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Original version of these opening sections, all installed.

Overview

This monograph uses the Maxwell and associated potential equations to determine the behavior of infinitely-long, straight transmission lines. The presentation is loosely based on R.W.P. King's book *Transmission-Line Theory*. No attempt is made to address non-straight geometries, bends, stubs and many other practical applications described by King. There is no discussion of discontinuities, reflections, standing wave ratios, Smith charts, or any of the traditional topics associated with transmission lines. The emphasis is more on how one derives the transmission line parameters R,L,G,C directly from electromagnetic theory, and what approximations are made in doing so. A key requirement is that the wavelength of a transmission line wave be significantly larger than the line's transverse dimensions, something we refer to as the "transmission line limit". Although the discussion generally concerns transmission lines with 2 conductors, comments here and there show how the conclusions can be extended to transmission lines with more than 2 conductors.

Unlike the case of waveguides, transmission lines are most easily analyzed using potentials rather than fields due to the nature of the boundary conditions. This then brings up the can of worms known as "the gauge condition". We show how a variant of the Lorenz gauge which we call "the King gauge" (since King uses it) serves to clarify the meaning of the Helmholtz integrals for the scalar and vector potentials over the surface and interior of the transmission line conductors. This subject is somewhat glossed over in King's highly compressed theoretical presentation, and we could not find clarification in his many other books on the subject. By the way, most books on "transmission lines" are concerned with the practical aspects of electrical power distribution and King's book is somewhat of a rarity.

An ancillary topic receiving much attention in this document is the description of the fields, potentials and currents *inside* a transmission line conductor operating at angular frequency ω. Mainly the discussion concerns round wires. A uniform round wire seems a simple physical situation, yet the analysis is quite complicated and involves the so-called Kelvin functions. The skin effect and surface impedance of such a wire are considered in detail.

There are very few "it can be shown" phrases in this document. Almost everything is derived in detail and the results verified against external sources. Simple examples are always presented and calculations for these examples are fully displayed, perhaps to a level of detail the reader will find annoying. Our view is that a piece of theory is useless if one cannot apply it to a simple case and get a reasonable result. This view is not shared by all practitioners in the sciences.

The reader is assumed to have some knowledge of ordinary and partial differential equations and associated calculus. Green's Functions (which we call propagators) appear frequently, since these are useful in solving differential equations, and details are provided for readers not familiar with this subject. In particular, our first major waypoint is the derivation of the transmission line potentials in the form of King's Helmholtz integrals as shown in (1.5.23). The propagators in these integrals are the 3D Helmholtz free-space fundamental solutions $e^{-j\beta R}/R$. This subject is fully laid out for the interested reader in

Appendices H and I for the 3D and 2D Helmholtz partial differential equations which are the frequency domain Fourier transforms of the more familiar 3D and 2D wave equations.

The document consists of six Chapters which are followed at the end by Appendices A through L which deal with issues we thought too detailed or perhaps too peripheral to the main topic to appear in the main text.

Maple is used as needed to compute analytic integrals, solve equations, do unpleasant algebra, and make graphs. The reader need not be a Maple expert to read and understand the presented Maple code.

To reduce clutter, derivatives that would normally be written $\frac{\partial f}{\partial x}$ or $\partial f/\partial x$ are written as $\partial_{\mathbf{x}} f$.

Rather than use exotic script fonts to distinguish various forms of the electric field E, we use an "overloaded" notation where the argument list determines which E function is implied.

When an equation is repeated after its first occurrence, the equation number is put in italics.

Chapter Summaries

Chapter 1 states Maxwell's Equations and various associated equations which extend Maxwell's theory from the vacuum to dielectric and magnetic media. After some comments, these equations are restated in integral form using the divergence theorem and Stokes's theorem, and then the behavior of field components at boundaries is obtained. Wave equations for both the fields and potentials are described, and the subject of gauges is dealt with. Starting with Section 1.5 the wave equations are transformed to the frequency domain and Helmholtz equations with parameter β^2 appear. King's Helmholtz integral solutions of these equations are then derived using what we call the King gauge. Finally, Section 1.6 clarifies the reasons for using complex fields when physical E and B fields are real.

Chapter 2 derives the E and B fields (and current $J = \sigma E$) inside a round wire which is assumed to have an axially symmetric current flow. The resulting fields are somewhat complicated and reveal the skin effect. The surface impedance is defined and various quantities are plotted. Assumptions are made about the vector directional nature of the E and B fields in this analysis. The same problem is treated without these assumptions and for an arbitrary current distribution in Appendix D. The main results of that lengthy Appendix appear in box (D.4.9). In this box, parameter $\beta'^2 = \beta^2 - \beta_d^2$ where β and β_d are the Helmholtz parameters for the conductor and dielectric media, as shown in (1.5.1).

Chapter 3 discusses odd topics such as dielectric loss tangent, the thickness of surface charge, and why there is no free charge inside a conductor or a dielectric. The chapter concludes with a qualitative description of the E and B fields of a transmission line, with some sketches of the fields.

Chapter 4 uses the Helmholtz integral form of the potentials to derive the well-known transmission line equations which are these,

$$\partial_{\mathbf{z}} V(z) = -\mathbf{z}\, i(z) \qquad \partial_{\mathbf{z}} i(z) = -\mathbf{y} V(z) \ . \qquad \textit{(4.11.11)}$$

$$\mathbf{z} = R + j\omega L \qquad \mathbf{y} = G + j\omega C \qquad \textit{(4.11.12)}$$

The analysis then yields precise meanings for the parameters R,L,G,C. L is in fact the sum of external and internal inductance contributions $L_e + L_i$ and it turns out that L_e, C and G are all related to each other in terms of a certain dimensionless real parameter K as shown in (4.11.30). Parameters R and L_i are the real and imaginary parts of the sum of the conductor surface impedances $Z_{s1} + Z_{s2}$. At this point, the transmission line parameters are clarified and are related to each other, but they are not "known" due to the fact that their solutions involve integral equations over the transmission line geometry. This is the typical chicken-and-egg problem one encounters in all real-world electromagnetic problems, a well-known example being the exact current distribution in a simple quarter wave radiating antenna. Apart from trivial cases (such as very thin transmission line conductors), further approximations must be made.

Chapter 5 describes the required approximation. It is basically a continuation of the "transmission line limit" mentioned earlier, along with a notion of "low-loss", which then allows the transmission line problem to be reformulated as a 2D potential theory problem which we call "the transverse problem". It is then basically a "capacitor problem" and then any geometry can be solved at least numerically. Basically the assumption that the conductors are very *good* conductors transforms the transverse Helmholtz equation into the 2D Laplace equation which is the basis of 2D potential theory.

Chapter 6 then gives a complete discussion of the exact solution, within the assumptions just mentioned, for transmission lines consisting of two circular conductors of arbitrary diameter and arbitrary relative (but not intersecting) location. This includes twin-lead lines with unequal conductor diameters as well as off-centered coaxial lines. Since an infinite radius cylinder is a plane, this discussion also obtains the exact solution for a transmission line consisting of a round wire over a ground plane.

Appendix Summaries

Appendix A discusses gauge invariance and proves the existence of gauges in which div **A** can be set to any arbitrary (but reasonable) scalar function, **A** being the vector potential appearing in **B** = curl **A**. A few passing comments are added regarding the connection to special relativity, covariance and quantum field theory.

Appendix B analyzes the situation in which the transmission line dielectric and conductors have different magnetic permeability μ, a situation not treated in King's TLT book. This causes a bound magnetization current density $\mathbf{J}_m$ to appear both at boundaries (as a surface current) and in the bulk conductors and dielectric (as a volume current). It is shown ("the J_m theorem") that the theory of Chapter 4 with its Helmholtz integrals for the potentials can be "rescued" by adding just the surface component of the magnetization current $\mathbf{J}_m$ to the true conduction current in the vector potential integrand. A simple method is given for computing this surface J_m current from the current distribution **J** in the conductor. As usual, the round wire serves as a calculational example.

Appendix C concerns the seemingly mundane subject: "DC properties of wires". The main issue here is the DC inductance of wires which we treat from a stored energy viewpoint. Internal inductances are computed for a round wire, a hollow round pipe, and a thin strip. It is shown that even for a simple rectangular cross section (including square), the internal inductance cannot be expressed analytically (at

least using our method) and a numerical calculation is beyond the capability of Maple (running on an old PC), though the calculation has been done recently (2009) by Holloway and Kuester.

Appendix D computes the **E** and **B** fields inside a round wire for an arbitrary current distribution. The wire is assumed to be one conductor of a transmission line down which a wave travels at frequency ω. The solution is obtained using azimuthal partial wave analysis. The **E** field Helmholtz equations are directly solved in cylindrical coordinates, and the **B** field is then computed from Maxwell's curl **E** equation. Boundary conditions at the wire surface are discussed. The results (D.4.9) are expressed in terms of the surface charge moment η_m in each partial wave. The results for the m=0 partial wave are compared with the results of Chapter 2 which assumed an azimuthal current distribution.

Appendix E ponders the thickness of the surface charge on a conductor. It is shown that the charge layer thickness is about 1/3 the radius of a copper atom for a copper conductor, and that this is 4000 times smaller than the skin depth at 100 GHz.

Appendix F is an elementary discussion of the waveguide modes of a stripline (parallel plates). It shows why there is a cutoff frequency below which no waveguide modes can operate, whereas the "transmission line mode" on the same structure operates all the way down to DC.

Appendix G computes the DC vector potential A_z inside and outside a round wire carrying uniform current. The computation is done three ways, the most difficult using the Helmholz (Poisson) integral. When the dielectric surrounding the wire has a μ different from that of the wire, a homogeneous solution must be added to the Helmholtz particular integral solution. It is this homogeneous solution that is synthesized by adding the surface magnetization current discussed in Appendix B.

Appendices H and I derive the Poisson and Helmholtz Green's Functions for the 3D and 2D Poisson and Helmholtz differential equations. These play a major role in the entire document.

Appendix J shows how the transmission line transverse analysis replaces 3D propagators with 2D propagators of the Helmholtz and Poisson equations. Results obtained blindly in the main document are interpreted in terms of these Green's function propagators.

Appendix K presents the standard network model of a transmission line as the limit of a set of lumped circuit components. By computing the characteristic impedance Z_0 both from this network model and from Maxwell's equations, it is shown that the R,L,G,C parameters of both models have the same meaning, and this makes the connection between these network-model parameters and those obtained in Chapter 4 from the Maxwell equations.

Appendix L considers a point charge located at the center of the cavity of a thick spherical dielectric shell. The problem is solved and limiting cases are obtained. The solution provides an interpretation of how bound charge is accounted for by the dielectric constant ε in $E_r = (1/4\pi\varepsilon)\,(q/r)$. This 3D analysis is then repeated in 2D for a line charge in the cavity of an infinite cylindrical shell.

References are then provided.