Paradox with Jackson PhL 10.13.13

The point here is that you get different A_z wave equations if you select different gauges. I think I am happy with all this stuff nowadays.

1. Statement of the Paradox

(a) **On the one hand,** in lines doc, I have this derivation of the potential wave equation. I have copied the derivation but have set $\mathbf{J_a} = 0$ everywhere.

Start with (1.1.1)

$$\mathbf{curl\ H} = \partial\mathbf{D}/\partial t + \mathbf{J}$$

Replace H by $B/\mu = curlA/\mu$ to get

$$(1/\mu)\ \mathbf{curl\ curl\ A} = \partial\mathbf{D}/\partial t + \mathbf{J}$$

or

$$(1/\mu)\ \mathbf{curl\ curl\ A} = \varepsilon\ \partial\mathbf{E}/\partial t + \sigma\ \mathbf{E}$$

Now use the identity

$$\text{curl curl A} = \text{grad div}A - \nabla^2 A \qquad \text{// verified against vector identities doc}$$

and we get

$$\text{grad div}\mathbf{A} - \nabla^2\mathbf{A} = \mu\varepsilon\ \partial\mathbf{E}/\partial t + \mu\sigma\ \mathbf{E}$$

Next replace on the right

$$\mathbf{E} = -\ \text{grad}\ \varphi - \partial\mathbf{A}/\partial t$$

to get

$$\text{grad div}\mathbf{A} - \nabla^2\mathbf{A} = \mu\varepsilon\ [-\ \text{grad}\ \partial_t\varphi - \partial_t^2\mathbf{A}\ ] + [\ \mu\sigma\ (-\ \text{grad}\ \varphi - \partial\mathbf{A}/\partial t)\ ]$$

or just changing signs,

$$-\text{grad div}\mathbf{A} + \nabla^2\mathbf{A} = \mu\varepsilon\ [\text{grad}\ \partial_t\varphi + \partial_t^2\mathbf{A}\ ] + [\ \mu\sigma\ (\ \text{grad}\ \varphi + \partial\mathbf{A}/\partial t)\ ]$$

or

$$\nabla^2\mathbf{A} - \mu\varepsilon\ \partial_t^2\mathbf{A} - \mu\sigma\ \partial_t\mathbf{A} = \text{grad div}\mathbf{A} + \mu\varepsilon\ \text{grad}\ \partial_t\varphi + \mu\sigma\ \text{grad}\ \varphi$$

or

$$(\nabla^2 - \mu\varepsilon \partial_t^2 - \mu\sigma \partial_t)\mathbf{A} = \text{grad div}\mathbf{A} + \mu\varepsilon \text{ grad } \partial_t\varphi + \mu\sigma \text{ grad } \varphi$$

or

$$(\nabla^2 - \mu\varepsilon \partial_t^2 - \mu\sigma \partial_t)\mathbf{A} = \text{grad} [\text{div}\mathbf{A} + \mu\varepsilon \partial_t\varphi + \mu\sigma \varphi] \qquad (1.3.3)$$

Then if I use my King gauge condition 1.3.4

$$\text{div } \mathbf{A} = - \mu\varepsilon \partial_t\varphi - \mu\sigma \varphi \quad . \qquad (1.3.4)$$

I then end up with

$$(\nabla^2 - \mu\varepsilon \partial_t^2 - \mu\sigma \partial_t)\mathbf{A} = 0.$$

If after doing the above I take the limit $\sigma \rightarrow 0$, I get

$$(\nabla^2 - \mu\varepsilon \partial_t^2)\mathbf{A} = 0$$

$$\text{div } \mathbf{A} = - \mu\varepsilon \partial_t\varphi \, .$$

(b) **On the other hand**, green Jackson page 180 (6.38) tells me that

$$(\nabla^2 - \mu\varepsilon \partial_t^2)\mathbf{A} = -4\pi\mathbf{J} \qquad \text{// gaussian units}$$

These results conflict with each other, hence the paradox. And I have derived the Jackson results in a separate document.

2. Discussion of paradox

This is a very subtle little deal here. I got the impression in the Jackson case that we had $\sigma = 0$ so no conductivity in our medium. If that were true, then $\mathbf{J} = \sigma\mathbf{E} = 0$ and so $\mathbf{J} = 0$ and then there is no paradox, both methods agree that $(\nabla^2 - \mu\varepsilon \partial_t^2)\mathbf{A} = 0$. In order to have a **J**, you have to have something that conducts, like a wire! Inside that wire, $\sigma > 0$. When you apply Ampere's Law you do $\int \mathbf{J} \bullet d\mathbf{A}$ over a surface that includes a cross section of a wire, and THAT is where **J** is located.

Fact 1: So my first misconception is that in the Jackson world we had $\sigma = 0$. That is just not so. He is assuming ε_0 and μ_0 but he is not assuming $\sigma = 0$!

Fact 2: Therefore, our two gauge conditions are NOT the same:

$\text{div } \mathbf{A} = - \mu\varepsilon \partial_t\varphi - \mu\sigma \varphi$ my gauge condition (King)

$\text{div } \mathbf{A} = - \mu\varepsilon \partial_t\varphi$ his gauge condition (Lorenz)

Both are "legal" gauge conditions. If you use the first, you obtain a version of **A**,φ that does not see real currents as a driving term! If you use the second, you obtain a different version of A,φ that DOES see the current as a driving term!

Somehow this same mechanism causes ρ to appear in the potential equation.

So, I guess I need to rewrite my derivations with the proper gauge conditions! And don't replace J by σE so early, just leave it as J.