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1.7 Reinterpretation of all equations in terms of complex functions

It seemed useful to defer the topic of this section so as not to clutter up the preceding six sections with this slight added level of complexity. The Fourier Transform has already been mentioned in the previous two sections, and here we shall discuss it more formally as a motivating factor in changing our point of view.

Up to this point, we have been regarding the following fields as representing real physical quantities,

$$\begin{array}{lllll} \mathbf{H}(\mathbf{x},t) & \mathbf{D}(\mathbf{x},t) & \mathbf{J}(\mathbf{x},t) & \rho(\mathbf{x},t) & \mathbf{A}(\mathbf{x},t) \\ \mathbf{B}(\mathbf{x},t) & \mathbf{E}(\mathbf{x},t) & \mathbf{J}_{\mathbf{a}}(\mathbf{x},t) & \rho_{\mathbf{a}}(\mathbf{x},t) & \varphi(\mathbf{x},t) \\ & & \mathbf{J}_{\mathbf{T}}(\mathbf{x},t) & \rho_{\mathbf{T}}(\mathbf{x},t) & \end{array} \qquad (1.7.1)$$

These fields exist in the real physical world and are related by equations involving real operators like curl and $\partial/\partial t$. We can represent such an equation as $L_{\mathbf{x},t} f(\mathbf{x},t) = g(\mathbf{x},t)$ where $L_{\mathbf{x},t}$ is some real differential operator and f and g are real fields.

One can extend f and g such that f and g are either both the real or both the imaginary parts of complex functions F and G. Then the equation $L_{\mathbf{x},t} F(\mathbf{x},t) = G(\mathbf{x},t)$ represents two distinct physical equations which we can write as

$$L_{\mathbf{x},t} F(\mathbf{x},t) = G(\mathbf{x},t) \quad \Rightarrow \quad L_{\mathbf{x},t}[f(\mathbf{x},t) + jf'(\mathbf{x},t)] = [g(\mathbf{x},t) + jg'(\mathbf{x},t)] \quad \Rightarrow$$

$$\begin{array}{ll} L_{\mathbf{x},t}\, f(x,t) = g(x,t) & F(\mathbf{x},t) = f(\mathbf{x},t) + jf'(\mathbf{x},t) \\ L_{\mathbf{x},t}\, f'(x,t) = g'(x,t) & G(\mathbf{x},t) = g(\mathbf{x},t) + jg'(\mathbf{x},t)\ . \end{array} \qquad (1.7.2)$$

It is convenient to regard all the fields listed above in (1.7.1) as complex fields like F and G. For example, we might write the Maxwell curl E equation (1.1.2) in this manner

$$\begin{array}{ll} \text{curl } \mathbf{E(x,t)} = -\,\partial \mathbf{B}(x,t)/\partial t & \mathbf{E}(x,t) = \mathbf{e}(x,t) + j\,\mathbf{e}'(x,t) \\ & \mathbf{B}(x,t) = \mathbf{b}(x,t) + j\,\mathbf{b}'(x,t)\ . \end{array} \qquad (1.7.3)$$

The left equation of (1.7.3) then represents these two different physical equations with real fields

$$\begin{array}{l} \text{curl } \mathbf{e(x,t)} = -\,\partial \mathbf{b}(x,t)/\partial t \\ \text{curl } \mathbf{e'(x,t)} = -\,\partial \mathbf{b'}(x,t)/\partial t\ . \end{array} \qquad (1.7.4)$$

The classic application of this idea is the assumption that some complex field is "monochrome" in its time dependence, meaning for example

$$\mathbf{E}(\mathbf{x},t) = e^{j\omega_1 t}\,\mathbf{e}(\mathbf{x})\ , \qquad (1.7.5)$$

where $\mathbf{e}(\mathbf{x})$ is a real function only of space. In this case, the corresponding physical assumption is either of these equations

$$\mathbf{e}(\mathbf{x},t) = \mathrm{Re}\{\,\mathbf{E}(\mathbf{x},t)\} = \cos(\omega_1 t)\,\mathbf{e}(x)$$
$$\mathbf{e}'(\mathbf{x},t) = \mathrm{Im}\{\,\mathbf{E}(\mathbf{x},t)\} = \sin(\omega_1 t)\,\mathbf{e}(x)\quad . \tag{1.7.6}$$

The reason for using a complex field like $\mathbf{E}(\mathbf{x},t)$ instead of the real field $\mathbf{e}(\mathbf{x},t)$ has to do with the Fourier Transform (or the Laplace Transform). This transform is almost always needed to solve a non-trivial problem involving Maxwell's equations, and we saw it in action in Section 1.5. In our somewhat sloppy notation, and with the conventions of Ref ***, we write this transform as :

$$\mathbf{E}(\mathbf{x},\omega) = \int_{-\infty}^{\infty} dt\ \mathbf{E}(\mathbf{x},t)\ e^{-j\omega t} \qquad \text{projection = transform} \tag{1.7.7}$$

$$\mathbf{E}(\mathbf{x},t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\ \mathbf{E}(\mathbf{x},\omega)\ e^{+j\omega t} \qquad \text{expansion = inverse transform = recovery} \tag{1.7.8}$$

Here $\mathbf{E}(\mathbf{x},t)$ is the original complex field whose real and imaginary parts are physical fields as in (1.7.3) or (1.7.6), while $\mathbf{E}(\mathbf{x},\omega)$ is the Fourier Transform of $\mathbf{E}(\mathbf{x},t)$. Since $\mathbf{E}(\mathbf{x},\omega)$ is a completely different complex function from $\mathbf{E}(\mathbf{x},t)$, one really should use some notation like $\boldsymbol{\mathcal{E}}(\mathbf{x},\omega)$ or $\mathbf{E}\hat{}(\mathbf{x},\omega)$, but we trust the reader to make the distinction when the ω argument is present. An obvious property of the Fourier Transform is this:

$$\partial_t \mathbf{E}(\mathbf{x},t) \;=\; (1/2\pi) \int_{-\infty}^{\infty} d\omega\ \mathbf{E}(\mathbf{x},\omega)\ \partial_t\, e^{+j\omega t} \;=\; (1/2\pi) \int_{-\infty}^{\infty} d\omega\ [j\omega\ \mathbf{E}(\mathbf{x},\omega)]\ e^{+j\omega t}$$

which we can write as

$$\mathbf{E}(\mathbf{x},t) \leftrightarrow \mathbf{E}(\mathbf{x},\omega) \qquad \Leftrightarrow \qquad \partial_t \mathbf{E}(\mathbf{x},t) \leftrightarrow\ j\omega\ \mathbf{E}(\mathbf{x},\omega) \tag{1.7.9}$$

which is just another way to state our rule (1.5.5). In the case of assumed monochrome time dependence of the form (1.7.5) and (1.7.6) one finds that

$$\mathbf{E}(\mathbf{x},t) = e^{j\omega_1 t}\,\mathbf{e}(\mathbf{x}) \tag{1.7.5}$$

$$\mathbf{E}(\mathbf{x},\omega) = \int_{-\infty}^{\infty} dt\ [e^{j\omega_1 t}\,\mathbf{e}(\mathbf{x})]\ e^{-j\omega t} = \mathbf{e}(\mathbf{x}) \int_{-\infty}^{\infty} dt\ e^{j(\omega_1-\omega)t}$$

$$= e(\mathbf{x})\ 2\pi\delta(\omega-\omega_1)\quad . \tag{1.7.10}$$

It is this simple form that motivates the use of complex fields as carriers of the real physical fields. One can of course transform the monochrome physical field directly, but the result is clumsy to deal with. For example,

$$\mathbf{e}(\mathbf{x},t) = \cos(\omega_1 t)\,\mathbf{e}(x)$$

$$\mathbf{e}(\mathbf{x},\omega) = \int_{-\infty}^{\infty} dt\ [\cos(\omega_1 t)\mathbf{e}(\mathbf{x})]\ e^{-j\omega t} = \mathbf{e}(\mathbf{x})\ (1/2)\ \int_{-\infty}^{\infty} dt\ [e^{j\omega_1 t} + e^{-j\omega_1 t}]\ e^{-j\omega t}$$

$$= \mathbf{e}(\mathbf{x})\ [\pi\ \delta(\omega-\omega_1) + \pi\ \delta(\omega+\omega_1)]\ . \qquad (1.7.11)$$

A directly related simplification in using the complex function approach is the fact that math with exponentials is so much simpler than the corresponding math with trig functions, as for example

$$e^{j(\omega t+\varphi)}\ e^{-j(\omega' t+\varphi')} = e^{j(\omega-\omega')t}\ e^{j(\varphi-\varphi')} \qquad \text{// dependence on t isolated to one factor}$$

$$\cos(\omega t+\varphi)\cos(\omega' t+\varphi') = (1/2)\ \{\ \cos[\ (\omega-\omega')t + (\varphi-\varphi')]\ +\ \cos[\ (\omega+\omega')t + (\varphi+\varphi')]\ \}\ . \qquad (1.7.12)$$

As a final observation, notice that

$$\mathbf{E}(\mathbf{x},t) = \mathbf{e(x,t)} + j\ \mathbf{e'(x,t)} \qquad \Rightarrow$$

$$\mathbf{E}(\mathbf{x},\omega) = \int_{-\infty}^{\infty} dt\ \mathbf{E}(\mathbf{x},t)\ e^{-j\omega t} = \int_{-\infty}^{\infty} dt\ [\mathbf{e}(\mathbf{x},t) + j\ \mathbf{e}'(\mathbf{x},t)]\ e^{-j\omega t}$$

$$= \mathbf{e}(\mathbf{x},\omega) + j\ \mathbf{e}'(\mathbf{x},\omega)\ . \qquad (1.7.13)$$

Whereas **e(x,t)** and **e'(x,t)** are the real and imaginary parts of $\mathbf{E}(\mathbf{x},t)$, the functions $\mathbf{e}(\mathbf{x},\omega)$ and $\mathbf{e}'(\mathbf{x},\omega)$ are *not* the real and imaginary parts of $\mathbf{E}(\mathbf{x},\omega)$ since in general $\mathbf{e}(\mathbf{x},\omega)$ and $\mathbf{e}'(\mathbf{x},\omega)$ are both complex functions. In this document we shall never deal with transforms of the type $\mathbf{e}(\mathbf{x},\omega)$ or $\mathbf{e}'(\mathbf{x},\omega)$ and one should interpret the ω-space functions already used such as appear in (1.5.1,2) and (1.6.1,2) as being complex functions.

In the Maxwell and related equations which include the ∂_t operator, if the fields are expanded onto their Fourier transformed representations using (1.7.8), then using the rule (1.7.9) one may instantly write the frequency domain version of these equations, just as in the example of Section 1.5. For example,

$$\text{curl}\ \mathbf{H(x,\omega)} = j\omega\mathbf{D}(\mathbf{x},\omega) + \mathbf{J}_T(\mathbf{x},\omega) \qquad (1.1.1)'$$

$$\text{curl}\ \mathbf{E(x,\omega)} = -j\omega\mathbf{B}(\mathbf{x},\omega) \qquad (1.1.2)'$$

$$\text{div}\ \mathbf{J}_T(\mathbf{x},\omega) = -j\omega\ \rho_T(\mathbf{x},\omega) \quad . \qquad (1.1.8)' \qquad (1.7.14)$$

Other equations in the Section 1.1 list have the same form but in terms of the frequency domain functions, for example,

$$\mathbf{J}(\mathbf{x},\omega) = \sigma(\mathbf{x})\ \mathbf{E(x,}\omega) \qquad (1.1.7)' \qquad (1.7.15)$$

where we momentarily allow $\sigma(\mathbf{x})$ to have spatial dependence but not time dependence.