Attempted rewrite of Section 2.5 now 2.6 I think PhL 11.27.13

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Solving the Wave Equations

First, write the two wave equations of interest

$$(\nabla^2 + \beta_1^2)\,\mathbf{E}(\mathbf{x},\omega) = 0$$
$$(\nabla^2 + \beta_1^2)\,\mathbf{B}(\mathbf{x},\omega) = 0 \qquad \text{// region 1} \qquad r>a$$

$$(\nabla^2 + \beta_2^2)\,\mathbf{E}(\mathbf{x},\omega) = 0$$
$$(\nabla^2 + \beta_2^2)\,\mathbf{B}(\mathbf{x},\omega) = 0 \qquad \text{// region 2} \qquad r<a$$

Next, assume that in both regions

$$\mathbf{E}(\mathbf{x},\omega) = E_z(\mathbf{x},\omega)\,\hat{\mathbf{z}} \quad \Rightarrow \quad \text{only } E_z$$
$$\mathbf{B}(\mathbf{x},\omega) = B_\theta(\mathbf{x},\omega)\,\hat{\boldsymbol{\theta}} \quad \Rightarrow \quad \text{only } B_\theta$$

Next, use the vector Laplacian in cylindrical coordinates

$$\nabla^2\mathbf{E} = \hat{\mathbf{r}}\,[\nabla^2 E_r - 2\,r^{-2}\partial_\theta E_\theta - r^{-2}E_r] + \hat{\boldsymbol{\theta}}\,[\nabla^2 E_\theta + 2\,r^{-2}\partial_\theta E_r - r^{-2}E_\theta] + \hat{\mathbf{z}}\nabla^2 E_z$$
$$= + \hat{\mathbf{z}}\nabla^2 E_z$$

$$\nabla^2\mathbf{B} = \hat{\mathbf{r}}\,[\nabla^2 B_r - 2\,r^{-2}\partial_\theta B_\theta - r^{-2}B_r] + \hat{\boldsymbol{\theta}}\,[\nabla^2 B_\theta + 2\,r^{-2}\partial_\theta B_r - r^{-2}B_\theta] + \hat{\mathbf{z}}\nabla^2 B_z$$
$$= \hat{\mathbf{r}}\,[-2\,r^{-2}\partial_\theta B_\theta] + \hat{\boldsymbol{\theta}}\,[\nabla^2 B_\theta - r^{-2}B_\theta]$$
$$= \hat{\boldsymbol{\theta}}\,[\nabla^2 B_\theta - r^{-2}B_\theta]$$

so that in region 1 we get

$$(\nabla^2 + \beta_1^2)\,\mathbf{E}(\mathbf{x},\omega) = 0$$

$$\nabla^2\mathbf{E}(\mathbf{x},\omega) + \beta_1^2\,\mathbf{E}(\mathbf{x},\omega) = 0$$

$$\hat{\mathbf{z}}\,\nabla^2 E_z(\mathbf{x},\omega) + \beta_1^2\,E_z(\mathbf{x},\omega)\,\hat{\mathbf{z}} = 0$$

$$(\nabla^2 + \beta_1^2)\, E_z(\mathbf{x},\omega) = 0$$

And for the B field also in region 1,

$$(\nabla^2 + \beta_1^2)\, \mathbf{B}(\mathbf{x},\omega) = 0$$

$$\nabla^2 \mathbf{B}(\mathbf{x},\omega) + \beta_1^2\, \mathbf{B}(\mathbf{x},\omega) = 0$$

$$\hat{\boldsymbol{\theta}}\, [\nabla^2 B_\theta - r^{-2} B_\theta] + \beta_1^2\, B_\theta(\mathbf{x},\omega)\, \hat{\boldsymbol{\theta}} = 0$$

$$(\nabla^2 - r^{-2} + \beta_1^2)\, B_\theta(\mathbf{x},\omega)$$

At this point then our four wave equations are

$$(\nabla^2 + \beta_1^2)\, E_z(\mathbf{x},\omega) = 0$$
$$(\nabla^2 - r^{-2} + \beta_1^2)\, B_\theta(\mathbf{x},\omega) = 0 \qquad \text{// region 1}$$

$$(\nabla^2 + \beta_2^2)\, E_z(\mathbf{x},\omega) = 0$$
$$(\nabla^2 - r^{-2} + \beta_2^2)\, B_\theta(\mathbf{x},\omega) = 0 \qquad \text{// region 2}$$

Next, assume that all fields have $\exp(-j\beta_1 z)$ z-dependence,

$$E_z(\mathbf{x},\omega) = \exp(-j\beta_1 z) E_z(x,y,\omega)$$
$$B_\theta(\mathbf{x},\omega) = \exp(-j\beta_1 z) B_\theta(x,y,\omega)$$

Then the four equations become, since $\partial_z^2 \rightarrow -\beta_1^2$,

$$(\nabla^2_{2D})\, E_z(x,y,\omega) = 0$$
$$(\nabla^2_{2D} - r^{-2})\, B_\theta(x,y,\omega) = 0 \qquad \text{// region 1}$$

$$(\nabla^2_{2D} + [\beta_2^2 - \beta_1^2])\, E_z(x,y,\omega) = 0$$
$$(\nabla^2_{2D} - r^{-2} + [\beta_2^2 - \beta_1^2])\, B_\theta(x,y,\omega) = 0 \qquad \text{// region 2}$$

Now invoke azimuthal symmetry to say

$$E_z(x,y,\omega) = E(r)$$
$$B_\theta(x,y,\omega) = B(r)$$

so the four equations are then

$$(\nabla^2_{2D})\, E(r) = 0$$
$$(\nabla^2_{2D} - r^{-2})\, B(r) = 0 \qquad \text{// region 1}$$

$(\nabla^2_{2D} + [\beta_2^2 - \beta_1^2])\ E(r) = 0$
$(\nabla^2_{2D} - r^{-2} + [\beta_2^2 - \beta_1^2])\ B(r) = 0$ // region 2

Now when acting on a function only of r we know that

$$\nabla^2_{2D} g(r) = r^{-1}\partial_r(r\partial_r g(r)) = r^{-1}[\, r\,\partial_r^2 g(r) + \partial_r g(r)] = g''(r) + r^{-1}g'(r)$$

Therefore our four equations become

$E''(r) + r^{-1}E'(r) = 0$
$B''(r) + r^{-1}B'(r) - r^{-2}B(r) = 0$ // region 1

$E''(r) + r^{-1}E'(r) + [\beta_2^2 - \beta_1^2]\ E(r) = 0$ // region 2
$B''(r) + r^{-1}B'(r) + [\beta_2^2 - \beta_1^2]\ B(r) - r^{-2}B(r) = 0$

Multiply by r^2 to get

$r^2E''(r) + r\,E'(r) = 0$
$r^2B''(r) + r\,B'(r) - B(r) = 0$ // region 1

$r^2\ E''(r) + rE'(r) + r^2\ [\beta_2^2 - \beta_1^2]\ E(r) = 0$
$r^2\ B''(r) + rB'(r) + r^2\ [\beta_2^2 - \beta_1^2]\ B(r) - B(r) = 0$ // region 2

Look now at just the exterior wave equations.

OK, let's stop the train and look at the region 1 equations

$r^2E''(r) + r\,E'(r) = 0$
$r^2B''(r) + r\,B'(r) - B(r) = 0$ // region 1

Here is what Maple has to say:

```
restart;
ode_E := r^2*diff(E(r),r,r) + r* diff(E(r),r) = 0;
```

$$ode_E := r^2\left(\frac{\partial^2}{\partial r^2}E(r)\right) + r\left(\frac{\partial}{\partial r}E(r)\right) = 0$$

```
dsolve(ode_E);
```

$$E(r) = _C1 + _C2\ln(r)$$

```
ode_B := r^2*diff(B(r),r,r) + r* diff(B(r),r) -B(r) = 0;
```

$$ode_B := r^2\left(\frac{\partial^2}{\partial r^2}B(r)\right) + r\left(\frac{\partial}{\partial r}B(r)\right) - B(r) = 0$$

```
dsolve(ode_B);
```

$$B(r) = \frac{_C1\,(r^2 - 1)}{r} + \frac{_C2\,(r^2 + 1)}{r}$$

Thus we conclude that

E(r) = A + B lnr // as expected since it is just Laplace azisym

B(r) = Cr + Dr^{-1} // not sure how to relate this to ang mom or whatever

I did check by hand that the B(r) works. So in the context of the round wire problem we really have

E(r) = A + B lnr
B(r) = Dr^{-1} // region 1

Does the exterior solution satisfy Maxwell's Equations?

Let's try a few consistency checks.

(1) What about div **E** = 0? That reads

0 = div **E** = $r^{-1}\partial_r(rE_r) + r^{-1}\partial_\theta E_\theta + \partial_z E_z$
= $\partial_z E_z$ = $-j\beta_1 E_z = -j\beta_1[A + B\ln r]$

so then

0 = $-j\beta_1[A + B\ln r]$

That does not look good! I know from the boundary match at r = a that A+Bln(a) = E(a) there which is some large value because J_z and E_z are both large in the wire just below the surface in the skin effect situation, and E_z is a tangential field with then is continuous through the surface.

(2) What about div **B** = 0?

$$0 = \text{div}\ \mathbf{B} = r^{-1}\partial_r(rB_r) + r^{-1}\partial_\theta B_\theta + \partial_z B_z$$
$$= 0$$

So this one is OK.

(3) What about curl $\mathbf{E} = -\partial_t \mathbf{B}$?

$$\text{curl}\ \mathbf{E} = -\partial_t \mathbf{B} \qquad\qquad \mathbf{E}(\mathbf{x},\omega)$$

$$\text{curl}\ \mathbf{E} = \hat{\mathbf{r}}\,[\, r^{-1}\partial_\theta E_z - \partial_z E_\theta] + \hat{\boldsymbol{\theta}}\,[\partial_z E_r - \partial_r E_z] + \hat{\mathbf{z}}\,[\, r^{-1}\partial_r(rE_\theta) - r^{-1}\partial_\theta E_r\,]$$
$$= \hat{\mathbf{r}}\,[\, r^{-1}\partial_\theta E_z] + \hat{\boldsymbol{\theta}}\,[-\partial_r E_z] = \hat{\boldsymbol{\theta}}\,[-\partial_r E_z]$$

The test here is then to see if

$$-\partial_r E_z = -j\omega B_\theta$$

$$\text{LHS} = -\partial_r E_z = -\partial_r E(r) = -\partial_r[A + B\ln r] = -B(1/r)$$
$$\text{RHS} = -j\omega B_\theta = -j\omega B(r) = -j\omega[Dr^{-1}]$$

The forms agree, and moreover we conclude that

$$B = j\omega D \qquad\qquad !$$

(4) Now let's try the other curl equation (in region 1 there is no free current J)

$$\text{curl}\ \mathbf{H} = \partial_t \mathbf{D} + \mathbf{J} \qquad \Rightarrow \qquad \text{curl}\ \mathbf{B} = \mu_1\, j\omega\, \varepsilon_1 \mathbf{E}$$

So

$$\text{curl}\ \mathbf{B} = \hat{\mathbf{r}}\,[\, r^{-1}\partial_\theta B_z - \partial_z B_\theta] + \hat{\boldsymbol{\theta}}\,[\partial_z B_r - \partial_r B_z] + \hat{\mathbf{z}}\,[\, r^{-1}\partial_r(rB_\theta) - r^{-1}\partial_\theta B_r\,]$$
$$= \hat{\mathbf{r}}\,[\partial_z B_\theta] + \hat{\mathbf{z}}\,[\, r^{-1}\partial_r(rB_\theta)]$$
$$= \hat{\mathbf{r}}\,[\partial_z B(r)] + \hat{\mathbf{z}}\,[\, r^{-1}\partial_r(rB(r))]$$
$$= \hat{\mathbf{z}}\,[\, r^{-1}\partial_r(rB(r))]$$

The test here then is to see if

$$r^{-1}\partial_r(rB(r)) = \mu_1\, j\omega\, \varepsilon_1 E(r)$$

$$\text{LHS} = r^{-1}\partial_r(rB(r)) = r^{-1}\partial_r(r\, Dr^{-1}) = 0$$
$$\text{RHS} = \mu_1\, j\omega\, \varepsilon_1 E(r) = \mu_1\, j\omega\, \varepsilon_1[A + B\ln r]$$

$$0 = \mu_1\, j\omega\, \varepsilon_1[A + B\ln r]$$

We have already shown that B = jωD so we cannot just set A = B = 0 to satisfy this equation. This is the paradox I ended up yesterday (which was 11.27.13) using the integral forms, but the curl equations are simpler to deal with, paradox obtained faster. Perhaps this second equation is roughly valid for small ω.

So let's back up and show how the two curl equations make the Helmholtz equation. Well, the derivations are all laid out in detail in Section 1.2 of lines. We find there that, let's say region 1 only,

$$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2)\mathbf{E} = \mu_1\partial_t\mathbf{J} + (1/\varepsilon) \text{ grad } \rho_1 \qquad (1.2.1)$$

$$(\nabla^2 - \mu_1\varepsilon_1 \partial_t^2)\mathbf{B} = - \mu_1 \text{ curl } \mathbf{J} \qquad (1.2.2)$$

or

$$(\nabla^2+\beta_1^2)\mathbf{E} = \mu_1\partial_t\mathbf{J} + (1/\varepsilon) \text{ grad } \rho_1$$

$$(\nabla^2+\beta_1^2)\mathbf{B} = - \mu_1 \text{ curl } \mathbf{J}$$

But in region 1 (away from boundaries) we have ρ = 0 and J = 0 so we get

$$(\nabla^2+\beta_1^2)\mathbf{E} = 0$$

$$(\nabla^2+\beta_1^2)\mathbf{B} = 0$$

So there is nothing wrong with these wave equations, they are exact in region 1.

So the problem is likely with our various **ansatz assumptions**. Recall they were 4 (the azisym form is not an assumption once you get to the f(x,y) forms)

$$\mathbf{E}(\mathbf{x},\omega) = E_z(\mathbf{x},\omega)\,\hat{\mathbf{z}} \quad \Rightarrow \quad \text{only } E_z$$

$$\mathbf{B}(\mathbf{x},\omega) = B_\theta(\mathbf{x},\omega)\,\hat{\boldsymbol{\theta}} \quad \Rightarrow \quad \text{only } B_\theta$$

$$E_z(\mathbf{x},\omega) = \exp(-j\beta_1 z)E_z(x,y,\omega)$$

$$B_\theta(\mathbf{x},\omega) = \exp(-j\beta_1 z)B_\theta(x,y,\omega)$$

I know that the first equation are *not* correct in an active transmission line because there are radial currents pumping out the surface charge. But in the current wire, there is no surface charge.

Possible Conclusion #1: Maxwell does not support an exact solution of my form!

If that conclusion is correct, you cannot have a wave going down a single wire as I have tried to make it out. I have never heard of a single wire transmission line.

Is there some limit in which my solution is approximately valid? In a wide spaced twinlead we might have the azisym situation for one of the wires. But we know there are charge densities active on the surfaces in that case. How are things above affected?

Pause to rewrite some of Chapter 2: I now realize that you can do this trick just below the wire surface,

$$2\pi a H_\theta = I \quad \Rightarrow \quad 2\pi a B_\theta/\mu_2 = I \quad \Rightarrow \quad B(a) = [\mu_2 I/(2\pi a)]$$

I added this to Chapter 2 first section, changed the box and ending 2.2.xx equation numbers

Plan A: Ignore the Maxwell violations and complete the solution

If I assume that ω is "small" in some sense, then my two Max violations can be ignored. But I don't know "how small" unless maybe I do some work. So let's go ahead and accept these violations for the moment and finish our solution. From the new box (2.2.31) we have the interior solution shown

interior solution: region 2

$$E(r) = [\mu_2 I/(2\pi a)] \frac{J_0(\beta_2 r)}{J_1(\beta_2 a)} (-j\omega/\beta_2) \qquad B(a) = [\mu_2 I/(2\pi a)]$$

$$B(r) = [\mu_2 I/(2\pi a)] \frac{J_1(\beta_2 r)}{J_1(\beta_2 a)} \qquad E(a) = [\mu_2 I/(2\pi a)] \frac{J_0(\beta_2 a)}{J_1(\beta_2 a)} (-j\omega/\beta_2)$$

exterior solution: region 1

$$E(r) = A' + B' \ln(r/a)$$

$$B(r) = D' r^{-1} = (B'/j\omega) r^{-1} \quad \text{since above we found that} \quad B' = j\omega D'$$

Matching boundary conditions (assuming for the moment that $\mu_1 = \mu_2$)for the fields then gives

$$[\mu_2 I/(2\pi a)] \frac{J_0(\beta_2 a)}{J_1(\beta_2 a)} (-j\omega/\beta_2) = A' \qquad \text{match } E_z$$

$$(1/\mu_2)\,[\mu_2 I/(2\pi a)] = (1/\mu_1)(B'/j\omega) a^{-1} \qquad \text{match } H_\theta$$

The two constants are found to be

$$A' = [\mu_2 I/(2\pi a)] \frac{J_0(\beta_2 a)}{J_1(\beta_2 a)} (-j\omega/\beta_2)$$

$$B' = [\mu_2 I/(2\pi a)] (j\omega a) \frac{\mu_1}{\mu_2}$$

Thus, the exterior fields are $\frac{\mu_1}{\mu_2}$

$$E(r) = [\mu_2 I/(2\pi a)] \frac{J_0(\beta_2 a)}{J_1(\beta_2 a)} (-j\omega/\beta_2) + [\mu_2 I/(2\pi a)] (j\omega a) \frac{\mu_1}{\mu_2} \ln(r/a)$$

$$= [\mu_2 I/(2\pi a)] \left\{ \frac{J_0(\beta_2 a)}{J_1(\beta_2 a)} (-j\omega/\beta_2) + (j\omega a) \frac{\mu_1}{\mu_2} \ln(r/a) \right\}$$

$$= [\mu_2 I/(2\pi a)](j\omega)\ \{\ \frac{J_0(\beta_2 a)}{J_1(\beta_2 a)}\ (-1/\beta_2)\ + a\,\frac{\mu_1}{\mu_2}\ \ln(r/a)\ \}$$

$$= [\mu_2 I/(2\pi a)](j\omega)\ \{\ \frac{J_0(\beta_2 a)}{J_1(\beta_2 a)}\ (-1/\beta_2)\ + a\,\frac{\mu_1}{\mu_2}\ \ln(r/a)\ \}$$

$$B(r) = (B'/j\omega)r^{-1} = (1/j\omega)B'\ r^{-1} = (1/j\omega)\ [\mu_2 I/(2\pi a)]\ (j\omega a)\,\frac{\mu_1}{\mu_2}\ r^{-1}$$

$$= [\mu_2 I/(2\pi a)]\ a\ \frac{\mu_1}{\mu_2}\ r^{-1} \quad = \quad [\mu_2 I/(2\pi a)]\ \frac{\mu_1}{\mu_2}\ (a/r) \quad = \quad [\mu_1 I/(2\pi r)]$$

This our result for B(r) comes out being the simple Ampere's law result, so in effect we have just neglected the displacement current in region 1. To summarize

Region 1

$$B(r) = [\mu_2 I/(2\pi a)]\ \frac{\mu_1}{\mu_2}\,(a/r) \qquad B(a+) = [\mu_2 I/(2\pi a)]\ \frac{\mu_1}{\mu_2}$$

$$E(r) = [\mu_2 I/(2\pi a)]\ (j\omega)\ \{\ \frac{J_0(\beta_2 a)}{J_1(\beta_2 a)}\ (-1/\beta_2)\ + a\,\frac{\mu_1}{\mu_2}\ \ln(r/a)\ \}$$

$$E(a) = [\mu_2 I/(2\pi a)]\ (j\omega)\ \{\ \frac{J_0(\beta_2 a)}{J_1(\beta_2 a)}\ (-1/\beta_2)$$

Region 2

$$B(r) = [\mu_2 I/(2\pi a)]\,\frac{J_1(\beta_2 r)}{J_1(\beta_2 a)} \qquad B(a-) = [\mu_2 I/(2\pi a)]$$

$$E(r) = [\mu_2 I/(2\pi a)]\ \frac{J_0(\beta_2 r)}{J_1(\beta_2 a)}\ (-j\omega/\beta_2) \qquad E(a) = [\mu_2 I/(2\pi a)]\ \frac{J_0(\beta_2 a)}{J_1(\beta_2 a)}\ (-j\omega/\beta_2)$$

This is then my complete solution given the "violations" mentioned above.

Can I now estimate the size of the violations in region 1?

Are there similar violations in region 2?

The quantities I need to be small are these

$$-j\beta_1[A' + B'\ln(r/a)]$$

$$\mu_1\, j\omega\, \varepsilon_1[A' + B'\ln(r/a)]$$

That is to say, I want both these quantities to be 0 in fact. But I just no "scale" for doing this. You need a larger picture somehow and then maybe you can throw out small terms relative to large terms.

Plan B. Try adding in an E_r radial field.

Certainly there is a transverse E field and that does not appear in the above. That might be the major issue. Perhaps near one of the wires write

$$\mathbf{E}(\mathbf{x},\omega) = E_z(\mathbf{x},\omega)\,\hat{\mathbf{z}} + E_r(\mathbf{x},\omega)\,\hat{\mathbf{r}}$$

What happens to everything above if I add this extra term in the E world and make no change in B

$$\begin{aligned}\nabla^2\mathbf{E} &= \hat{\mathbf{r}}\,[\nabla^2 E_r - 2\,r^{-2}\partial_\theta E_\theta - r^{-2}E_r] + \hat{\boldsymbol{\theta}}\,[\nabla^2 E_\theta + 2\,r^{-2}\partial_\theta E_r - r^{-2}E_\theta] + \hat{\mathbf{z}}\nabla^2 E_z \\ &= \hat{\mathbf{r}}\,[\nabla^2 E_r - r^{-2}E_r] + \hat{\boldsymbol{\theta}}\,[2\,r^{-2}\partial_\theta E_r] + \hat{\mathbf{z}}\nabla^2 E_z \\ &= \hat{\mathbf{r}}\,[\nabla^2 E_r - r^{-2}E_r] + \hat{\mathbf{z}}\nabla^2 E_z\end{aligned}$$

Then

$$(\nabla^2 + \beta_1^2)\,\mathbf{E}(\mathbf{x},\omega) = 0$$

$$\nabla^2\mathbf{E}(\mathbf{x},\omega) + \beta_1^2\,\mathbf{E}(\mathbf{x},\omega) = 0$$

$$\hat{\mathbf{r}}\,[\nabla^2 E_r - r^{-2}E_r] + \hat{\mathbf{z}}\nabla^2 E_z + \beta_1^2\,E_z\,\hat{\mathbf{z}} + \beta_1^2 E_r(\mathbf{x},\omega)\,\hat{\mathbf{r}} = 0$$

This then leads to two equations instead of one:

$$\begin{aligned}\nabla^2 E_r(\mathbf{x},\omega) + \beta_1^2 E_r(\mathbf{x},\omega) - r^{-2}E_r(\mathbf{x},\omega) &= 0 \\ \nabla^2 E_z(\mathbf{x},\omega) + \beta_1^2 E_z(\mathbf{x},\omega) &= 0\end{aligned}$$

or

$$\begin{aligned}(\nabla^2 + \beta_1^2 - r^{-2})E_r(\mathbf{x},\omega) &= 0 \\ (\nabla^2 + \beta_1^2)E_z(\mathbf{x},\omega) &= 0\end{aligned}$$

Now let's make the ansatz that both components have the same z behavior,

$$\begin{aligned}E_z(\mathbf{x},\omega) &= \exp(-j\beta_1 z)E_z(x,y,\omega) = \exp(-j\beta_1 z)E(r) \\ E_r(\mathbf{x},\omega) &= \exp(-j\beta_1 z)\,E_r(r)\end{aligned}$$

The two equations then become

$$\begin{aligned}(\nabla^2_{2D} - r^{-2})\,E_r(r) &= 0 \\ (\nabla^2_{2D})\,E(r) &= 0\end{aligned}$$

The second equation is the same as obtained last time without the E_r field, but now we have added in the E_r field and its governing equation. From above, I know we have these two solution types:

$$E(r) = A + B\ln r$$

$E_r(r) = Fr^{-1}$

Pause just one moment to check divergence equations.

$$0 = \text{div } \mathbf{E} = r^{-1}\partial_r(rE_r) + r^{-1}\partial_\theta E_\theta + \partial_z E_z \quad = r^{-1}\partial_r(rE_r) + \partial_z E_z$$

$$= r^{-1}\partial_r(rE_r) - j\beta_1 E_z$$

$$= r^{-1}\partial_r(rE_r(r)) - j\beta_1 E(r)$$

$$= 0 - j\beta_1 F\, r^{-1}$$

so entirely in the E world we have a **violation** in region 1 since no charge there! But now I might claim that E_r is very small so F is small, so this is then a small violation. OK, I like that.

Now onto the curl equations. Let's assume that B still has this form

$$B(r) = Dr^{-1} \qquad\qquad \mathbf{B}(\mathbf{x},\omega) = B_\theta(\mathbf{x},\omega)\,\hat{\boldsymbol{\theta}}$$

Then here goes for the **curl E equation**:

$$\text{curl } \mathbf{E} = -\partial_t \mathbf{B} = -j\omega \mathbf{B}$$

$$\text{curl } \mathbf{E} = \hat{\mathbf{r}}\,[\, r^{-1}\partial_\theta E_z] + \hat{\boldsymbol{\theta}}\,[\partial_z E_r - \partial_r E_z] + \hat{\mathbf{z}}\,[- r^{-1}\partial_\theta E_r\,]$$

$$= + \hat{\boldsymbol{\theta}}\,[-j\beta_1 E_r - \partial_r E_z]$$

The curl E equation then has

$$\text{LHS} = \hat{\boldsymbol{\theta}}\,[-j\beta_1 E_r - \partial_r E_z]$$
$$\text{RHS} = -j\omega B_\theta \hat{\boldsymbol{\theta}}$$

so we must then have

$$-j\beta_1 E_r - \partial_r E_z = -j\omega B_\theta$$

or

$$-j\beta_1 E_r(r) - \partial_r E(r) = -j\omega B(r)$$

Now use the forms above to get

$$-j\beta_1[Fr^{-1}] - \partial_r[A + B \ln r] = -j\omega\,[Dr^{-1}]$$

$$-j\beta_1 Fr^{-1} - Br^{-1} = -j\omega\, Dr^{-1}$$

$$-j\beta_1 F - B = -j\omega D$$

This is fairly encouraging, maybe we can make this work.

Then here goes for the **curl B equation.** Since B has not changed, we still have

$$\text{curl } \mathbf{B} = \hat{\mathbf{z}} \left[r^{-1}\partial_r(rB(r))\right]$$

and so the curl equation

$$\text{curl } \mathbf{B} = \mu_1 j\omega \varepsilon_1 \mathbf{E}$$

now says

$$\hat{\mathbf{z}} \left[r^{-1}\partial_r(rB(r))\right] = \mu_1 j\omega \varepsilon_1 [E_z(\mathbf{x},\omega)\, \hat{\mathbf{z}} + E_r(\mathbf{x},\omega)\, \hat{\mathbf{r}}]$$

which then says

$$\hat{\mathbf{z}} \left[r^{-1}\partial_r(rB(r))\right] = \mu_1 j\omega \varepsilon_1 [E(r)\, \hat{\mathbf{z}} + E_r(r)\, \hat{\mathbf{r}}]$$

which gives two equations

$$r^{-1}\partial_r(rB(r)) = \mu_1 j\omega \varepsilon_1 E(r)$$

$$0 = \mu_1 j\omega \varepsilon_1 E_r(r)$$

We can dismiss the second violation by arguing small E_r, just as we did in the div E = 0 case. The first equation then says

$$r^{-1}\partial_r(r\, Dr^{-1}) = \mu_1 j\omega \varepsilon_1 [A + B \ln r]$$

$$0 = \mu_1 j\omega \varepsilon_1 [A + B \ln r]$$

Therefore, adding in the extra E_r field component does not repair the problem we had without it! Unless it says that both A and B must be very small and we then can ignore the violation as with other examples above. But here there is no E_r field sitting there which we can claim is small!

Now consider what happens when we match the BC at r = a. We get

$$A + B \ln(a) = E(a)$$

I know that E(a) is large from the inside analysis, so I don't thing both A and B can be small!

So the problem persists!

Statement of Problem: In the curl equation curl $\mathbf{B} = \mu_1 \, j\omega \, \varepsilon_1 \mathbf{E}$, we cannot get a balance in the z component. The displacement current is required to feed the curl of B. There IS non-zero a displacement current for sure since we know E_z exists in region 1, but the curl B is always 0 since $B_\theta = D/r$. So something is then wrong with B!

Plan C: Have all 6 fields active -- but this is Appendix D!

If I draw the known twin-lead pattern, I see clearly that there is some B_r component! It must be this extra component that fixes the curl equation. I guess I could try to add in such a term. What would such an extra term do? I have to start ALL over I guess

First, write the two wave equations of interest

$$(\nabla^2 + \beta_1^2)\, \mathbf{E}(\mathbf{x},\omega) = 0$$
$$(\nabla^2 + \beta_1^2)\, \mathbf{B}(\mathbf{x},\omega) = 0 \qquad \text{// region 1} \qquad r>a$$

$$(\nabla^2 + \beta_2^2)\, \mathbf{E}(\mathbf{x},\omega) = 0$$
$$(\nabla^2 + \beta_2^2)\, \mathbf{B}(\mathbf{x},\omega) = 0 \qquad \text{// region 2} \qquad r<a$$

Let's just assume that ALL components exist and see what happens.

$$\mathbf{E}(\mathbf{x},\omega) = E_r(\mathbf{x},\omega)\, \hat{\mathbf{r}} + E_\theta(\mathbf{x},\omega)\, \hat{\boldsymbol{\theta}} + E_z(\mathbf{x},\omega)\, \hat{\mathbf{z}}$$
$$\mathbf{B}(\mathbf{x},\omega) = B_r(\mathbf{x},\omega)\, \hat{\mathbf{r}} + B_\theta(\mathbf{x},\omega)\, \hat{\boldsymbol{\theta}} + B_z(\mathbf{x},\omega)\, \hat{\mathbf{z}}$$

Next, use the vector Laplacian in cylindrical coordinates

$$\nabla^2\mathbf{E} = \hat{\mathbf{r}}\,[\nabla^2 E_r - 2\, r^{-2}\partial_\theta E_\theta - r^{-2}E_r] + \hat{\boldsymbol{\theta}}\,[\nabla^2 E_\theta + 2\, r^{-2}\partial_\theta E_r - r^{-2}E_\theta] + \hat{\mathbf{z}}\nabla^2 E_z$$
$$\nabla^2\mathbf{B} = \hat{\mathbf{r}}\,[\nabla^2 B_r - 2\, r^{-2}\partial_\theta B_\theta - r^{-2}B_r] + \hat{\boldsymbol{\theta}}\,[\nabla^2 B_\theta + 2\, r^{-2}\partial_\theta B_r - r^{-2}B_\theta] + \hat{\mathbf{z}}\nabla^2 B_z$$

The region 1 Helmholtz equations are then

$$\hat{\mathbf{r}}\,[\nabla^2 E_r - 2\, r^{-2}\partial_\theta E_\theta - r^{-2}E_r] + \hat{\boldsymbol{\theta}}\,[\nabla^2 E_\theta + 2\, r^{-2}\partial_\theta E_r - r^{-2}E_\theta] + \hat{\mathbf{z}}\nabla^2 E_z$$
$$= -\beta_1^2\,[E_r(\mathbf{x},\omega)\, \hat{\mathbf{r}} + E_\theta(\mathbf{x},\omega)\, \hat{\boldsymbol{\theta}} + E_z(\mathbf{x},\omega)\, \hat{\mathbf{z}}]$$

$$\hat{\mathbf{r}}\,[\nabla^2 B_r - 2\, r^{-2}\partial_\theta B_\theta - r^{-2}B_r] + \hat{\boldsymbol{\theta}}\,[\nabla^2 B_\theta + 2\, r^{-2}\partial_\theta B_r - r^{-2}B_\theta] + \hat{\mathbf{z}}\nabla^2 B_z$$
$$= -\beta_1^2\,[B_r(\mathbf{x},\omega)\, \hat{\mathbf{r}} + B_\theta(\mathbf{x},\omega)\, \hat{\boldsymbol{\theta}} + B_z(\mathbf{x},\omega)\, \hat{\mathbf{z}}]$$

This gives us the following 6 equations:

$$\nabla^2 E_r - 2\, r^{-2}\partial_\theta E_\theta - r^{-2}E_r = -\beta_1^2 E_r(\mathbf{x},\omega)$$
$$\nabla^2 E_\theta + 2\, r^{-2}\partial_\theta E_r - r^{-2}E_\theta = -\beta_1^2\, E_\theta(\mathbf{x},\omega)$$

$$\nabla^2 E_z = -\beta_1^2 E_z(\mathbf{x},\omega)$$

$$\nabla^2 B_r - 2 r^{-2}\partial_\theta B_\theta - r^{-2} B_r = -\beta_1^2 B_r(\mathbf{x},\omega)$$
$$\nabla^2 B_\theta + 2 r^{-2}\partial_\theta B_r - r^{-2} B_\theta = -\beta_1^2 B_\theta(\mathbf{x},\omega)$$
$$\nabla^2 B_z = -\beta_1^2 B_z(\mathbf{x},\omega)$$

Now haven't I done all this somewhere before? Some appendix? It is Appendix D where I do the "general case" including non-azisym! I assume the $\exp(-j\beta_d z)$ dependence for all components in (D.1.9) where I do only E and ignore B. I use the cylin coords vector Laplacian explicitly. I then first look at the z equation which is $[\nabla^2\mathbf{E}]_z + \beta^2 E_z = 0$. Here β is generic, I don't pick any region yet. I end up with this radial equation

$$[r^2\partial_r^2 + r\,\partial_r - m^2 + r^2(\beta^2 - \beta_d^2)]\, E_z(r,m) = 0$$

I then do this for each of the three component equations and I end up with this summary:

Summary of Examples:

$[\nabla^2\mathbf{E}]_z + \beta^2 E_z = 0$:
$$[r^2\partial_r^2 + r\,\partial_r - m^2 + r^2(\beta^2 - \beta_d^2)]\, E_z(r,m) = 0 \qquad \text{(D.1.16)}$$

$[\nabla^2\mathbf{E}]_r + \beta^2 E_r = 0$:
$$[r^2\partial_r^2 + r\partial_r - (m^2+1) + r^2(\beta^2-\beta_d^2)]\, E_r(r,m) - 2jm\, E_\varphi(r,m) = 0 \qquad \text{(D.1.17)}$$

$[\nabla^2\mathbf{E}]_\varphi + \beta^2 E_\varphi = 0$:
$$[r^2\partial_r^2 + r\partial_r - (m^2+1) + r^2(\beta^2-\beta_d^2)]\, E_\varphi(r,m) + 2jmE_r(r,m) = 0 \qquad \text{(D.1.18)}$$

div $\mathbf{E} = 0$:
$$\partial_r [r\, E_r(r,m)] + jmE_\varphi(r,m) - j\beta_d\, r\, E_z(r,m) = 0 \qquad \text{(D.1.19)}$$

Again, this is all done in light of the full partial wave expansion. Glad I did that!

I think I go on to completely solve the problem ! Wow. And matching boundary conditions too. Here are my results,

E Fields Inside a Round Wire

$x = \beta' r$ $\quad \beta'^2 = \beta^2 - \beta_d^2 \approx \beta^2$ $\quad \beta = \omega\sqrt{\mu\xi}$ $\quad \xi = [\varepsilon + \sigma/(j\omega)] \approx \sigma/(j\omega)$ conductor

$x_a = \beta' a$ $\quad \beta_d = \omega\sqrt{\mu_d\xi_d}$ $\quad \xi_d = [\varepsilon_d + \sigma_d/(j\omega)] \approx \varepsilon_d$ dielectric

$\mathbf{E}(r,\varphi,z,t) = e^{j(\omega t - \beta_d z)}\,\mathbf{E}(r,\varphi)$ *(D.1.2)* $\quad \mathbf{E} = E_r\hat{\mathbf{r}} + E_\varphi\hat{\boldsymbol{\varphi}} + E_z\hat{\mathbf{z}}$ (for any arguments)

$$\mathbf{E}(r,\varphi) = \mathbf{E}(r,0) + 2\sum_{m=1}^{\infty}[\mathrm{Re}\{\mathbf{E}(r,m)\}\cos(m\varphi) - \mathrm{Im}\{\mathbf{E}(r,m)\}\sin(m\varphi)] = \text{real} \qquad (D.1.4)$$

where:

$$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\left[\frac{x_a J_m(x)}{J_{m+1}(x_a)} - \frac{x_a J_m(x)}{J_{m-1}(x_a)}\right] \qquad a = \text{radius} \qquad \eta_m \equiv \frac{N_m}{N_0}$$

$$E_z(r,0) = (1/2)\, I\, R_{dc}\left[\frac{x_a\, J_0(x)}{J_1(x_a)}\right] \qquad // = (2.2.22) \text{ for } \beta_d = 0 \qquad \eta_0 = 1$$

$$E_r(r,m) = (j/4)\,(a\beta_d)\,\eta_m\, I\, R_{dc}\left[\frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m+1}(x)}{J_{m-1}(x_a)}\right]$$

$$E_r(r,0) = (j/2)\,(a\beta_d)\, I\, R_{dc}\left[\frac{J_1(x)}{J_1(x_a)}\right]$$

$$E_\varphi(r,m) = (1/4)\,(a\beta_d)\,\eta_m\, I\, R_{dc}\left[-\frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m+1}(x)}{J_{m-1}(x_a)}\right]$$

$$E_\varphi(r,0) = 0$$

(D.4.6)

Good heavens! But do I know these solutions satisfy the two curl equations, or is my z dependence ansatz too strong? Well, I only have the E solution! Where is the B solution? Maybe I get that from the curl E equation! My entire effort here is "inside the wire only". Hmmm.

Comments: I have been ignoring this Appendix D for a long time, just a painful messy thing. But now I see that it is basically a complete solution to the round wire problem, as claimed. I even then have a special section on the *outside* of the wire, which I have been struggling with here in this doc. I found:

$\beta' = \beta_d^2 - \beta_d^2 = 0$ so that:

$[\nabla^2\mathbf{E}]_z + \beta^2 E_z = 0$:

$$[r^2\partial_r^2 + r\,\partial_r - m^2]\,E_z(r,m) = 0 \qquad (D.1.16)_o$$

$[\nabla^2\mathbf{E}]_r + \beta^2 E_r = 0$:

$$[r^2\partial_r^2 + r\partial_r - (m^2+1)]\,E_r(r,m) - 2jm\,E_\varphi(r,m) = 0 \qquad (D.1.17)_o$$

$[\nabla^2\mathbf{E}]_\varphi + \beta^2 E_\varphi = 0$:

$$[r^2\partial_r^2 + r\partial_r - (m^2+1)]\,E_\varphi(r,m) + 2jmE_r(r,m) = 0 \qquad (D.1.18)_o$$

div $\mathbf{E} = 0$:

$$\partial_r\,[r\,E_r(r,m)] + jmE_\varphi(r,m) - j\beta_d\, r\, E_z(r,m) = 0 \qquad (D.1.19)_o$$

OK, I am ending this doc now, I have seen enough.