **E** and **B**. The motivation for using the King gauge is made clear.

Unlike most sources on this subject, we allow for the possibility that the conductors' μ_i might differ from that of the dielectric.

KING GAUGE

Wave equation for **A**

Consider the following general cross section of a transmission line which happens to be of coaxial cable type,



The gray regions 2 and 3 are conductors, while the white region 1 is the (possibly conducting) dielectric. Currents $\mathbf{J}_1$, $\mathbf{J}_2$ and $\mathbf{J}_3$ are conduction currents.

We start by selecting the **King gauge for region 1** and we apply it to all three regions,

$$\text{div } \mathbf{A} = - \mu_1\varepsilon_1 \partial_t\varphi - \mu_1\sigma_1\varphi \qquad \text{// King gauge, applied to all of R .} \qquad (1.3.18)$$

We first obtain the wave equation for **A** in **region 1**. Start with (1.3.3) which gives the wave equation for **A** before any gauge choice is made

$$(\nabla^2 - \mu_1\varepsilon_1\,\partial_t^2)\,\mathbf{A} = \text{grad}\,[\mu_1\varepsilon_1\,\partial_t\varphi + \text{div}\mathbf{A}\,] - \mu_1\mathbf{J} \quad . \qquad \text{// region 1} \qquad \textit{(1.3.3)}$$

Now insert the King gauge (1.3.18) to get

$$(\nabla^2 - \mu_1\varepsilon_1\,\partial_t^2)\,\mathbf{A} = \text{grad}\,[\mu_1\varepsilon_1\,\partial_t\varphi + (-\mu_1\varepsilon_1\,\partial_t\varphi - \mu_1\sigma_1\varphi)\,] - \mu_1\mathbf{J}$$

$$= -\mu_1\sigma_1\,\text{grad}\,\varphi - \mu_1\mathbf{J}$$

$$= -\mu_1\sigma_1\,(-\mathbf{E} - \partial_t\mathbf{A}) - \mu_1(\sigma_1\mathbf{E}) \quad . \qquad \text{// from (1.3.1) and } \mathbf{J} = \sigma_1\mathbf{E}$$

$$= -\mu_1\sigma_1\,(-\partial_t\mathbf{A})\,.$$

Thus the wave equation for **A** in region 1 is

$$(\nabla^2 - \mu_1\varepsilon_1\,\partial_t^2 - \mu_1\sigma_1\partial_t)\,\mathbf{A} = 0 \qquad \text{// region 1} \qquad (1.3.19)$$

This is a damped wave equation with no driving source; the equation is homogeneous.

Now we start over with (1.3.3) for **region 2:**

$$(\nabla^2 - \mu_2\varepsilon_2\,\partial_t^2)\,\mathbf{A} = \text{grad}\,[\mu_2\varepsilon_2\,\partial_t\varphi + \text{div}\mathbf{A}\,] - \mu_2\mathbf{J}_2 \qquad \text{// region 2} \qquad \textit{(1.3.3)}$$

As before, we insert the region 1 King gauge expression (1.3.18) for div **A**, even though we are now working in region 2,

$$(\nabla^2 - \mu_2\varepsilon_2\,\partial_t^2)\,\mathbf{A} = \text{grad}\,[\mu_2\varepsilon_2\,\partial_t\varphi + (-\mu_1\varepsilon_1\,\partial_t\varphi - \mu_1\sigma_1\varphi)\,] - \mu_2\mathbf{J}_2$$

$$= [\mu_2\varepsilon_2\,\partial_t + (-\mu_1\varepsilon_1\,\partial_t - \mu_1\sigma_1)\,]\,\text{grad}\varphi - \mu_2\mathbf{J}_2$$

$$= [\,(\mu_2\varepsilon_2 - \mu_1\varepsilon_1)\,\partial_t - \mu_1\sigma_1)\,]\,\text{grad}\varphi - \mu_2\mathbf{J}_2$$

$$= [\,(\mu_2\varepsilon_2 - \mu_1\varepsilon_1)\,\partial_t - \mu_1\sigma_1)\,]\,(-\mathbf{E} - \partial_t\mathbf{A}) - \mu_2\mathbf{J}_2 \qquad \text{// using (1.3.1)}$$

$$= [\,(\mu_2\varepsilon_2 - \mu_1\varepsilon_1)\,\partial_t - \mu_1\sigma_1)\,]\,(-\mathbf{J}_2/\sigma_2 - \partial_t\mathbf{A}) - \mu_2\mathbf{J}_2 \qquad \text{// } \mathbf{J}_2 = \sigma_2\mathbf{E}$$

$$\approx [\,(\mu_2\varepsilon_2 - \mu_1\varepsilon_1)\,\partial_t - \mu_1\sigma_1)\,]\,(-\partial_t\mathbf{A}) - \mu_2\mathbf{J}_2 \qquad \text{// since } \sigma_2 \text{ is very large in conductor 2}$$

$$= -[\,(\mu_2\varepsilon_2 - \mu_1\varepsilon_1)\,\partial_t^2 - \mu_1\sigma_1\partial_t\,]\,\mathbf{A} - \mu_2\mathbf{J}_2$$

Notice that we have chosen *not* to set $\mathbf{J}_2 = \sigma_2\mathbf{E}$ in region 2 for the last term, we just leave it as $\mathbf{J}_2$. Moving the first term on the right to the left we get

$$(\ \nabla^2 - \mu_2\varepsilon_2\,\partial_t^2 + [\,(\mu_2\varepsilon_2 - \mu_1\varepsilon_1)\,\partial_t^2 - \mu_1\sigma_1\partial_t\,]\ )\,\mathbf{A} = -\mu_2\mathbf{J}_2$$

or

$$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2 - \mu_1\sigma_1\partial_t) \mathbf{A} = - \mu_2\mathbf{J}_2 \qquad \text{// region 2} \qquad (1.3.20)$$

On the left side we see the same region 1 damped wave operator although we are in region 2, and $\mathbf{J}_2$ is the conduction current density in region 2. A similar result will be obtained for region 3. Thus we have shown that

$$\begin{aligned} &(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2 - \mu_1\sigma_1\partial_t) \mathbf{A} = 0 && \text{region 1} \\ &(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2 - \mu_1\sigma_1\partial_t) \mathbf{A} = - \mu_2\mathbf{J}_2 && \text{region 2} \\ &(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2 - \mu_1\sigma_1\partial_t) \mathbf{A} = - \mu_3\mathbf{J}_3 && \text{region 3} \end{aligned} \qquad (1.3.21)$$

We can combine these into a single equation which is then valid over all of region R,

$$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2 - \mu_1\sigma_1) \mathbf{A} = - \mu_2\mathbf{J}_2 - \mu_3\mathbf{J}_3 \qquad \text{all of region R} \qquad (1.3.22)$$

with the understanding that the conduction current in region 1 has already been accounted for and $\mathbf{J}_i$ represents conduction currents in conductor i .We could generalize this result for a region R containing any number N of conductors labeled i = 2,3...N+1

$$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2 - \mu_1\sigma_1) \mathbf{A} = - \Sigma_{i=2}^{N+1}\mu_i\mathbf{J}_i \qquad \text{all of region R} \qquad (1.3.23)$$

Wave equation for φ

We first obtain the wave equation for φ in region 1. Start with (1.3.2) which gives the wave equation for φ before any gauge choice is made

$$\nabla^2\varphi + \partial_t[\text{div } \mathbf{A}] = -\rho/\varepsilon_1 \quad . \qquad \textit{(1.3.2)}$$

Now use the same global region R King gauge (1.3.18) for div **A**.

$$\nabla^2\varphi + \partial_t[- \mu_1\varepsilon_1 \partial_t\varphi - \mu_1\sigma_1\varphi] = - \rho_1/\varepsilon_1$$

$$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2 - \mu_1\sigma_1\partial_t)\varphi = - (1/\varepsilon_1)\rho_1 \qquad \text{// region 1} \qquad (1.3.24)$$

Again the same damped region 1 wave operator appears on the left side. Since the King gauge is the same in all three regions, we can write

$$\begin{aligned} &(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2 - \mu_1\sigma_1\partial_t)\varphi = - (1/\varepsilon_1)\rho_1 && \text{// region 1} \\ &(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2 - \mu_1\sigma_1\partial_t)\varphi = - (1/\varepsilon_2)\rho_2 && \text{// region 2} \\ &(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2 - \mu_1\sigma_1\partial_t)\varphi = - (1/\varepsilon_3)\rho_3 && \text{// region 3} \end{aligned} \qquad (1.3.25)$$

where ρ always means free charge. The three equations are basically the same because the pre-gauge equation (1.3.2) has no region-specific parameters apart from ε_1, in contrast with (1.3.3) quoted above.

Now the only actual free charge present is the surface charge on the outside surfaces of the conductors and we regard this as being in region 1. There is no free charge inside the dielectric or inside the conductors. We can then combine the above three equations into a single equation for all of region R

$$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2 - \mu_1\sigma_1\partial_t)\varphi = - (1/\varepsilon_1)\rho_s \qquad \text{all of region R} \qquad (1.3.26)$$

where now ρ_s is the surface charge on the conductors.

Conclusion for wave equations in the King gauge

Here then are the wave equations for φ and **A** in region R using the region 1 King gauge:

Potential Wave Equations in the King Gauge (1.3.27)

$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2 - \mu_1\sigma_1\partial_t)\varphi = - (1/\varepsilon_1)\rho_s$	all of region R	*(1.3.26)*
$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2 - \mu_1\sigma_1\partial_t)\mathbf{A} = - \Sigma_{i=2}^{N} \mu_i\mathbf{J}_i$	all of region R	*(1.3.23)*
$\text{div } \mathbf{A} = - \mu_1\varepsilon_1 \partial_t\varphi - \mu_1\sigma_1\varphi$	King gauge	*(1.3.18)*

1 = dielectric 2,3,4.... = conductors
ρ_s = total free surface charge on all conductors
$\mathbf{J}_i$ = current in conductor i (current in dielectric exists but does not appear in $\Sigma_i \mu_i\mathbf{J}_i$)

King never writes these wave equations in his transmission-line theory book, so it is difficult to find verification of our logic pathway in his book. However, Panofsky and Phillips *do* show the equations and we quote the relevant section from p 241 of their book where we identify their parameters μ,ε,σ with our region 1 parameters μ_1, ε_1, σ_1 :

14–1] THE WAVE EQUATION FOR THE POTENTIALS 241

We obtain the symmetrical set of equations:

$$\nabla^2\mathbf{A} - \mu\epsilon\frac{\partial^2\mathbf{A}}{\partial t^2} - \mu\sigma\frac{\partial\mathbf{A}}{\partial t} = -\mu\mathbf{j}', \qquad (14\text{–}4)$$

$$\nabla^2\phi - \mu\epsilon\frac{\partial^2\phi}{\partial t^2} - \mu\sigma\frac{\partial\phi}{\partial t} = -\rho/\epsilon. \qquad (14\text{–}5)$$

Here $\mathbf{j}'$ represents a current given by $\mathbf{j}' = \sigma\mathbf{E}'$, that part of the current density that is produced by the external electromotive forces. It does not contain any part of the current induced by the electric fields in conducting media and represented by the last term on the left side of Eq. (14–4).

Their last sentence says that $\mathbf{J} = \sigma\mathbf{E}$ in the conducting dielectric has been incorporated into the $-\mu\sigma\partial_t\mathbf{A}$ term in their first equation, just as we have done above. These authors have assumed that the μ's of the dielectric and the conductors are all the same (normally $\mu = \mu_0$). In order to obtain the above equations,

Panofsky and Phillips use the King gauge (1.3.18) but they refer to this gauge simply as "the Lorentz condition" (illustrating Comments 1and 2 above) . From their page 240,

within an additive arbitrary function of position. Let us define the divergence of **A** by what is called the Lorentz condition,

$$\nabla \cdot \mathbf{A} + \mu\epsilon \frac{\partial \phi}{\partial t} + \mu\sigma\phi = 0, \qquad (14\text{–}2)$$

which in free space becomes

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0. \qquad (14\text{–}3)$$

LORENZ GAUGE

If we carry out exact same program with respect to Fig ** using a global Lorenz gauge for all of R,

$$\text{div}\mathbf{A} = -\mu_1\varepsilon_1\partial_t\varphi \ , \qquad (1.3.28)$$

we obtain these results for **A**, where in region 1 the conduction current is *not* absorbed into a damping term on the left side,

$$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2)\mathbf{A} = -\mu_1\mathbf{J}_1 \qquad \text{region 1}$$
$$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2)\mathbf{A} = -\mu_2\mathbf{J}_2 \qquad \text{region 2}$$
$$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2)\mathbf{A} = -\mu_3\mathbf{J}_3 \ . \qquad \text{region 3}$$

As before, all three equations have the same wave operator on the left side. Again assuming N conductors, we combine these into a single equation as follows

$$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2)\mathbf{A} = -\Sigma_{i=1}^{N} \mu_i\mathbf{J}_i \qquad \text{all of region R} \qquad (1.3.28)$$

Meanwhile, the results for φ are

$$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2)\varphi = -\rho/\varepsilon_1 \qquad \text{region 1}$$
$$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2)\varphi = -\rho/\varepsilon_2 \qquad \text{region 2}$$
$$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2)\varphi = -\rho/\varepsilon_3 \qquad \text{region 3}$$

so that with the same comments made earlier (ρ_s includes all surface charges bordering region 1)

$$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2)\varphi = -\rho_s/\varepsilon_1 \qquad \text{all of region R} \qquad (1.3.29)$$

COMPARISON

We can now do a side by side comparison:

King Gauge:

$(\nabla^2 - \mu_1\varepsilon_1\,\partial_t^2 - \mu_1\sigma_1\partial_t)\varphi = -\rho_s/\varepsilon_1$ all of region R *(1.3.26)*

$(\nabla^2 - \mu_1\varepsilon_1\,\partial_t^2 - \mu_1\sigma_1)\,\mathbf{A} = -\Sigma_{i=2}^{N}\,\mu_i\mathbf{J}_i$ all of region R *(1.3.23)*

$\text{div}\,\mathbf{A} = -\mu_1\varepsilon_1\,\partial_t\varphi - \mu_1\sigma_1\varphi$ King gauge region 1 *(1.3.18)*

Lorenz Gauge:

$(\nabla^2 - \mu_1\varepsilon_1\,\partial_t^2)\varphi = -\rho_s/\varepsilon_1$ all of region R

$(\nabla^2 - \mu_1\varepsilon_1\,\partial_t^2)\mathbf{A} = -\Sigma_{i=1}^{N}\,\mu_i\mathbf{J}_i$ all of region R

$\text{div}\mathbf{A} = -\mu_1\varepsilon_1\partial_t\varphi$ Lorenz gauge region 1

In the Lorenz gauge, we get undamped wave operators, but the first sum on the right of the **A** equation includes a sum over the current in the dielectric, whereas this is not the case in the King gauge. In a situation where we have prescribed currents $\mathbf{J}_i$ in the conductors, it is inconvenient to have to worry about the dielectric conduction current $\mathbf{J}_1$ which complicates the solution of the problem (this will become more obvious later). In the King gauge, we get damped wave operators but we have to include only the current in the conductors since the current in the dielectric has been incorporated into the damping term. When we transform to the frequency domain and write the Helmholtz equation for **A** and its Helmholtz Integral solution, we need only integrate over the conductors which makes life easier.

E AND B WAVE EQUATIONS

Meanwhile, the **E** and **B** *field* wave equations of course don't know anything about gauges and from (1.2.1) and (1.2.2) we have, with respect to Fig. ** ,

$(\nabla^2 - \mu_1\varepsilon_1\,\partial_t^2)\mathbf{E} = \mu_1\partial_t\mathbf{J}_1 + (1/\varepsilon_1)\,\text{grad}\,\rho_1 = \mu_1\partial_t\mathbf{J}_1 + (1/\varepsilon_1)\,\text{grad}\,\rho_s$ // region 1

$(\nabla^2 - \mu_2\varepsilon_2\,\partial_t^2)\mathbf{E} = \mu_2\partial_t\mathbf{J}_2 + (1/\varepsilon_2)\,\text{grad}\,\rho_2 = \mu_2\partial_t\mathbf{J}_2$ // region 2

$(\nabla^2 - \mu_3\varepsilon_3\,\partial_t^2)\mathbf{E} = \mu_3\partial_t\mathbf{J}_3 + (1/\varepsilon_3)\,\text{grad}\,\rho_3 = \mu_3\partial_t\mathbf{J}_3$ // region 3

(1.3.30)

$(\nabla^2 - \mu_1\varepsilon_1\,\partial_t^2)\mathbf{B} = -\mu_1\,\text{curl}\,\mathbf{J}_1$ // region 1

$(\nabla^2 - \mu_2\varepsilon_2\,\partial_t^2)\mathbf{B} = -\mu_2\,\text{curl}\,\mathbf{J}_2$ // region 2

$(\nabla^2 - \mu_3\varepsilon_3\,\partial_t^2)\mathbf{B} = -\mu_3\,\text{curl}\,\mathbf{J}_3$ // region 3

where $\mathbf{J}_i = \sigma_i\mathbf{E}$. We cannot unify each group of three equations into a single region R equation as we could in the potential case since the wave operators are different in each region. Using $\mathbf{J}_i = \sigma_i\mathbf{E}$ and curl $\mathbf{E} = -\partial_t\mathbf{B}$ we get

$(\nabla^2 - \mu_1\varepsilon_1\,\partial_t^2 - \mu_1\sigma_1\partial_t)\mathbf{E} = (1/\varepsilon_1)\,\text{grad}\,\rho_s$ // region 1

$(\nabla^2 - \mu_2\varepsilon_2\,\partial_t^2 - \mu_2\sigma_2\partial_t)\mathbf{E} = 0$ // region 2

$(\nabla^2 - \mu_3\varepsilon_3\,\partial_t^2 - \mu_3\sigma_3\partial_t)\mathbf{E} = 0$ // region 3

(1.3.31)

$(\nabla^2 - \mu_1\varepsilon_1\,\partial_t^2 - \mu_1\sigma_1\partial_t)\mathbf{B} = 0$ // region 1

$(\nabla^2 - \mu_2\varepsilon_2\,\partial_t^2 - \mu_2\sigma_2\partial_t)\mathbf{B} = 0$ // region 2

$(\nabla^2 - \mu_3\varepsilon_3\,\partial_t^2 - \mu_3\sigma_3\partial_t)\mathbf{B} = 0$ // region 3

Again the three damped wave operators are different. The solution of these equations requires solving the first for its particular solution, finding all possible homogenous solutions to all 6 equations using appropriate harmonic forms with "constants to be determined", then matching these conditions at the two boundaries to evaluate the constants. In contrast, in the *potential* problem of (1.3.27),

$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2 - \mu_1\sigma_1\partial_t)\varphi = -(1/\varepsilon_1)\rho_s$	all of region R	*(1.3.26)*	
$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2 - \mu_1\sigma_1\partial_t)\mathbf{A} = -\Sigma_{i=2}^{N} \mu_i \mathbf{J}_i$	all of region R	*(1.3.23)*	
$\text{div}\ \mathbf{A} = -\mu_1\varepsilon_1 \partial_t\varphi - \mu_1\sigma_1\varphi$	King gauge	*(1.3.18)*	*(1.3.27)*

one worries about a single unified region R and there is only one damped wave operator. The method of solution is to find the particular solutions of the φ and **A** equations, add in homogenous solutions and match boundary conditions.