Section 1_5 (e)

This section is installed as 1.5 (e), here is where it was first written.

(e) Self Consistency of Helmholtz Integral Solutions

The various Helmholtz partial differential equations encountered in the previous sections have particular solutions expressed as "Helmholtz integrals". In particular, our King gauge Helmholtz integrals for the potentials have this form

$$\varphi(\mathbf{x},\omega) = \frac{1}{4\pi\xi}\ \Sigma_i \int n_{ci}(\mathbf{x}',\omega)\frac{e^{-j\beta R}}{R}\,dS' \qquad R = |\mathbf{x} - \mathbf{x}'| \qquad (1.5.13)$$

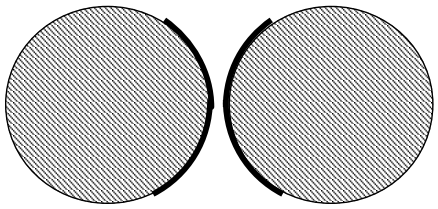
$$\mathbf{A}(\mathbf{x},\omega) = \frac{1}{4\pi}\ \Sigma_i \int \mu_i \mathbf{J}_i(\mathbf{x}',\omega)\,\frac{e^{-j\beta R}}{R}\,dV' \qquad (1.5.9)$$

These equations sometimes give the impression that one can willy-nilly specify an arbitrary charge distribution n_{ci} and an arbitrary current distribution $\mathbf{J}_i$ for a set of transmission line conductors and then these Helmholtz integrals will generate the correct potentials **A** and φ from which the correct fields **E** and **B** may be obtained using (1.3.1),

$$\mathbf{B} = \text{curl}\ \mathbf{A} \qquad \mathbf{E} = -\ \text{grad}\ \varphi - \partial_t \mathbf{A}\ . \qquad (1.3.1)$$

This is a false impression for one to infer from the discussion of the previous sections.

For example, in a "fat twinlead" transmission line of the kind to be mentioned in Section 2.5 below,



Fat twinlead *Fig 2.16*

the charge and current densities are extremely non-uniform. One cannot arbitrarily specify for this problem a uniform n and J_z distribution in each conductor and expect the resultant **E** and **B** fields to be correct.

The issue here is that solutions have to be self-consistent. Suppose one *were* to specify for the above fat twin-lead problem a uniform n and J_z. That is to say, one specifies that surface charge n is uniform around each circular cross section perimeter, and J_z is uniform across each disk area. The Helmholtz integrals shown above would then yield some **A** and φ and that in turn would yield some **E** and **B** for the fields in the dielectric between the conductors. One could then compute from the **E** field the value of surface charge n on each conductor using (1.1.47) $n = \varepsilon E_n$, where E_n is the normal **E** field just above the conductor surface. Similarly, one could compute conduction currents in the conductors perhaps from **J** =

(1/μ)curl **B** - $\varepsilon\partial_t$**E**, which is Maxwell (1.1.1). One would find, unfortunately, that the resulting n and **J** did not agree with the initially assumed values of n and **J**. Such a "solution" is then meaningless because it is not self-consistent.

All real-world Maxwell equation problems tend to have this circular aspect which makes solutions more difficult than the solution of idealized problems. A problem mentioned elsewhere in this document is that of a radiating dipole antenna. One can assume a certain sine shaped current pattern in the antenna, compute from it the potentials and fields, and one will find when the antenna current is back-computed from those fields that the pattern is not quite a sine pattern unless the wire is infinitely thin.

There are then two useful conclusions to be drawn here.

First, if transmission line conductors *are* very thin relative to their spacing, it is probably just fine to assume a uniform charge and current distribution in those wires, since the actual non-uniformity will have only a small effect on the solutions.

Second, a general method of solution is to start with some charge and current distributions that seem reasonable based on one's general analysis of a problem. One can then find the back-computed charges and currents, and adjust the input model accordingly. This would be the basis of either an analytic iterative procedure, where the model has some adjustable parameters, or of a numerical procedure where the model is the set of values that comprise the charge and current distribution and some kind of "relaxation" method then produces self-consistent solutions.

We note that the exact solution of the "fat twin lead" transmission line is derived in Chapter 6 by a method which bypasses this iterative process, and which works only due to the simple nature of the geometry.