New Retarded path not taken PhL 9.19.13

I think this is my working doc for updating this section. It is now installed.

1.4 Retarded Solutions: a path not taken

In a medium with $\sigma = 0$ and in which μ and ε are time-independent, equations (1.3.7) and (1.3.8) apply,

$$(\nabla^2 - \mu\varepsilon\,\partial_t^2)\mathbf{A} = -\mu\mathbf{J}_T \qquad (1.3.7)$$

$$(\nabla^2 - \mu\varepsilon\,\partial_t^2)\varphi = -(1/\varepsilon)\rho_T \qquad (1.3.8)$$

One approach to solving these equations for **A** and φ is the method of retarded solutions. Although we shall not use this approach, we outline the method here. We seek to solve an equation of this form

$$(\nabla^2 - \mu\varepsilon\,\partial_t^2)\,u = -f \qquad (1.4.1)$$

where for example in (1.3.8) we have $u = \varphi$ and $f = \rho_T/\varepsilon$. Since $\mu\varepsilon = 1/v^2$ where v is the wave velocity in the medium, write (1.4.1) as

$$(\partial_t^2 - v^2\nabla^2)\,u = v^2 f$$

or

$$\Box\, u = f \quad \text{where} \quad \Box \equiv \frac{1}{v^2}\partial_t^2 - \nabla^2 \ . \qquad (1.4.2)$$

This last equation is similar to (A.7.2) of Appendix A and we solve it in the same manner. Define a Green's function g as the solution of

$$v^2\ \Box\ g(\mathbf{x},t;\,\mathbf{x}',t') = \delta(\mathbf{x}-\mathbf{x}')\delta(t-t') \ . \qquad (1.4.3)$$

As Appendix A section 7 shows, the solution is given by

$$v^2\, g(\mathbf{x},t;\,\mathbf{x}',t') = (1/4\pi R)\delta(t-t'-R/v) \quad \text{with} \quad R = |\mathbf{x}-\mathbf{x}'| \ . \qquad (1.4.4)$$

Jackson (6.41) and (6.44) uses $G^{(+)} = 4\pi v^2 g$ with $v = c$ and refers to the solution as a "retarded Green function". See also Stakgold references in Appendix A. Then the solution to (1.4.1) is

$$u(\mathbf{x},t) = \int d^3x' \int dt'\, v^2\, g(\mathbf{x},t;\,\mathbf{x}',t')\, f(\mathbf{x}',t')$$

as can be verified by applying $\Box$ to both sides and making use of (1.4.3). Inserting (1.4.4) then gives

$$u(\mathbf{x},t) = \int d^3x' \int dt'\ (1/4\pi R)\delta(t-t'-R/v)\, f(\mathbf{x}',t')$$

$$= \int d^3x' \, (1/4\pi R) \, f(\mathbf{x}', t-R/v)$$

$$= \frac{1}{4\pi} \int d^3x' \, \frac{f(\mathbf{x}', t-R/v)}{R} \tag{1.4.5}$$

Thus, the solutions to (1.3.7) and (1.3.8) are

$$\mathbf{A}(\mathbf{x},t) = \frac{\mu}{4\pi} \int d^3x' \, \frac{\mathbf{J}_T(\mathbf{x}', t-R/v)}{R} \tag{1.4.6}$$

$$\varphi(\mathbf{x},t) = \frac{1}{4\pi\varepsilon} \int d^3x' \, \frac{\rho_T(\mathbf{x}', t-R/v)}{R} \tag{1.4.7}$$

The potentials at time t are generated by the values the sources *had* at time t - R/v since the influence of the sources travels at finite velocity v through the medium. Compare (1.4.7) to (A.0.2) which is the solution to the electrostatic Poisson equation.

With the medium σ loss term present, and with frequency dependent (hence time dependent) μ and ε, it is better to work in the frequency domain instead of the time domain. Thus, although they are interesting, we shall not use these retarded solutions.