retired at 11:10 AM on 12/5/13

1.6 Reinterpretation of all equations in terms of complex functions

It seemed useful to defer the topics of this section to avoid cluttering up the preceding five sections. The Fourier Transform has already been used in the previous two sections, and here we shall discuss it more formally as a motivating factor in changing our point of view from real to complex functions. The general nature of the Fourier transform of complex monochrome ($e^{j\omega t}$) fields sets the stage for the analysis of the round wire in Section 2.

(a) Complex Functions

Up to this point, we have been regarding the following fields as representing real physical quantities,

$$\begin{array}{llll} \mathbf{H}(\mathbf{x},t) & \mathbf{D}(\mathbf{x},t) & \mathbf{J}(\mathbf{x},t) & \mathbf{A}(\mathbf{x},t) \\ \mathbf{B}(\mathbf{x},t) & \mathbf{E}(\mathbf{x},t) & \rho(\mathbf{x},t) & \varphi(\mathbf{x},t) \end{array} \qquad (1.6.1)$$

The fields, potentials and sources exist in the real physical world and are related by equations involving real operators like curl and $\partial/\partial t$. We can represent such an equation as $L_{\mathbf{x},t} f(\mathbf{x},t) = g(\mathbf{x},t)$ where $L_{\mathbf{x},t}$ is some real differential operator and f and g are real fields (g does not mean Green's function here).

One can extend f and g such that f and g are either both the real or both the imaginary parts of complex functions F and G. Then the equation $L_{\mathbf{x},t} F(\mathbf{x},t) = G(\mathbf{x},t)$ represents two distinct physical equations which we can write as

$$L_{\mathbf{x},t} F(\mathbf{x},t) = G(\mathbf{x},t) \quad \Rightarrow \quad L_{\mathbf{x},t}[f(\mathbf{x},t) + jf'(\mathbf{x},t)] = [g(\mathbf{x},t) + jg'(\mathbf{x},t)] \quad \Rightarrow$$

$$\begin{array}{ll} L_{\mathbf{x},t}\, f(x,t) = g(x,t) & F(\mathbf{x},t) = f(\mathbf{x},t) + jf'(\mathbf{x},t) \\ L_{\mathbf{x},t}\, f'(x,t) = g'(x,t) & G(\mathbf{x},t) = g(\mathbf{x},t) + jg'(\mathbf{x},t)\ . \end{array} \qquad (1.6.2)$$

It is convenient to regard all the mathematical fields listed above in (1.6.1) as complex fields like F and G. For example, we might write the Maxwell curl **E** equation (1.1.2) in this manner

$$\begin{array}{ll} \text{curl } \mathbf{E(x,t)} = -\,\partial \mathbf{B}(x,t)/\partial t & \mathbf{E}(x,t) = \mathbf{e}(x,t) + j\,\mathbf{e}'(x,t) \\ & \mathbf{B}(x,t) = \mathbf{b}(x,t) + j\,\mathbf{b}'(x,t)\ . \end{array} \qquad (1.6.3)$$

where $\mathbf{e} = \mathrm{Re}(\mathbf{E})$ and $\mathbf{e}' = \mathrm{Im}(\mathbf{E})$ and similarly for the **B** field.

The single left equation of (1.6.3) then represents these two different physical equations with real fields

$$\begin{array}{l} \text{curl } \mathbf{e(x,t)} = -\,\partial \mathbf{b}(x,t)/\partial t \\ \text{curl } \mathbf{e'(x,t)} = -\,\partial \mathbf{b'}(x,t)/\partial t\ . \end{array} \qquad (1.6.4)$$

(b) Monochrome time

The classic application of this idea is the assumption that some complex field is "monochrome" in its time dependence, meaning for example,

$$\mathbf{E}(\mathbf{x},t) = e^{j[\omega_1 t+\varphi(\mathbf{x},\omega_1)]}\, \boldsymbol{E}(\mathbf{x},\omega_1) \;=\; e^{j\omega_1 t}\, e^{j\varphi(\mathbf{x},\omega_1)}\, \boldsymbol{E}(\mathbf{x},\omega_1) \;, \qquad (1.6.5)$$

where $\boldsymbol{E}(\mathbf{x},\omega_1)$ is real. All time dependence is in the $e^{j\omega_1 t}$ factor and all spatial dependence is in the factor $[e^{j\varphi(\mathbf{x},\omega_1)}\, \boldsymbol{E}(\mathbf{x},\omega_1)]$ -- separation of variables. This monochrome field might be regarded as a probe or driver of some system and the solution function $\boldsymbol{E}(\mathbf{x},\omega_1)$ and phase $\varphi(\mathbf{x},\omega_1)$ might depend parametrically on the probe frequency ω_1 as well as on position $\mathbf{x}$.

For (1.6.5) the corresponding physical field assumption is either of these equations,

$$\mathbf{e(x,t)} = \mathrm{Re}\{\, \mathbf{E}(\mathbf{x},t)\} = \cos[\omega_1 t + \varphi(\mathbf{x},\omega_1)]\, \boldsymbol{E}(\mathbf{x},\omega_1)$$
$$\mathbf{e'(x,t)} = \mathrm{Im}\{\, \mathbf{E}(\mathbf{x},t)\} = \sin[\omega_1 t \; + \varphi(\mathbf{x},\omega_1)]\, \boldsymbol{E}(\mathbf{x},\omega_1) \quad . \qquad (1.6.6)$$

We stress again that the phase $\varphi(\mathbf{x},\omega_1)$ might depend on both $\mathbf{x}$ and ω_1. A good prototype example for the ω_1 dependence of phase $\varphi(\mathbf{x},\omega_1)$ is a damped harmonic oscillator with resonant frequency ω_0 which is driven at frequency ω_1. The solution is:

$$x(t) = x(0)\, \sin[\omega_1 t + \varphi(\omega_1)] \qquad \tan\varphi(\omega_1) = -(\omega_1/\tau)/(\omega_0^2 - \omega_1^2)$$

Of course the solution function x(t) is not a field over R^3, so in this case the phase φ has no $\mathbf{x}$ dependence.

Comments:

(1) The assumed form (1.6.5) is *not* the most general form possible for monochromatic $\mathbf{E}(\mathbf{x},t)$. One could, for example, allow each field component $E_i(\mathbf{x},t)$ to have a separate phase $\varphi_i(\mathbf{x},t)$. The form (1.6.5) is really an "ansatz" form meaning that one assumes an $\mathbf{E}$ field of form (1.6.5), and then one tries to find the solution to a specific problem with the $\mathbf{E}$ field so restricted. If a solution is found, then the assumed form is justified. Often one makes a more restrictive ansatz where $\varphi(\mathbf{x},\omega) = \varphi(\omega)$, or even more restrictive with $\varphi(\mathbf{x},\omega) = \varphi$, a constant.

(2) For a different kind of problem, one might make an ansatz different from (1.6.5). One might for example write

$$\mathbf{E}(\mathbf{x},t) = e^{j(\omega_1 t - kz)}\, [e^{j\varphi(\mathbf{x},y,\omega_1)} \boldsymbol{E}(\mathbf{x},\omega_1)] \qquad (1.6.7)$$

if one were searching for a solution representing some kind of wave traveling in the z direction. This $\boldsymbol{E}(\mathbf{x},\omega_1)$ is then different from that shown in (1.6.5).

(c) Why complex fields?

The reason for using a complex field like $\mathbf{E}(\mathbf{x},t)$ instead of the real field $\mathbf{e}(\mathbf{x},t)$ has to do with the Fourier Transform (or the Laplace Transform). This transform is almost always needed to solve a non-trivial

problem involving Maxwell's equations, and we saw it in action in Section 1.5. In our somewhat sloppy notation, and with the convention that the $(1/2\pi)$ goes in the expansion formula along with $e^{+j\omega t}$, we write this transform as :

$$\mathbf{E}(\mathbf{x},\omega) = \int_{-\infty}^{\infty} dt\, \mathbf{E}(\mathbf{x},t)\, e^{-j\omega t} \qquad \text{projection = transform} \qquad (1.6.8a)$$

$$\mathbf{E}(\mathbf{x},t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, \mathbf{E}(\mathbf{x},\omega)\, e^{+j\omega t}\ . \qquad \text{expansion = inverse transform = recovery} \qquad (1.6.8b)$$

Here $\mathbf{E}(\mathbf{x},t)$ is the original complex field whose real and imaginary parts are physical fields as in (1.6.3) or (1.6.6), while $\mathbf{E}(\mathbf{x},\omega)$ is the Fourier Transform of $\mathbf{E}(\mathbf{x},t)$. Since $\mathbf{E}(\mathbf{x},\omega)$ is a completely different complex function from $\mathbf{E}(\mathbf{x},t)$, one really should use some notation like $\boldsymbol{\mathcal{E}}(\mathbf{x},\omega)$ or $\mathbf{E}\hat{}(\mathbf{x},\omega)$, but we trust the reader to make the distinction when the ω argument is present or in the general context of some discussion.

As (1.6.7) shows, the dimensional units of the Fourier transform of some quantity have an extra sec factor. For example, since dim[$\mathbf{E}(\mathbf{x},t)$] = volt/m, it follows that dim[$\mathbf{E}(\mathbf{x},\omega)$] = volt-sec/m.

An obvious property of the Fourier Transform is this:

$$\partial_t \mathbf{E}(\mathbf{x},t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, \mathbf{E}(\mathbf{x},\omega)\, \partial_t\, e^{+j\omega t} = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, [j\omega\, \mathbf{E}(\mathbf{x},\omega)]\, e^{+j\omega t}$$

which we can write as (symbol $\leftrightarrow$ means "corresponds to")

$$\mathbf{E}(\mathbf{x},t) \leftrightarrow \mathbf{E}(\mathbf{x},\omega) \qquad \Leftrightarrow \qquad \partial_t \mathbf{E}(\mathbf{x},t) \leftrightarrow j\omega\, \mathbf{E}(\mathbf{x},\omega) \qquad (1.6.9)$$

which is just another way to state our rule (1.5.2). In the case of assumed monochrome time dependence of the form (1.6.5) (reflected in (1.6.6)) one finds that

$$\mathbf{E}(\mathbf{x},t) = e^{j[\omega_1 t + \varphi(\mathbf{x},\omega_1)]}\, \boldsymbol{E}(\mathbf{x},\omega_1) \qquad \textit{(1.6.5)}$$

$$\mathbf{E}(\mathbf{x},\omega) = \int_{-\infty}^{\infty} dt\, \left[e^{j\omega_1 t}\, e^{j\varphi(\mathbf{x},\omega_1)} \boldsymbol{E}(\mathbf{x},\omega_1)\right] e^{-j\omega t} = \boldsymbol{E}(\mathbf{x},\omega_1)\, e^{j\varphi(\mathbf{x},\omega_1)} \int_{-\infty}^{\infty} dt\, e^{j(\omega_1-\omega)t}$$

$$= \boldsymbol{E}(\mathbf{x},\omega_1)\, e^{j\varphi(\mathbf{x},\omega_1)}\, 2\pi\delta(\omega-\omega_1)\ . \qquad (1.6.10)$$

It is this simple single-δ-function form that motivates the use of complex fields as carriers of the real physical fields. One can of course Fourier-transform the monochrome physical field directly, but the result is clumsy to deal with. For example,

$$\mathbf{e}(\mathbf{x},t) = \cos[\omega_1 t + \varphi_1(\mathbf{x},\omega_1)]\, \boldsymbol{E}(\mathbf{x},\omega_1)$$

$$\mathbf{e}(\mathbf{x},\omega) = \int_{-\infty}^{\infty} dt\, \{\cos[\omega_1 t + \varphi_1(\mathbf{x},\omega_1)]\, \boldsymbol{E}(\mathbf{x},\omega_1)\}\, e^{-j\omega t}$$

$$= \boldsymbol{E}(\mathbf{x},\omega_1)\,(1/2)\int_{-\infty}^{\infty} dt\ \{\, e^{j[\omega_1 t+\varphi(\mathbf{x},\omega_1)]} + e^{-j[\omega_1 t+\varphi(\mathbf{x},\omega_1)]}\,\}\, e^{-j\omega t}$$

$$= \boldsymbol{E}(\mathbf{x},\omega_1)\ [e^{j\varphi(\mathbf{x},\omega_1)}\pi\delta(\omega-\omega_1) + e^{-j\varphi(\mathbf{x},\omega_1)}\pi\delta(\omega+\omega_1)\,] \ . \qquad (1.6.11)$$

A directly related benefit of using the complex function approach is the fact that math with exponentials is so much simpler than the corresponding math with trig functions, as for example

$e^{j(\omega t+\varphi)}\, e^{-j(\omega' t+\varphi')} = e^{j(\omega-\omega')t}\, e^{j(\varphi-\varphi')}$ // dependence on t isolated to one factor

$$\cos(\omega t+\varphi)\cos(\omega' t+\varphi') = (1/2)\ \{\cos[(\omega-\omega')t + (\varphi-\varphi')] + \cos[(\omega+\omega')t + (\varphi+\varphi')]\ \} \ . \qquad (1.6.12)$$

Another benefit of using the Fourier transform is its close connection with the Laplace Transform.

Comment: Using the real cosine form shown as the first line of (1.6.6) along with the Fourier Cosine Transform is not viable because $\cos[\omega_1 t + \varphi(\mathbf{x},\omega_1)]\ \boldsymbol{E}(\mathbf{x},\omega_1)$ is not an even function of t.

(d) Monochrome E and B fields

One might seek to solve a system using monochrome fields of the form (1.6.5) for both the electric and magnetic fields. Those forms would be ($\boldsymbol{E}$ and $\boldsymbol{B}$ are real)

$$\mathbf{E}(\mathbf{x},t) = e^{j[\omega_1 t+\varphi_e(\mathbf{x},\omega_1)]}\ \boldsymbol{E}(\mathbf{x},\omega_1)$$
$$\mathbf{B}(\mathbf{x},t) = e^{j[\omega_1 t+\varphi_b(\mathbf{x},\omega_1)]}\ \boldsymbol{B}(\mathbf{x},\omega_1) \qquad (1.6.13)$$

where we assume the same frequency ω_1 for both fields, but allow the fields to have different phase functions φ_e and φ_b. In this case (1.6.10) becomes

$$\mathbf{E}(\mathbf{x},\omega) = \boldsymbol{E}(\mathbf{x},\omega_1)\, e^{j\varphi_e(\mathbf{x},\omega_1)}\ 2\pi\delta(\omega-\omega_1)$$
$$\mathbf{B}(\mathbf{x},\omega) = \boldsymbol{B}(\mathbf{x},\omega_1)\, e^{j\varphi_b(\mathbf{x},\omega_1)}\ 2\pi\delta(\omega-\omega_1) \ . \qquad (1.6.14)$$

Suppose the directions of the **E** and **B** fields are $\hat{\mathbf{n}}_e$ and $\hat{\mathbf{n}}_b$. Then (1.6.14) says

$$E_{n_e}(\mathbf{x},\omega) = E_{n_e}(\mathbf{x},\omega_1)\, e^{j\varphi_e(\mathbf{x},\omega_1)}\ 2\pi\delta(\omega-\omega_1)$$
$$B_{n_b}(\mathbf{x},\omega) = B_{n_b}(\mathbf{x},\omega_1)\, e^{j\varphi_b(\mathbf{x},\omega_1)}\ 2\pi\delta(\omega-\omega_1) \qquad (1.6.15)$$

and one finds that

$$\frac{E_{n_e}(\mathbf{x},\omega)}{B_{n_b}(\mathbf{x},\omega)} = \frac{E_{n_e}(\mathbf{x},\omega_1)}{B_{n_b}(\mathbf{x},\omega_1)}\ e^{j[\varphi_e(\mathbf{x},\omega_1)-\varphi_b(\mathbf{x},\omega_1)]} \ . \qquad (1.6.16)$$

Since $\boldsymbol{E}$ and $\boldsymbol{B}$ are real, the phase of the ratio E_{n_e}/B_{n_b} is determined by the last factor and will in general be a function of both position **x** and frequency ω_1.

We shall see this situation arise in the Chapter 2 when we calculate the fields inside a conducting round wire (see (2.3.14) and (2.3.15)) .

(e) A Pitfall to Avoid

Notice that

$$\mathbf{E}(\mathbf{x},t) = \mathbf{e}(\mathbf{x},t) + j\,\mathbf{e}'(\mathbf{x},t) \quad \Rightarrow$$

$$\mathbf{E}(\mathbf{x},\omega) = \int_{-\infty}^{\infty} dt\, \mathbf{E}(\mathbf{x},t)\, e^{-j\omega t} = \int_{-\infty}^{\infty} dt\, [\mathbf{e}(\mathbf{x},t) + j\,\mathbf{e}'(\mathbf{x},t)]\, e^{-j\omega t}$$

$$= \mathbf{e}(\mathbf{x},\omega) + j\,\mathbf{e}'(\mathbf{x},\omega)\,. \tag{1.6.17}$$

Whereas $\mathbf{e}(\mathbf{x},t)$ and $\mathbf{e}'(\mathbf{x},t)$ are the real and imaginary parts of $\mathbf{E}(\mathbf{x},t)$, the functions $\mathbf{e}(\mathbf{x},\omega)$ and $\mathbf{e}'(\mathbf{x},\omega)$ are *not* the real and imaginary parts of $\mathbf{E}(\mathbf{x},\omega)$ since in general $\mathbf{e}(\mathbf{x},\omega)$ and $\mathbf{e}'(\mathbf{x},\omega)$ are both complex functions. In this document we shall never deal with transforms of the type $\mathbf{e}(\mathbf{x},\omega)$ or $\mathbf{e}'(\mathbf{x},\omega)$.

(f) Maxwell's Equations in ω space

In the Maxwell and related equations which include the ∂_t operator, if the fields are expanded onto their Fourier transformed components using (1.6.8), then using the rule (1.6.9) one may instantly write the frequency-domain version of these equations, just as in the example of Section 1.5. For example,

$$\text{curl}\ \mathbf{H}(\mathbf{x},\omega) = j\omega\mathbf{D}(\mathbf{x},\omega) + \mathbf{J}(\mathbf{x},\omega) \tag{1.6.18}$$

$$\text{curl}\ \mathbf{E}(\mathbf{x},\omega) = -j\omega\mathbf{B}(\mathbf{x},\omega) \tag{1.6.19}$$

$$\text{div}\ \mathbf{J}(\mathbf{x},\omega) = -j\omega\,\rho(\mathbf{x},\omega) \quad . \tag{1.6.20}$$

Other equations in the Section 1.1 list have the same form but in terms of the frequency-domain functions. For example,

$$\mathbf{J}(\mathbf{x},\omega) = \sigma(\mathbf{x})\,\mathbf{E}(\mathbf{x},\omega) \tag{1.6.21}$$

where we momentarily allow $\sigma(\mathbf{x})$ to have spatial dependence but not time dependence.