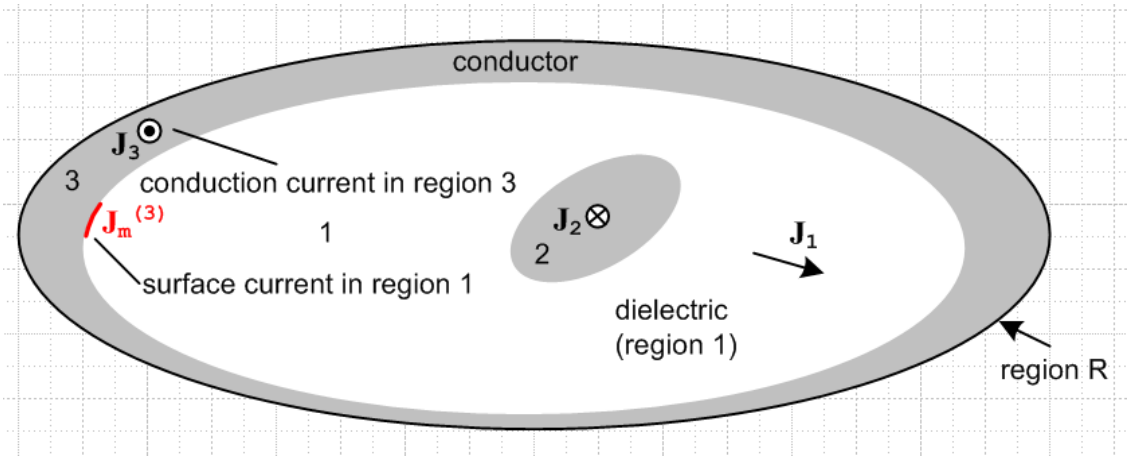
Repair Section 1.3 (c) PhL 11.16.13

Here I try a different King Gauge and try to still obtain a unified wave equation in all region R.

KING GAUGE

Wave equation for **A**

Consider the following general cross section of a transmission line which happens to be of coaxial cable type,



The gray regions 2 and 3 are conductors, while the white region 1 is the (possibly conducting) dielectric. Currents $\mathbf{J}_1$, $\mathbf{J}_2$ and $\mathbf{J}_3$ are conduction currents, while $\mathbf{J}_m^{(3)}$ represents a possible surface current in region 1 which runs on the surface of conductor 3. It runs in a direction opposite $\mathbf{J}_3$ and only exists in the case that $\mu_1 \neq \mu_3$.

We start by selecting the **King gauge for region 1**

$$\text{div}\,\mathbf{A} = -\,\mu_1\varepsilon_1\,\partial_t\varphi - \mu_1\sigma_1\varphi \qquad (1.3.18)$$

We first obtain the wave equation for **A** in **region 1**. Start with (1.3.3) which gives the wave equation for **A** before any gauge choice is made

$$\nabla^2(\mathbf{A} - \mu_0\varepsilon_1\,\partial_t^2)\mathbf{A} = \text{grad}\,[\text{div}\mathbf{A} + \mu_0\varepsilon_1\,\partial_t\,\varphi] - \mu_0(\mathbf{J}_{c1}+\mathbf{J}_{m1}) \qquad \text{// region 1} \qquad \textit{(1.3.3)}$$

where $\mathbf{J}_{c1}$ and $\mathbf{J}_{m1}$ are the conduction and magnetization currents in region 1.

Now insert the King gauge (1.3.18) to get

$$(\nabla^2 - \mu_0\varepsilon_1\,\partial_t^2)\,\mathbf{A} = \text{grad}\,[\mu_0\varepsilon_1\,\partial_t\varphi + (-\,\mu_1\varepsilon_1\,\partial_t\varphi - \mu_1\sigma_1\varphi)\,] - \mu_0\,(\mathbf{J}_{c1}+\mathbf{J}_{m1})$$

$$= [\mu_0\varepsilon_1\,\partial_t + (-\,\mu_1\varepsilon_1\,\partial_t - \mu_1\sigma_1)\,]\,\text{grad}\varphi - \mu_0\,(\mathbf{J}_{c1}+\mathbf{J}_{m1})$$

$$= [\mu_0\varepsilon_1\,\partial_t + (-\,\mu_1\varepsilon_1\,\partial_t - \mu_1\sigma_1)\,]\,(-\mathbf{E} - \partial_t\mathbf{A}) - \mu_0\,(\sigma_1\mathbf{E} + \mathbf{J}_{m1})$$

$$= [(\mu_0-\mu_1)\varepsilon_1\,\partial_t - \mu_1\sigma_1]\,(-\mathbf{E} - \partial_t\mathbf{A}) - \mu_0\,(\sigma_1\mathbf{E} + \mathbf{J}_{m1})$$

$$= [(\mu_0-\mu_1)\varepsilon_1\,\partial_t - \mu_1\sigma_1](-\mathbf{E}) - \mu_0\sigma_1\mathbf{E} - [(\mu_0-\mu_1)\varepsilon_1\,\partial_t - \mu_1\sigma_1]\,\partial_t\mathbf{A} - \mu_0\,\mathbf{J}_{m1}$$

$$= [(\mu_0-\mu_1)\varepsilon_1\,\partial_t - \mu_1\sigma_1 + \mu_0\sigma_1](-\mathbf{E}) - [(\mu_0-\mu_1)\varepsilon_1\,\partial_t^2\mathbf{A} - \mu_1\sigma_1\,\partial_t\mathbf{A}] - \mu_0\,\mathbf{J}_{m1}$$

$$= [(\mu_0-\mu_1)\varepsilon_1\,\partial_t + (\mu_0-\mu_1)\sigma_1](-\mathbf{E}) - [(\mu_0-\mu_1)\varepsilon_1\,\partial_t^2\mathbf{A} - \mu_1\sigma_1\,\partial_t\mathbf{A}] - \mu_0\,\mathbf{J}_{m1}$$

$$= [(\mu_0-\mu_1)(\varepsilon_1\,\partial_t + \sigma_1)](-\mathbf{E}) - [(\mu_0-\mu_1)\varepsilon_1\,\partial_t^2\mathbf{A} - \mu_1\sigma_1\,\partial_t\mathbf{A}] - \mu_0\,\mathbf{J}_{m1}$$

The E does not cancel and we have a complete mess! Stop here

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so our final result is then

$$(\nabla^2 - \mu_0\varepsilon_1\,\partial_t^2 - \mu_0\sigma_1\partial_t)\,\mathbf{A} = -\,\mu_0\mathbf{J}_{m1} \qquad \text{// region 1} \qquad (1.3.19)$$

This is a damped wave equation driven by surface current $\mathbf{J}_{m1}$. If there is no surface current, this is a homogeneous damped wave equation that **A** must satisfy within region 1.
ok edit pass to here
Now we start over with (1.3.3) for **region 2:**

$$(\nabla^2 - \mu_0\varepsilon_2\,\partial_t^2)\,\mathbf{A} = \text{grad}\,[\mu_0\varepsilon_2\,\partial_t\varphi + \text{div}\mathbf{A}\,] - \mu_0(\mathbf{J}_{c2}+\mathbf{J}_{m2}) \qquad \text{// region 2} \qquad \textit{(1.3.3)}$$

As before, we insert the region 1 King gauge expression (1.3.18) for div **A**, even though we are now working in region 2,

$$(\nabla^2 - \mu_0\varepsilon_2\,\partial_t^2)\,\mathbf{A} = \text{grad}\,[\mu_0\varepsilon_2\,\partial_t\varphi + (-\,\mu_0\varepsilon_1\,\partial_t\varphi - \mu_0\sigma_1\varphi)\,] - \mu_0(\mathbf{J}_{c2}+\mathbf{J}_{m2})$$

$$= [\mu_0\varepsilon_2\,\partial_t + (-\,\mu_0\varepsilon_1\,\partial_t - \mu_0\sigma_1)\,]\,\text{grad}\,\varphi - \mu_0(\mathbf{J}_{c2}+\mathbf{J}_{m2})$$

$$= [\mu_0(\varepsilon_2-\varepsilon_1)\,\partial_t - \mu_0\sigma_1)\,]\,\text{grad}\,\varphi - \mu_0(\mathbf{J}_{c2}+\mathbf{J}_{m2}) \quad)$$

$$= [\mu_0(\varepsilon_2-\varepsilon_1)\,\partial_t - \mu_0\sigma_1)\,]\,(-\mathbf{E}-\partial_t\mathbf{A}) - \mu_0(\mathbf{J}_{c2}+\mathbf{J}_{m2})$$

Since region 2 is a conductor, we assume a perfect conductor and set **E** = 0 so that

$$(\nabla^2 - \mu_0\varepsilon_2\,\partial_t^2)\,\mathbf{A} = [\mu_0(\varepsilon_2 - \varepsilon_1)\,\partial_t - \mu_0\sigma_1)\,]\,(-\partial_t\mathbf{A}) - \mu_0(\mathbf{J}_{c2}+\mathbf{J}_{m2})$$

$$(\nabla^2 - \mu_0\varepsilon_2\,\partial_t^2)\,\mathbf{A} = [\mu_0(\varepsilon_2 - \varepsilon_1)\,\partial_t\,(-\partial_t\mathbf{A}) - \mu_0\sigma_1\,(-\partial_t\mathbf{A}))\,] - \mu_0(\mathbf{J}_{c2}+\mathbf{J}_{m2})$$

$$(\nabla^2 - \mu_0\varepsilon_2\,\partial_t^2)\,\mathbf{A} = [-\mu_0(\varepsilon_2 - \varepsilon_1)\,\partial_t^2\mathbf{A} + \mu_0\sigma_1\,(\partial_t\mathbf{A})\,] - \mu_0(\mathbf{J}_{c2}+\mathbf{J}_{m2})$$

$$(\nabla^2 - \mu_0\varepsilon_1\,\partial_t^2 - \mu_0\sigma_1\partial_t)\,\mathbf{A} = -\mu_0(\mathbf{J}_{c2}+\mathbf{J}_{m2}) \qquad \text{// region 2}$$

On the left side we see the same region 1 damped wave operator although we are in region 2 A similar result will be obtained for region 3. Thus we have shown that

$$(\nabla^2 - \mu_0\varepsilon_1\,\partial_t^2 - \mu_0\sigma_1\partial_t)\,\mathbf{A} = -\mu_0\mathbf{J}_{m1} \qquad \text{region 1}$$
$$(\nabla^2 - \mu_0\varepsilon_1\,\partial_t^2 - \mu_0\sigma_1\partial_t)\,\mathbf{A} = -\mu_0(\mathbf{J}_{c2}+\mathbf{J}_{m2}) \qquad \text{region 2}$$
$$(\nabla^2 - \mu_0\varepsilon_1\,\partial_t^2 - \mu_0\sigma_1\partial_t)\,\mathbf{A} = -\mu_0(\mathbf{J}_{c3}+\mathbf{J}_{m3}) \qquad \text{region 3}$$

old:

$$(\nabla^2 - \mu_1\varepsilon_1\,\partial_t^2 - \mu_1\sigma_1\partial_t)\,\mathbf{A} = -\mu_1\mathbf{J}_m = -\mu_1[\,\mathbf{J}_m^{(2)} + \mathbf{J}_m^{(3)}] \qquad \text{region 1}$$
$$(\nabla^2 - \mu_1\varepsilon_1\,\partial_t^2 - \mu_1\sigma_1\partial_t)\,\mathbf{A} = -\mu_2\mathbf{J}_2 \qquad \text{region 2}$$
$$(\nabla^2 - \mu_1\varepsilon_1\,\partial_t^2 - \mu_1\sigma_1\partial_t)\,\mathbf{A} = -\mu_3\mathbf{J}_3 \qquad \text{region 3} \qquad (1.3.21)$$

We can combine these into a single equation which is then valid over all of region R,

$$(\nabla^2 - \mu_0\varepsilon_1\,\partial_t^2 - \mu_0\sigma_1\partial_t)\,\mathbf{A} = -\mu_0\mathbf{J}_{m1} - \mu_0(\mathbf{J}_{c2}+\mathbf{J}_{m2}) - \mu_0(\mathbf{J}_{c3}+\mathbf{J}_{m3}) \qquad \text{all of region R} \qquad (1.3.22)$$

$$(\nabla^2 - \mu_1\varepsilon_1\,\partial_t^2 - \mu_1\sigma_1)\,\mathbf{A} = -\mu_1[\,\mathbf{J}_m^{(2)} + \mathbf{J}_m^{(3)}] - \mu_2\mathbf{J}_2 - \mu_3\mathbf{J}_3 \qquad \text{all of region R} \qquad (1.3.22)$$

with the understanding that the conduction current in region 1 has already been accounted for We could generalize this result for a region R containing any number N of conductors labeled i = 2,3...N+1

$$(\nabla^2 - \mu_0\varepsilon_1\,\partial_t^2 - \mu_0\sigma_1\partial_t)\,\mathbf{A} = -\mu_0\,[\,\Sigma_{i=1}^{N}\mathbf{J}_m^{(i)} + \Sigma_{i=2}^{N}\mathbf{J}_c^{(i)}] \qquad \text{all of region R} \qquad (1.3.23)$$

Wave equation for φ

We first obtain the wave equation for φ in region 1. Start with (1.3.2) which gives the wave equation for φ before any gauge choice is made

$$\nabla^2\varphi + \partial_t[\text{div}\,\mathbf{A}] = -\rho/\varepsilon_1 \qquad (1.3.2)$$

Now use the same global region R King gauge (1.3.8) for div **A**.

$$\nabla^2\varphi + \partial_t[-\mu_0\varepsilon_1\,\partial_t\varphi - \mu_0\sigma_1\varphi] = -\rho_1/\varepsilon_1$$

$$(\nabla^2 - \mu_0\varepsilon_1\,\partial_t^2 - \mu_0\sigma_1\partial_t)\varphi = -(1/\varepsilon_1)\rho_1 \qquad \text{// region 1} \qquad (1.3.24)$$

Again the same damped region 1 wave operator appears on the left side. Since the King gauge is the same in all three regions, we can write

$(\nabla^2 - \mu_0\varepsilon_1 \partial_t^2 - \mu_0\sigma_1\partial_t)\varphi = -(1/\varepsilon_1)\rho_1$ // region 1

$(\nabla^2 - \mu_0\varepsilon_1 \partial_t^2 - \mu_0\sigma_1\partial_t)\varphi = -(1/\varepsilon_2)\rho_2$ // region 2

$(\nabla^2 - \mu_0\varepsilon_1 \partial_t^2 - \mu_0\sigma_1\partial_t)\varphi = -(1/\varepsilon_3)\rho_3$ // region 3 (1.3.25)

where ρ always means free charge. The three equations are basically the same because the pre-gauge equation (1.3.2) has no region-specific parameters apart from ε_1, in contrast with (1.3.3) quoted above.

Now the only actual free charge present is the surface charge on the outside surfaces of the conductors and we regard this as being in region 1. There is no free charge inside the dielectric or inside the conductors. We can then combine the above three equations into a single equation for all of region R

$(\nabla^2 - \mu_0\varepsilon_1 \partial_t^2 - \mu_0\sigma_1\partial_t)\varphi = -(1/\varepsilon_1)\rho_s$ all of region R (1.3.26)

where now ρ_s is the surface charge on the conductors.

Conclusion for wave equations in the King gauge

Here then are the wave equations for φ and **A** in region R using the region 1 King gauge:

$(\nabla^2 - \mu_0\varepsilon_1 \partial_t^2 - \mu_0\sigma_1\partial_t)\varphi = -(1/\varepsilon_1)\rho_s$ all of region R *(1.3.26)*

$(\nabla^2 - \mu_0\varepsilon_1 \partial_t^2 - \mu_0\sigma_1\partial_t)\mathbf{A} = -\mu_0 [\ \Sigma_{i=1}^{N} \mathbf{J}_m^{(i)} + \Sigma_{i=2}^{N} \mathbf{J}_c^{(i)}]$ all of region R *(1.3.23)*

$\text{div}\ \mathbf{A} = -\mu_0\varepsilon_1 \partial_t\varphi - \mu_0\sigma_1\varphi$ King gauge *(1.3.18)*

where ρ_s is the total surface charge on all conductors and $\mathbf{J}_i$ is the current in the interior of conductor i.

King never writes the inhomogeneous wave equations (1.3.23) and (1.3.26) in his transmission-line theory book, so it is difficult to find verification of our logic pathway in his book. However, Panofsky and Phillips *do* show the equations and we quote the relevant section from their book where we identify their parameters μ,ε,σ with our region 1 parameters μ_1, ε_1, σ_1 :

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We obtain the symmetrical set of equations:

$$\nabla^2\mathbf{A} - \mu\epsilon\frac{\partial^2\mathbf{A}}{\partial t^2} - \mu\sigma\frac{\partial\mathbf{A}}{\partial t} = -\mu\mathbf{j}', \qquad (14\text{–}4)$$

$$\nabla^2\phi - \mu\epsilon\frac{\partial^2\phi}{\partial t^2} - \mu\sigma\frac{\partial\phi}{\partial t} = -\rho/\epsilon. \qquad (14\text{–}5)$$

Here $\mathbf{j}'$ represents a current given by $\mathbf{j}' = \sigma\mathbf{E}'$, that part of the current density that is produced by the external electromotive forces. It does not contain any part of the current induced by the electric fields in conducting media and represented by the last term on the left side of Eq. (14–4).

Their last sentence says that usual $\mathbf{J} = \sigma\mathbf{E}$ in the conducting dielectric has been incorporated into the $-\mu\sigma\partial_t\mathbf{A}$ term in their first equation, just as we have done above. These authors have assumed that the μ's of

the dielectric and the conductors are all the same (normally $\mu = \mu_0$). In this case, there are no surface currents and our (1.3.23a) can be written

$$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2 - \mu_1\sigma_1)\,\mathbf{A} = -\mu_1 \Sigma_{i=2}^{N}\, \mathbf{J}_i = -\mu_1\, \mathbf{j}' \qquad \text{all of region R} \qquad \textit{(1.3.23a)}$$

and then we can identify their current $\mathbf{j}'$ with the sum over the current densities in the conductors. In order to obtain the above equations, Panofsky and Phillips use the King gauge (1.3.18) but they refer to this gauge simply as "the Lorentz condition" (illustrating Comment 1 above) . From their page 240,

> within an additive arbitrary function of position. Let us define the divergence of **A** by what is called the Lorentz condition,
>
> $$\nabla \cdot \mathbf{A} + \mu\epsilon \frac{\partial \phi}{\partial t} + \mu\sigma\phi = 0, \qquad (14\text{–}2)$$
>
> which in free space becomes
>
> $$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0. \qquad (14\text{–}3)$$