Confusion with complex E and B PhL 9.24.13

I was truly confused, and the notes below led me to get things straightened out. The subject is now fully documented in Section 1.7 of lines.doc. This little working doc turned out to be very useful. There is no need to read it ever again, but I will keep it for the record.

Big Bug just found. In my Fourier Transform stuff in general one must have

$$\mathbf{B}(\mathbf{x},\omega) = \int_{-\infty}^{\infty} dt\, e^{-j\omega t}\, \mathbf{B}(\mathbf{x},t)$$

and although $\mathbf{B}(\mathbf{x},t)$ is real, we know that $\mathbf{B}(\mathbf{x},\omega)$ is in general complex, so my little arrows in my pictures showing real quantities don't mean much! On the other hand, one could say

$$\mathbf{B}(\mathbf{x},t) = e^{i\omega t}\, \mathbf{B}(\mathbf{x})$$

but then again $\mathbf{B}(\mathbf{x},t)$ is complex. I have not clarified this always-murky idea. So today will be lost getting this mess cleaned up. But now it is time for caulk and paint! // Am back. We like the $e^{i\omega t}$ idea because it makes for simple rules on ∂_t and so on. The theory of course is that

$$\mathbf{B}(\mathbf{x},t) = \mathrm{Re}[\, e^{i\omega t}\, \mathbf{B}(\mathbf{x})\,] = \cos(\omega t)\, \mathbf{B}(\mathbf{x}) \quad = \text{real times real.}$$

But how does this pass through differential equations like those of Maxwell. Consider:

$$\mathrm{curl}\, \mathbf{E}(\mathbf{x},t) = -\, \partial \mathbf{B}(\mathbf{x},t)/\partial t \qquad \text{ok}$$

In full FT this does say

$$\mathrm{curl}\, \mathbf{E}(\mathbf{x},\omega) = i\omega \mathbf{B}(\mathbf{x},\omega) \qquad \text{ok}$$

Ouch. This has the feel of a whole new Appendix once I get this cleaned up which will take a while.

As I look through lines before section 2.1, I am fine with equations having $\mathbf{E}(\mathbf{x},\omega)$ which is a complex entity. Where I have trouble is in those Stoke's loops. Look at (2.1.1) and (2.1.2). I think as equations these are all just fine, and express the relations between complex vectors.

$$\mathrm{curl}\, \mathbf{B}(\mathbf{x},\omega) = \mu\, (j\omega\varepsilon\, \mathbf{E}(\mathbf{x},\omega) + \mathbf{J}(\mathbf{x},\omega)\,) \qquad \oint \mathbf{B}(\mathbf{x},\omega)\bullet\mathbf{ds} = \mu \int (j\omega\varepsilon \mathbf{E}(\mathbf{x},\omega) + \mathbf{J}(\mathbf{x},\omega)\,)\bullet\mathbf{dA} \qquad (2.1.1)$$

$$\mathrm{curl}\, \mathbf{E}(\mathbf{x},\omega) = -\, j\omega \mathbf{B}(\mathbf{x},\omega) \qquad \oint \mathbf{E}(\mathbf{x},\omega)\bullet\mathbf{ds} = -j\omega \int \mathbf{B}(\mathbf{x},\omega)\bullet\mathbf{dA} \quad . \quad \text{all ok} \qquad (2.1.2)$$

Good Question: But how do you draw $\mathbf{E}(\mathbf{x},\omega)$ in a diagram if it is a complex vector? I don't think I have done this before. It has a direction like $\hat{\mathbf{z}}$, that part is OK, but the component in the z direction is complex.

So perhaps my drawings really show Re(**B**) and not **B** [no], say. But in (2.1.2) things are mixed. Go back to

$$\oint \mathbf{E}(\mathbf{x},t)\bullet\mathbf{ds} = -\partial_t \int \mathbf{B}(\mathbf{x},t)\bullet\mathbf{dA}$$

Now the vectors are real, but time derivative sits there. This is such an elementary level subject, I wonder if I have dealt with it somewhere before. If not, time to produce some notes.

Note: I think I resolved this issue of what the little arrows mean, added it to lines doc. Perhaps it will come up below in this doc. A complex vector can still have a direction!!! Like $\mathbf{E}(\mathbf{x},t) = E_z(\mathbf{x},t)\hat{\mathbf{z}}$.

Suppose we *assume* this for FT electric field vector: $\mathcal{E}(\mathbf{x},\omega) = \mathcal{E}(\mathbf{x})2\pi\delta(\omega-\omega_1)$

[but this form is not correct for the round wire which is more $\mathcal{E}(\mathbf{x},\omega) = \mathcal{E}_1(\mathbf{x},\omega)2\pi\delta(\omega-\omega_1)$]

Then: $\mathbf{E}(\mathbf{x},t) = e^{j\omega t}\,\mathcal{E}(\mathbf{x}) = (\cos\omega t + j\sin\omega t)(\text{Re}\,\mathcal{E}(\mathbf{x}) + j\,\text{Im}\,\mathcal{E}(\mathbf{x}))$

$$[\cos\omega t\,\text{Re}\,\mathcal{E}(\mathbf{x}) - \sin\omega t\,\text{Im}\,\mathcal{E}(\mathbf{x})] + j\,[\cos\omega t\,\text{Im}\,\mathcal{E}(\mathbf{x}) + \sin\omega t\,\text{Re}\,\mathcal{E}(\mathbf{x})]$$

If $\mathbf{E}(\mathbf{x},t)$ is real, this implies a relationship between Re and Im parts of $\mathcal{E}(\mathbf{x})$. [but don't want this E(x,t) to be real!]

Maybe I have started at the wrong point. In time domain write

$\mathbf{E}(\mathbf{x},t) = e^{j\omega_1 t}\,\mathbf{E}(\mathbf{x})$ (1) ok, a special assumed form

Note that $\mathbf{E}(\mathbf{x})$ might depend parametrically on ω_1

Think now of $\mathbf{E}(x)$ as being real [ok], and then $\mathbf{E}(\mathbf{x},t)$ is the thing that is complex [ok] , and then

actual field = Re $\mathbf{E}(\mathbf{x},t) = \cos\omega t\,\mathbf{E}(x)$ by fiat ok

Now do the FT on both sides of (1) to get

$\mathbf{E}(\mathbf{x},\omega) = 2\pi\delta(\omega-\omega_1)\,\mathbf{E}(x)$ (2) ok

Now you cannot give **B** the same phase because of Maxwell:

curl $\mathbf{E}(\mathbf{x},t) = -\,\partial\mathbf{B}(\mathbf{x},t)/\partial t$ claim stated is true, you cannot

So start over once again and postulate that

$\mathbf{B}(\mathbf{x},t) = e^{j\omega_1 t}\,\mathbf{B}(\mathbf{x})$ $\mathbf{B}(\mathbf{x})$ = real ok, but a completely new form

Then

curl $\mathbf{E}(\mathbf{x},t) = -i\omega_1 \mathbf{B}(\mathbf{x},t) = -i\omega_1 e^{j\omega_1 t} \mathbf{B}(\mathbf{x}) = \omega_1 e^{i\omega_1 t - \pi/2} \mathbf{B}(\mathbf{x})$ ok

I will continue this a bit. Suppose it were this simpler 1D equation

$\partial_x E(x,t) = -j\omega e^{i\omega t} B(x)$

What conclusions can one reach about E(x,t) just from this equation? After a short digression into PDE theory (see new folder in math on same), the general solution to the above is

$E(x,t) = \int^x dx\, [-j\omega e^{i\omega t} B(x)] + g(t) = -j\omega e^{i\omega t} \int^x dx\, B(x) + g(t)$.

One conclusion is that one possible solution of the equation is this

$E(x,t) = -j\omega e^{i\omega t} \int^x dx\, B(x)$

So I crudely claim that, apart from a possible arbitrary time function **G**(t), for which curl **G** = 0, the curl **E** equation above does imply that **E** is monochrome.

****

$$\frac{\partial E(r)}{\partial r} = j\omega B(r) . \qquad (2.1.9)$$

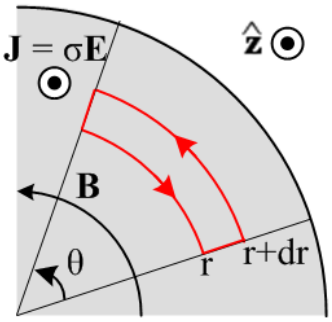
This does NOT imply that

$\frac{\partial |E(r)|}{\partial r} = \omega |B(r)|$ correct, ∂_r and | | do not commute

Example: $\partial_r |e^{jr}| = \partial_r 1 = 0$, whereas $|\partial_r e^{jr}| = |j\, e^{jr}| = 1$.

so I really cannot make any clean statement regarding skin effect. yes

Question: In this picture,



one might say " the vector **J** points in the $\hat{\mathbf{z}}$ direction " . What do those words mean when **J** has complex components? Even with real components, you don't really know whether J_z is positive or negative when you say this. It has to do with my right hand rules I think. $\mathbf{J} = J(r)\hat{\mathbf{z}}$ same issue raised earlier and resolved in lines doc.

****

OK, I have resolved this issue and the resolution is explained in a simple comment in transmission lines doc. The phrase" **B** points in the z direction" is meaningful even if **B** is complex. See that comment. The FT has survived, the world is back in joint. yes!

********** ok to here

Let's try again, here is an imagined new section to stuff somewhere. One quickly runs out of symbols to represent different quantities, I will give it a try though.

Appendix I: Dealing with Fourier Transforms of electric and magnetic fields.

1. Electric Field

Let the real, physical electric field be $\mathbf{e}(\mathbf{x},t)$. We can write

$$\mathbf{e}(\mathbf{x},\omega) = \int_{-\infty}^{\infty} dt\, e^{-j\omega t}\, \mathbf{e}(\mathbf{x},t) \qquad \mathbf{e}(\mathbf{x},\omega) = \text{volt-sec/m} \qquad \text{(I.1.1)}$$ ok

$$\mathbf{e}(\mathbf{x},t) = (1/2\pi) \int_{-\infty}^{\infty} dt\, e^{j\omega t}\, \mathbf{e}(\mathbf{x},\omega) \qquad \mathbf{e}(\mathbf{x},t) = \text{volts/m} \qquad \text{(I.1.2)}$$ ok

In this transform, $\mathbf{e}(\mathbf{x},\omega)$ is complex as a general rule (ie, for some general real $\mathbf{e}(\mathbf{x},t)$).

As a special case, one can consider a special $\mathbf{e}(\mathbf{x},t)$ that is sinusoidal in time (monochromatic) with a phase that may depend on spatial location (as I think in the round wire situation)

$$\mathbf{e}(\mathbf{x},t) = \cos[\omega_e t + \varphi_e(\mathbf{x})]\, \mathbf{e}(\mathbf{x}) \qquad \mathbf{e}(\mathbf{x}) = \text{volts/m} \qquad \text{(I.1.3)}$$ ok

where all functions are real. Now suppose we define *complex* $\mathbf{E}(\mathbf{x},t)$ as,

$$\mathbf{E}(\mathbf{x},t) \equiv e^{j(\omega_e t+\varphi_e(\mathbf{x}))}\,\mathbf{e}(\mathbf{x})\ . \qquad \mathbf{E}(\mathbf{x},t)\ = \text{volts/m} \qquad \text{(I.1.4)}\quad \text{ok}$$

If one is given $\mathbf{E}(\mathbf{x},t)$, one can always recover the physical field using

$$\mathbf{e}(\mathbf{x},t) = \text{Re}\{\,\mathbf{E}(\mathbf{x},t)\}. \qquad \text{(I.1.5)}\quad \text{ok}$$

Now we can consider this Fourier Transform,

$$\mathbf{E}(\mathbf{x},\omega) = \int_{-\infty}^{\infty} dt\, e^{-j\omega t}\,\mathbf{E}(\mathbf{x},t) \qquad \mathbf{E}(\mathbf{x},\omega) = \text{volt-sec/m} \qquad \text{(I.1.6)}\quad \text{ok}$$

$$\mathbf{E}(\mathbf{x},t) = (1/2\pi)\int_{-\infty}^{\infty} dt\, e^{j\omega t}\,\mathbf{E}(\mathbf{x},\omega) \qquad \mathbf{E}(\mathbf{x},t)\ = \text{volts/m} \qquad \text{(I.1.7)}\quad \text{ok}$$

where now both $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{E}(\mathbf{x},\omega)$ are (in general) complex. Were we to somehow compute $\mathbf{E}(\mathbf{x},\omega)$, we could use (I.1.7) to get $\mathbf{E}(\mathbf{x},t)$ and then (I.1.5) to get the physical $\mathbf{e}(\mathbf{x},t)$.

In the special monochromatic case (I.1.4),

$$\mathbf{E}(\mathbf{x},t) \equiv e^{j(\omega_e t+\varphi_e(\mathbf{x}))}\,\mathbf{e}(\mathbf{x})\ . \qquad \mathbf{E}(\mathbf{x},t)\ = \text{volts/m} \qquad \textit{(I.1.4)}\quad \text{ok}$$

we find that

$$\mathbf{E}(\mathbf{x},\omega) = \int_{-\infty}^{\infty} dt\, e^{-j\omega t}\,\mathbf{E}(\mathbf{x},t)\ = e^{j\varphi_e(\mathbf{x})} \qquad \mathbf{e}(\mathbf{x}) \qquad 2\pi\delta(\omega-\omega_1) \qquad \text{(I.1.9)}\quad \text{ok}$$

volt-sec/m		volt/m	sec
complex	complex	real	real

The above line now appears in lines doc as (1.7.10) but with ω_1 dependence of φ and **e** shown.

We shall now repeat this section for the magnetic field, putting in appropriate units. The monochromatic frequency will be taken ad ω_b and phase φ_b.

2. Magnetic Field

Let the real, physical electric field be $\mathbf{b}(\mathbf{x},t)$. We can write

$$\mathbf{b}(\mathbf{x},\omega) = \int_{-\infty}^{\infty} dt\, e^{-j\omega t}\,\mathbf{b}(\mathbf{x},t) \qquad \mathbf{b}(\mathbf{x},\omega) = \text{tesla-sec} \qquad \text{(I.2.1)}\quad \text{ok}$$

$$\mathbf{b}(\mathbf{x},t) = (1/2\pi)\int_{-\infty}^{\infty} dt\, e^{j\omega t}\,\mathbf{b}(\mathbf{x},\omega) \qquad \mathbf{b}(\mathbf{x},t) = \text{tesla} \qquad \text{(I.2.2)}\quad \text{ok}$$

In this transform, $\mathbf{b}(\mathbf{x},\omega)$ is complex as a general rule (ie, for some general real $\mathbf{b}(\mathbf{x},t)$).

As a special case, one can consider a special $\mathbf{b}(\mathbf{x},t)$ that is sinusoidal in time (monochromatic) with a phase that may depend on spatial location (as I think in the round wire situation)

$$\mathbf{b}(\mathbf{x},t) = \cos(\omega_b t + \varphi_b(x))\, \mathbf{b}(\mathbf{x}) \qquad \mathbf{b}(\mathbf{x}) \text{ = tesla} \qquad \text{(I.2.3)} \quad \text{ok}$$

where here all three functions are real. Now suppose we define complex $\mathbf{B}(\mathbf{x},t)$ as,

$$\mathbf{B}(\mathbf{x},t) \equiv e^{j(\omega_b t+\varphi_b(x))}\, \mathbf{b}(\mathbf{x}) \ . \qquad \mathbf{B}(\mathbf{x},t) \text{ = tesla} \qquad \text{(I.2.4)} \quad \text{ok}$$

If one is given $\mathbf{B}(\mathbf{x},t)$, one can always recover the physical field using

$$\mathbf{b}(\mathbf{x},t) = \mathrm{Re}\{\, \mathbf{B}(\mathbf{x},t)\, \}. \qquad \text{(I.2.5)} \quad \text{ok}$$

Now we can consider this Fourier Transform,

$$\mathbf{B}(\mathbf{x},\omega) = \int_{-\infty}^{\infty} dt\, e^{-j\omega t}\, \mathbf{B}(\mathbf{x},t) \qquad \mathbf{B}(\mathbf{x},\omega) \text{ = tesla-sec} \qquad \text{(I.2.6)} \quad \text{ok}$$

$$\mathbf{B}(\mathbf{x},t) = (1/2\pi) \int_{-\infty}^{\infty} dt\, e^{j\omega t}\, \mathbf{B}(\mathbf{x},\omega) \qquad \mathbf{B}(\mathbf{x},t) \text{ = tesla} \qquad \text{(I.2.7)} \quad \text{ok}$$

where now both $\mathbf{B}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},\omega)$ are (in general) complex. Were we to somehow compute $\mathbf{B}(\mathbf{x},\omega)$, we could use (I.2.7) to get $\mathbf{B}(\mathbf{x},t)$ and then (I.2.5) to get the physical $\mathbf{b}(\mathbf{x},t)$.

In the special monochromatic case (I.2.4),

$$\mathbf{B}(\mathbf{x},t) \equiv e^{j(\omega_b t+\varphi_b(x))}\, \mathbf{b}(\mathbf{x}) \ . \qquad \mathbf{B}(\mathbf{x},t) \text{ = tesla} \qquad \textit{(I.2.4} \quad \text{ok}$$

we find that

$$\mathbf{B}(\mathbf{x},\omega) = \int_{-\infty}^{\infty} dt\, e^{-j\omega t}\, \mathbf{B}(\mathbf{x},t) \ = e^{j\varphi_b(x)} \quad \mathbf{b}(\mathbf{x}) \quad 2\pi\delta(\omega-\omega_1) \qquad \text{(I.2.9)} \quad \text{ok}$$

tesla-sec			tesla	sec
complex		complex	real	real

Comment: In the above two sections I have just given names to things so we can tell what is what, and know what is real and what is not real. I am at this point unsure whether you can insist that both these complex field relations be true at the same time when σ is present [you can!]

$$\mathbf{B}(\mathbf{x},t) \equiv e^{j(\omega_b t+\varphi_b(x))}\, \mathbf{b}(\mathbf{x}) \qquad \mathbf{b} \text{ real}$$
$$\mathbf{E}(\mathbf{x},t) \equiv e^{j(\omega_e t+\varphi_e(x))}\, \mathbf{e}(\mathbf{x}) \qquad \mathbf{e} \text{ real} \qquad \text{(I.2.10)}$$

But we could assume both these forms are valid and see if we can find a solution. If a solution exists, then these claims are both justified at the same time. Notice that the above forms allow B_θ/E_z to have a phase which varies with space **x** (with radius r), something we observe in the round wire solution as now presented in lines doc.

3. Implications of the electric field Maxwell curl equation

Now consider the physical Maxwell curl equation:

$$\text{curl } \mathbf{e}(\mathbf{x},t) = -\partial_t \mathbf{b}(\mathbf{x},t) \qquad \text{(I.3.1)} \quad \text{ok}$$

If we apply $\int_{-\infty}^{\infty} dt\, e^{-j\omega t}$ to both sides, we obtain [might want to show his this works with parts]

$$\text{curl } \mathbf{e}(\mathbf{x},\omega) = -j\omega\, \mathbf{b}(\mathbf{x},\omega) \qquad \text{(I.3.2)} \quad \text{ok}$$

Now we wonder if there is meaning to the curl equation (I.3.1) for our "complex fields":

$$\text{curl } \mathbf{E}(\mathbf{x},t) = -\partial_t \mathbf{B}(\mathbf{x},t) \qquad ?? \qquad \text{(I.3.3)}$$

If we decompose **E** and **B** into real and imaginary part we have

$$\text{curl } \{ \text{Re}[\mathbf{E}(\mathbf{x},t)] + j\, \text{Re}[\mathbf{E}(\mathbf{x},t)] \} = -\partial_t \{\text{Re}[\mathbf{E}(\mathbf{x},t)] + j\, \text{Re}[\mathbf{E}(\mathbf{x},t)] \}$$

Since multiplication by j commutes with curl and ∂_t, we can equate real and imaginary parts to get

$$\begin{aligned} &\text{curl } \{ \text{Re}[\mathbf{E}(\mathbf{x},t)] \} = -\partial_t \{\text{Re}[\mathbf{B}(\mathbf{x},t)] \} \\ &\text{curl } \{ \text{Im}[\mathbf{E}(\mathbf{x},t)] \} = -\partial_t \{\text{Im}[\mathbf{B}(\mathbf{x},t)] \} \end{aligned} \qquad \text{(I.3.4)}$$

These are both "physical" curl equations based on our choice of the physical fields. Perhaps write these as

$$\begin{aligned} &\text{curl } \mathbf{e}(\mathbf{x},t) = -\partial_t \mathbf{b}(\mathbf{x},t) && \mathbf{e}(\mathbf{x},t) = \text{Re}[\mathbf{E}(\mathbf{x},t)] && \mathbf{b}(\mathbf{x},t) = \text{Re}[\mathbf{B}(\mathbf{x},t)] \\ &\text{curl } \mathbf{e}'(\mathbf{x},t) = -\partial_t \mathbf{b}'(\mathbf{x},t) && \mathbf{e}'(\mathbf{x},t) = \text{Im}[\mathbf{E}(\mathbf{x},t)] && \mathbf{b}'(\mathbf{x},t) = \text{Im}[\mathbf{B}(\mathbf{x},t)] \end{aligned} \qquad \text{(I.3.5)}$$

Thus our "complex field curl equation" (I.3.3) can be regarded as a combination of "physical field curl equations" for two different sets of physical fields [yes]. Thus, (I.33) is completely justified in terms of the complex fields shown, and the same is then true for the other Maxwell equations and in fact any other equations involving E and B fields such as $J = \sigma E$. [yes]

So we know exactly how to get from the complex fields E and B to either set of physical fields.

Now what happens when we do a Fourier transform to the complex fields?

Consider

$$\int_{-\infty}^{\infty} dt\, e^{-j\omega t}\ \mathbf{E}(\mathbf{x},t) \;=\; \int_{-\infty}^{\infty} dt\, e^{-j\omega t}\ [\mathbf{e}(\mathbf{x},t) + j\ \mathbf{e}'(\mathbf{x},t)]$$

which tells us that

$$\mathbf{E}(\mathbf{x},\omega) = \mathbf{e}(x,\omega) + j\ \mathbf{e}'(x,\omega)$$
$$\mathbf{B}(\mathbf{x},\omega) = \mathbf{b}(x,\omega) + j\ \mathbf{b}'(x,\omega)$$

These equations have the same form as

$$\mathbf{E}(\mathbf{x},t) = \mathbf{e}(x,t) + j\ \mathbf{e}'(x,t)$$
$$\mathbf{B}(\mathbf{x},t) = \mathbf{b}(x,t) + j\ \mathbf{b}'(x,t)$$

but there is a big difference. In the second set of equations, we can identify the pieces with the real and imaginary part of the $\mathbf{F}(x,t)$ fields, whereas in the first set of equations since all functions are in general complex, we cannot say for example that $\mathbf{e}(x,\omega)$ is the real part of $\mathbf{E}(\mathbf{x},\omega)$. Just a small warning to be careful. [correct]

Now we can compare different versions of the FT of a Maxwell equation. For example, apply $\int_{-\infty}^{\infty} dt\, e^{-j\omega t}$ to both sides of (I.3.3) to get

$$\text{curl}\ \mathbf{E}(\mathbf{x},\omega) = -\,j\omega\mathbf{B}(\mathbf{x},\omega) \qquad \textit{(I.3.6)}$$

which we can write as

$$\text{curl}\ \{\ \mathbf{e}(x,\omega) + j\ \mathbf{e}'(x,\omega)\} = -\,j\omega\{\ \mathbf{b}(x,\omega) + j\ \mathbf{b}'(x,\omega)\}$$

or

$$\text{curl}\ \mathbf{e}(\mathbf{x},\omega) + j\ \text{curl}\ \mathbf{e}'(\mathbf{x},\omega) = -j\omega\ \mathbf{b}(x,\omega) + \omega\ \mathbf{b}'(x,\omega)$$

We cannot at this point claim that curl $\mathbf{e}(\mathbf{x},\omega) = -j\omega\ \mathbf{b}(x,\omega)$ for example! So just be careful! I think there will never be any need to use the functions like $\mathbf{e}(x,\omega)$. We will use only $\mathbf{E}(\mathbf{x},\omega)$. [correct]

Now let's assume the monochromatic situation for both electric and magnetic fields,

$$\mathbf{E}(\mathbf{x},t) \equiv e^{j(\omega_e t+\varphi_e)}\ \mathbf{e}(\mathbf{x})\ . \qquad \mathbf{E}(\mathbf{x},t) = \text{volts/m} \qquad \textit{(I.1.4)}$$
$$\mathbf{B}(\mathbf{x},t) \equiv e^{j(\omega_b t+\varphi_b)}\ \mathbf{b}(\mathbf{x})\ . \qquad \mathbf{B}(\mathbf{x},t) = \text{tesla} \qquad \textit{(I.2.4)}$$

Inserting these into (I.3.3) gives

$$e^{j(\omega_e t+\varphi_e)}\ \text{curl}\ \mathbf{e}(\mathbf{x}) \;=\; -\,\partial_t[e^{j(\omega_b t+\varphi_b)}\ \mathbf{b}(\mathbf{x})\ ] = -j\omega_b\, e^{j(\omega_b t+\varphi_b)}\ \mathbf{b}(\mathbf{x})$$

[wrong! In general $\varphi_e = \varphi_e(\mathbf{x})$ so does not pass through the curl !]

$$= \omega_b \, e^{j(\omega_b t + \varphi_b - \pi/2)} \, \mathbf{b}(\mathbf{x})$$

or

$$\text{curl } \mathbf{e}(\mathbf{x}) = \omega_b \, e^{j([\omega_b - \omega_e]t + [\varphi_b - \varphi_e] - \pi/2)} \, \mathbf{b}(\mathbf{x}) \qquad \text{// wrong}$$

Since curl $\mathbf{e}(\mathbf{x})$, ω_b and $\mathbf{b}(\mathbf{x})$ are all real, the phasor must be unity. For this to be true for all time, we clearly need to have $\omega_b = \omega_e$ and $\varphi_b - \varphi_e = \pi/2$. Without loss of generality, we shall set $\varphi_e = 0$ and let the frequencies both be ω_1 and $\varphi_b = \pi/2$. We then have this monochromatic set of complex fields

$$\mathbf{E}(\mathbf{x},t) \equiv e^{j\omega_1 t} \, \mathbf{e}(\mathbf{x}) \qquad \mathbf{E}(\mathbf{x},t) = \text{volts/m}$$
$$\mathbf{B}(\mathbf{x},t) \equiv e^{j(\omega_1 t + \pi/2)} \, \mathbf{b}(\mathbf{x}) \; . \qquad \mathbf{B}(\mathbf{x},t) = \text{tesla} \qquad \text{(I.3.5)}$$

and the curl equation may be written

$$\text{curl } \mathbf{e}(\mathbf{x}) = \omega_1 \, \mathbf{b}(\mathbf{x}) \qquad \text{(I.3.6)}$$

where $\mathbf{e}(\mathbf{x})$ and $\mathbf{b}(\mathbf{x})$ are real fields. The physical fields are these

$$\mathbf{e}(\mathbf{x},t) = \cos(\omega_1 t) \, \mathbf{e}(\mathbf{x}) \qquad \mathbf{e}(\mathbf{x}) = \text{volts/m} \qquad \text{(I.1.3)}$$
$$\mathbf{b}(\mathbf{x},t) = \cos(\omega_1 t + \pi/2) \, \mathbf{b}(\mathbf{x}) \qquad \mathbf{b}(\mathbf{x}) = \text{tesla} \qquad \text{(I.2.3)}$$

The magnetic field leads the electric field by phase of $\pi/2$. [wrong]

Fact: In any monochromatic electromagnetic field situation such as free space waves, reflection, boundary conditions, transmission lines, etc. the time-domain electric and magnetic fields at any point always have this rigid phase relationship due to the electric curl equation. [wrong!] (I.3.7)

What about ω domain stuff?

Question: How does all this affect my Section 2.1 and 2.2?

Here are Maxwell's equations in terms of the complex fields.

$$\text{curl } \mathbf{b}(\mathbf{x},t) = \mu \, \mathbf{j}(\mathbf{x},t) \qquad \text{// these are for the actual physical fields}$$

$$\text{curl } \mathbf{e}(\mathbf{x},t) = -\partial \mathbf{b}(\mathbf{x},t)/\partial t$$

$$\mathbf{j}(\mathbf{x},t) = \sigma \mathbf{e}(\mathbf{x},t)$$

$$\text{curl } \mathbf{B}(\mathbf{x},t) = \mu \, \mathbf{J}(\mathbf{x},t) \qquad \text{// these are for the complex fields}$$

$$\text{curl } \mathbf{E}(\mathbf{x},t) = -\partial \mathbf{B}(\mathbf{x},t)/\partial t$$

$$\mathbf{J}(\mathbf{x},t) = \sigma \mathbf{E}(\mathbf{x},t)$$

where we have [these are not general since possible phases $\varphi_i(\mathbf{x},\omega_1)$ are not included.]

but ok, suppose you DID have these limited forms, what conclusions would you reach?

$\mathbf{E}(\mathbf{x},t) \equiv e^{j\omega_1 t}\,\mathbf{e}(\mathbf{x})$ $\qquad$ $\mathbf{e}(\mathbf{x},t) = \mathrm{Re}\{\mathbf{E}(\mathbf{x},t)\} = \cos(\omega_1 t)\,\mathbf{e}(\mathbf{x})$

$\mathbf{J}(\mathbf{x},t) \equiv e^{j\omega_1 t}\,\mathbf{j}(\mathbf{x})$ $\qquad$ $\mathbf{J}(\mathbf{x},t) = \mathrm{Re}\{\mathbf{J}(\mathbf{x},t)\} = \cos(\omega_1 t)\,\mathbf{j}(\mathbf{x})$

$\mathbf{B}(\mathbf{x},t) \equiv e^{j(\omega_1 t+\pi/2)}\,\mathbf{b}(\mathbf{x})$ $\qquad$ $\mathbf{B}(\mathbf{x},t) = \mathrm{Re}\{\mathbf{B}(\mathbf{x},t)\} = \cos(\omega_1 t+\pi/2)\,\mathbf{j}(\mathbf{x})$

Also we know that

$\mathbf{E}(\mathbf{x},\omega) = \mathbf{e}(\mathbf{x})\,2\pi\delta(\omega-\omega_1)$ (I.1.9)

$\mathbf{B}(\mathbf{x},\omega) = e^{j\pi/2}\,\mathbf{b}(\mathbf{x})\,2\pi\delta(\omega-\omega_1) = j\,\mathbf{b}(\mathbf{x})\,2\pi\delta(\omega-\omega_1)$ (I.2.9)

Now try to translate this result from the round wire analysis,

$$\frac{\partial E(r)}{\partial r} = j\omega B(r)\,. \qquad (2.1.9)$$

$\partial_r\,\mathbf{E}(\mathbf{x},\omega) = j\omega\,\mathbf{B}(\mathbf{x},\omega)$ $\quad$ where $\mathbf{E}(\mathbf{x},\omega) = E(r)\hat{\mathbf{z}}$ $\quad$ and $\quad$ $\mathbf{B}(\mathbf{x},\omega) = B(r)\hat{\boldsymbol{\theta}}$

Now put in the results above to get

$\partial_r\,[\mathbf{e}(\mathbf{x})\,2\pi\delta(\omega-\omega_1)] = j\omega\,[j\;\mathbf{b}(\mathbf{x})\,2\pi\delta(\omega-\omega_1)\,]$

or

$\partial_r\,\mathbf{e}(\mathbf{x}) = -\omega\,\mathbf{b}(\mathbf{x})$

or

$\partial_r\,e(r) = -\omega\,b(r)$

Now everything is real and maybe I can really say that $\partial_r e(r) < 0$ which seems the wrong conclusion!

Given our limited assumed forms, yes, $\partial_r e(r) < 0$ if $b(r) > 0$, but those limited forms simply are not compatible with the round wire problem!

This is leading to many new questions and problems. Consider my solution

$E(r) = f(x) = C\,J_0(\beta r)$ $\qquad$ $\beta = e^{j3\pi/4}\sqrt{\omega\mu\sigma}$ (2.1.18)

I now think this has a constant phase! [wrong]

Look at these results:

$E(r) = E(a)\,[J_0(\beta r) / J_0(\beta a)]$ $\qquad$ *(2.1.22)*

$B(r) = -(\beta/j\omega)\, E(a)\, [J_1(\beta r) / J_0(\beta a)]$

$B(r)/E(r) = -(\beta/j\omega)\, [J_1(\beta r) / J_0(\beta a)] / [J_0(\beta r) / J_0(\beta a)]$

$= -(\beta/j\omega)\, [J_1(\beta r)/ J_0(\beta r)]$

I CLAIM, but want to verify it, that the ratio $[J_1(\beta r)/ J_0(\beta r)]$ must have a constant phase as a function of ω. [wrong] It just seems unlikely [yup!], but Maple can tell me. Let's write $\beta = e^{j3\pi/4}\, b$ where b is real. What does Maple have to say here?

Aside: Let

$\beta = e^{j3\pi/4}\, b$
$J_\nu(\beta r) = J_\nu(e^{j3\pi/4}\, br)$

The $I_\nu(z) = (-i)^\nu\, J_\nu(iz)$ so this is NOT the right phase! So we don't just have I or K functions.

Well, here is what Maple says about my so-called constant phase:

```
restart; with(plots):
beta := exp(I*3*Pi/4)*b;
```

$$\beta := \left(-\frac{1}{2}\sqrt{2} + \frac{1}{2} I \sqrt{2}\right) b$$

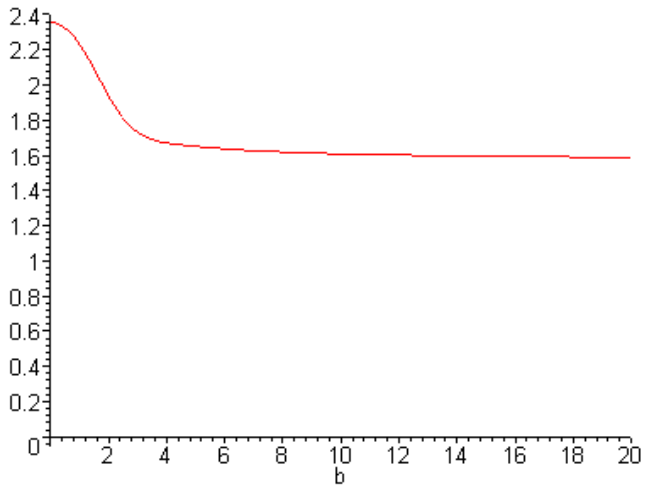
```
rat := BesselJ(1,beta*r)/BesselJ(0,beta*r);
```

$$rat := \frac{\mathrm{BesselJ}\left(1, \left(-\frac{1}{2}\sqrt{2} + \frac{1}{2} I \sqrt{2}\right) b\, r\right)}{\mathrm{BesselJ}\left(0, \left(-\frac{1}{2}\sqrt{2} + \frac{1}{2} I \sqrt{2}\right) b\, r\right)}$$

```
r := 1;
```

$$r := 1$$

```
plot(argument(evalf(rat)),b = 0..20);
```



So now I have a serious contradiction.

- E and B in ω space must have a constant phase shift of 90 degrees [wrong!]
- E/B in the above specific case shows this is not true. [right!]

Is there something wrong with my set of assumptions for the round wire? Consider for example the divergence equation inside the conductor (recall no charge there).

$$\text{div } \mathbf{E} = 0$$ ok

Recall $\mathbf{E} = \mathbf{E}(\mathbf{x},\omega) = E(r)\hat{\mathbf{z}}$. Looking at the cyl formula for div we really do get div $\mathbf{E} = 0$ with our assumptions because only E_z exists, but we have no variation in z. correct

What about the other divergence equation?

$$\text{div } \mathbf{B} = 0$$

This is also perfectly respected in our assumption set. So I think our wire solution inside the wire solves all four Maxwell's, so why do I get this discrepancy?

At ω = 0 for the wire, it seems the phase between E and B would be 0? But pretty vague.

Go back to these monochrome equations

$$\mathbf{E}(\mathbf{x},\omega) = \mathbf{e}(\mathbf{x})\, 2\pi\delta(\omega-\omega_1) \qquad \text{(I.1.9)}$$
$$\mathbf{B}(\mathbf{x},\omega) = e^{j\pi/2}\ \mathbf{b}(\mathbf{x})\, 2\pi\delta(\omega-\omega_1) = j\ \mathbf{b}(\mathbf{x})\, 2\pi\delta(\omega-\omega_1) \qquad \text{(I.2.9)}$$

I can translate these to the round wire to read

$$E(r)\,\hat{\mathbf{z}} = e(r)\,\hat{\mathbf{z}}\, 2\pi\delta(\omega-\omega_1) \qquad E(r) = e(r)\, 2\pi\delta(\omega-\omega_1)$$
$$B(r)\,\hat{\boldsymbol{\theta}} = e^{j\pi/2}\, b(r)\,\hat{\boldsymbol{\theta}}\, 2\pi\delta(\omega-\omega_1) \qquad B(r) = e^{j\pi/2} b(r)\, 2\pi\delta(\omega-\omega_1)$$

and therefore

$$B(r)/E(r) = e^{j\pi/2}$$

$$\text{argument}(B(r)/E(r)) = \pi/2$$

and this is what Maple is showing but not for small ω. [I am persisting in this π/2 error]

Maple says that for small ω, the phase goes to 3π/4 instead of π/2. I can surely show this as follows:

$$B(r)/E(r) = (\beta/j\omega)\,[J_1(\beta r)/\, J_0(\beta r)\,]$$

$\beta = e^{j3\pi/4}\sqrt{\omega\mu\sigma} = e^{j3\pi/4}\, b$

$J_0(\beta r) \approx 1$

$J_1(\beta r) \approx \beta r/2$

$B(r)/E(r) \approx (\beta/j\omega)\,[\beta r/2] \quad = \beta^2 r/(2j\omega) = [-(2j/\delta^2)]r/(2j\omega) = -(r/[\omega\delta^2])$

But here I get a phase if $-\pi$, which disagrees with everything else!

OK, after writing this I realized the business of phase $\varphi(x)$ being spatially dependent in the round wire problem and that removed all the above paradoxes. Today 9/30/13 I read through this entire doc and I think all mysteries are resolved, and Section 1.7 I hope clarifies everything.