Finding a Meaning for Chapter 2 PhL 11.28.13

Reviewed with some red line on 11.30.13. Moved bulk App D stuff to Actions 1 doc.

This is a fairly complicated subject, be prepared!

History

1. In Chapter 2 I have this rather nice treatment of the round wire which then shows the skin effect. The treatment is made under a certain set of assumptions regarding the various fields and symmetry. I had always been a little uneasy one of the conditions (weak z dependence), but it has really implicitly been there since 1991 when I first wrote Chapter 2. I think I confused E(**x**,ω) with E(x,y,ω) which are different functions.

2. Somewhere along the way I wrote **Appendix D** doing a "general treatment" of the round wire, but after writing it, I ignored it as just so much math fiddling.

3. **Doc 2.5 and the Exterior Mistake.** As part of my cleanup of lines doc, I then decided to clean up Chapter 2 a bit. My first action on 11.26 was to write a final section for Chapter 2 which computed the fields outside the wire, so they I would have a complete solution within my set of assumptions. I would even match the boundary conditions and set all the constants that way. I went ahead and wrote this up in " Section 2_5 on round wire solution.doc" [renamed "Exterior solution for round wire as Section 2_5 REVIEWED.doc"] (call it **Doc 2.5)**. In this working doc, I first start off playing with things, then I start more seriously where it says Back up in bold. I assumed that **E** = E(r) $\hat{\mathbf{z}}$ and **B** = B(r) $\hat{\boldsymbol{\theta}}$ [as in my stated Chap 2 assumptions list] , and I plugged these forms into my **E** and **B** Helmholtz wave equations which had a generic unspecified β^2 constant. I was not doing region 1 and region 2 yet, just a generic region with β. I then assumed that this general form was applicable inside and out (region 2 and 1) and I solved the resulting ODE in each region. In the interior region, I replicated the results of Chapter 2, not surprising since Chap 2 assumed the same things I assumed here. But then I went ahead and solved the *exterior* region assuming that the Helmholtz equation for that region simply had β_1^2 in place of β_2^2. That led to $H_1^{(1)}(\beta_1 r)$ as the exterior solution, and I even made some plots. I thought all was well and done.

4. But then on 11.27 I realized that **something was not right**. I wanted to interpret the Chap 2 problem as described there (with assumptions) as some kind of long wavelength *wave* going down the wire. In Doc 2.5, I never once thought about the z dependence of anything, I just said "there is no z dependence".

5. Sisyphus Paradox and Exterior Mistake Identified. Next, on 11.27 I wrote "Sisyphus paradox for 11_27.13.doc" where I summarized my exterior solution from Doc 2.5 My only Paradox here is that I could not come up with a "physical situation". I tried to assume exp(-jkz) z-dependence, and I then got $(\nabla^2_{2D} + \beta_1^2 - k^2)\mathbf{E}_0(x,y,\omega) = 0$. That is because ∂_z^2 in ∇^2 generates a term $-\beta_1^2$ inside the Helmholtz equation I then realized that probably a true wave solution must have exp(-jβ1z) dependence in both regions 1 and 2, in order for it to really be a "travelling wave", where β_1 is of course the dielectric value. I argued that at r = a when you match BC's, you will find that the internal solution must have this exact same z dependence. I noticed that I had already made this assumption in the long-forgotten Appendix D.

The implication here was that the exterior Helm equation is really $(\nabla^2_{2D} + 0)\mathbf{E}_0(x,y,\omega) = 0$ whereas in Doc 2.5 I had incorrectly used $(\nabla^2_{2D} + \beta_1^2)\mathbf{E}_0(x,y,\omega) = 0$.

6. **Doc 2.5a Repair, but then Maxwell's violated.** I saw this as a minor error of my Doc 2.5 which I would then correct. So my next doc was "attempted rewrite of section 2_5.doc" (call it **Doc 2.5a**) where I intended to simply correct this exterior solution, rematch BC's, and be done with it. Along the way I noted that the interior solution of Doc 2.5 would not be much altered because I would be replacing β_2^2 by $\beta_2^2 - \beta_1^2$ but since $|\beta_2| >> \beta_1$, this really made no difference in the interior solution. That meant the interior work of Chapter 2 is still OK.

I started working in Doc 2.5a. With the usual assumptions of Chap 2 that $\mathbf{E} = E(r)\,\hat{\mathbf{z}}$ and $\mathbf{B} = B(r)\,\hat{\boldsymbol{\theta}}$ I set out to solve $\nabla^2\mathbf{E} = 0$ and $\nabla^2\mathbf{B} = 0$ in the exterior region. I came up with simple exterior solutions of the form $E(r) = A + B\ln r$ and $B(r) = Dr^{-1}$. They differ because in cyl coordinates there is an extra $-r^{-2}$ term in the Helm equation for B.

I then thought it would be good to see if these solutions satisfied the **Maxwell equations**. When I impose div $\mathbf{E} = 0$ the result is then $0 = -j\beta_1[A + B\ln r]$ which is a contradiction. Since E(r) has to match large E(a), we cannot just have A and B = 0. This was the first sign of trouble. The div $\mathbf{B} = 0$ equation was OK. I next looked at curl $\mathbf{E} = -j\omega\mathbf{B}$ and found that it could be satisfied if $B = j\omega D$, so that one was OK. Finally, when I looked at curl $\mathbf{B} = \mu_1\, j\omega\, \varepsilon_1\mathbf{E}$, that implied that $0 = \mu_1\, j\omega\, \varepsilon_1[A + B\ln r]$ which is very similar to the div $\mathbf{E} = 0$ problem just quoted.

So I was then in this situation: The Helm equations are derived from the Max equations, but two of the Max equations are violated. I was forced to conclude that one or more of my *ansatz* assumptions was not viable. One assumption was the $\exp(-j\beta_1 z)$ z dependence, the others were $\mathbf{E} = E(r)\,\hat{\mathbf{z}}$ and $\mathbf{B} = B(r)$. [I think this was a correct thing to conclude.]

I then started to flail inside Doc 2.5a. I could say that $0 = -j\beta_1[A + B\ln r]$ is OK in some sense when ω is very small since $\beta_1^2 = \omega^2\mu_1\varepsilon_1$. This would also "fix" the other violation $0 = \mu_1 j\omega\varepsilon_1[A + B\ln r]$. But then the question is "how small" must ω be to have OK violations?

I then deferred this "how small" question and did **Plan A** which uses the exterior solution just discussed and matches the boundary conditions. While doing this, I rewrote the Chapter 2 first summary box because I realized that both B(a) and E(a) are determined by the interior solution alone, given the current I in the wire. I then got a final answer for the Plan A, and I just ignore the violations. I wonder if there are violations both for interior and exterior solutions?

In **Plan B** I tried adding an $E_r\hat{\mathbf{r}}$ field to see if that would quench my violations. This complicated the analysis a bit, but in the end I got the exact same violation as with not having this extra term, since it turns out that this extra term does not affect the $\hat{z}$ violation for the curl B equation.

In **Plan C** I bit the bullet and said I should really allow all 6 field components. But then I suddenly realized that I have done all this before, and that was in **Appendix D**! So then I have come full circle in this little history.

My next actions will be these:

(1) see if I can estimate how small ω has to be in the simpler model of Doc 2.5a
(2) do the same thing in the full solution of Appendix D.

I don't really know where to start for item (1) so I guess I will attempt item (2).

[I now know that **E** = E(r) $\hat{\mathbf{z}}$ and **B** = B(r) is over-restrictive and if you expect to satisfy Maxwell's equations, you must allow more general forms for **E** and **B**.]

Fri Nov 29,2013.

7. Regarding the Meaning Issue. I think I understand now. The wire in isolation does not allow the exact Maxwell equation solution I tried to impose on it with my ansatz. [but neither does the coax situation]. That is WHY the two Max equations are violated [well, maybe related to WHY] . You have to find a physically realizable situation in order to have all Maxwell's equations respected [true]. The appropriate situation here is that the round wire is the center conductor of a large diameter coaxial cable where the current return occurs on a metal cylindrical surface far out from the central wire. In this situation, the capacitance of the cable is C = 2πε ln(b/a), the inner wire radius, if b >> a. In the true wave situation, then, charge must exist on the round wire surface which my ansatz ignores [correct]. This charge generates an internal and external radial E_r field which the ansatz model does not include. [correct, and this will affect divE = 0.]

When I added such a possible E_r field in Plan B of Doc 2.5a, I found $E_r = Fr^{-1}$ for my outside solution which seems very reasonable. However, adding this E_r field to the picture failed to cure the problem that div **E** ≠ 0. So adding just this extra field, though very reasonable, does not seem to make Maxwell happy. I need to return to this point soon.

Meanwhile, the "meaning" of the wire as a large coax center conductor has the extra advantage of realizing the assumed **axial symmetry**. This means m = 0 in Appendix D.

This then suggests the following solution method using Appendix D. Recall that App D solves only for the field E using the same method of Doc 2.5a (assumes $k = \beta_d$ for all z dependence). If App D is correct, then it will produce a certain m = 0 solution for E. Then I can compute B from curlE = -jωB, and THEN maybe I have a solution that makes Max happy! Recall that App D makes a special point of being sure that div E = 0 is respected. My full result is in a box (D.4.6).

8. What does App D have to say for m = 0 ?

Here is the m=0 version of that E field solution:

E Fields Inside a Round Wire

$x = \beta' r \qquad \beta'^2 = \beta^2 - \beta_d^2 \approx \beta^2 \qquad \beta = \omega\sqrt{\mu\xi} \qquad \xi = [\varepsilon + \sigma/(j\omega)] \approx \sigma/(j\omega)$ conductor

$x_a = \beta' a \qquad\qquad\qquad\qquad \beta_d = \omega\sqrt{\mu_d\xi_d} \qquad \xi_d = [\varepsilon_d + \sigma_d/(j\omega)] \approx \varepsilon_d$ dielectric

$\mathbf{E}(r,\varphi,z,t) = e^{j(\omega t - \beta_d z)}\,\mathbf{E}(r,\varphi)$ *(D.1.2)* $\qquad \mathbf{E} = E_r\hat{\mathbf{r}} + E_\varphi\hat{\boldsymbol{\varphi}} + E_z\hat{\mathbf{z}}$ (for any arguments)

$$\mathbf{E}(r,\varphi) = \mathbf{E}(r,0) + 2\sum_{m=1}^{\infty}[\mathrm{Re}\{\mathbf{E}(r,m)\}\cos(m\varphi) - \mathrm{Im}\{\mathbf{E}(r,m)\}\sin(m\varphi)] = \text{real} \qquad (D.1.4)$$

where:

$$E_z(r,0) = (1/2)\, I\, R_{dc}\left[\frac{x_a\, J_0(x)}{J_1(x_a)}\right] \qquad // = (2.2.22) \text{ for } \beta_d = 0 \qquad \eta_0 = 1$$

$$E_r(r,0) = (j/2)\,(a\beta_d)\,I\,R_{dc}\,\left[\frac{J_1(x)}{J_1(x_a)}\right] \qquad R_{dc} = \frac{1}{\sigma\pi a^2}$$

$$E_\varphi(r,0) = 0$$

(D.4.6)

Note:

(1) Chapter 2 assumes $(j\omega\varepsilon + \sigma) \approx \sigma$ in its development.
(2) Chapter 2 assumes β_d on the outside and β on the inside where $\beta = (j - 1)\sqrt{\omega\mu\sigma/2}$

We can **compare the E_z solution to the Chapter 2 solution**

$$E(r) = -(j\omega/\beta)\,[\mu I/(2\pi a)]\,\frac{J_0(\beta r)}{J_1(\beta a)} \qquad (2.2.30)$$

$$E_z(r,0) = (1/2)\,I\,R_{dc}\,\left[\frac{x_a\,J_0(x)}{J_1(x_a)}\right] \qquad \text{above box m = 0}$$

$$= (1/2)\,I\,\frac{1}{\sigma\pi a^2}\,\beta' a\,\frac{J_0(\beta r)}{J_1(\beta a)}$$

Now *assuming* $\beta' = \beta$, I do know that in the conductor $\beta^2 = \omega^2\mu\xi \approx \omega^2\mu[\sigma/j\omega] = -j\omega\mu\sigma$ which then says

$$\beta = (1/\beta)[-j\omega\mu\sigma] = (-j\omega/\beta)\,\mu\sigma \qquad (-j\omega/\beta) = \beta/(\mu\sigma)$$

If I use this in the m=0 box result I get

$$E_z(r,0) = (1/2)\,I\,\frac{1}{\sigma\pi a^2}\,(1/\beta)\,[-j\omega\mu\sigma]\,a\,\frac{J_0(\beta r)}{J_1(\beta a)}$$

$$= I\,\frac{1}{2\pi a}\,(1/\beta)\,[-j\omega\mu]\,\frac{J_0(\beta r)}{J_1(\beta a)} = -(j\omega/\beta)\,[\mu I/(2\pi a)]\,\frac{J_0(\beta r)}{J_1(\beta a)}$$

and I am **VERY HAPPY** to report that this then exactly matches the Chapter 2 result.

The conclusions at this juncture are:

(1) The E_z field for the coaxial center conductor in my "meaning model" exactly agrees with the result of a detailed calculation in Appendix D.

(2) Only fields E_z and E_r exist inside the wire (for m = 0)

(3) The E_r field also has a complex Bessel form as shown.

$$E_r(r,0) = (j/2)\,(a\beta_d)\,I\,R_{dc}\,\left[\ \frac{J_1(x)}{J_1(x_a)}\ \right]$$

and this is indeed "small" in the transmission line limit, a new and useful fact. BUT, below I will show that this is the wrong thing to say about smallness!

Discussion of the m = 0 solution: We get these fields

$$E_z(r,0) = (1/2)\,(a\beta')\,I\,R_{dc}\,[\,J_0(\beta' r)/\,J_1(\beta' a)\,]$$
$$E_r(r,0) = (j/2)\,(a\beta_d)\,I\,R_{dc}\,[J_1(\beta' r)/\,J_1(\beta' a)]$$
$$E_\varphi(r,0) = C_{\varphi 0}\,J_1(\beta' r)$$

Two of the components are fully nailed down! I think that part of the calculation is OK, though we may still have Maxwell equation violations I suppose. For the round wire in Chapter 2 I would just set the constant $C_{\varphi 0} = 0$ as ansatz. Then the E_r seems to be "pumping charge" to the surface of the wire. I think my boundary condition $E_r(r=a,m) = (j\omega/\sigma)\,N_m$ is probably still valid, so I then have

$$(j\omega/\sigma)\,N_0 = (j/2)\,(a\beta_d)\,I\,R_{dc}$$

In this scenario with no other partial waves active, $n = N_0$ from (D.1.6) so this equation is actually telling me the physical surface charge on the conductor :

$$n = N_0 = (\sigma/2\omega)\,(a\beta_d)\,I\,R_{dc} = (\sigma/2\omega)\,(a\beta_d)\,I\,\frac{1}{\sigma\pi a^2} = (\beta_d/\omega)\,I/(2\pi a) = \sqrt{\mu_d\xi_d}\ I/(2\pi a)$$

$$n(\varphi,z,t) = e^{j(\omega t-\beta_d z)}\ n(\varphi)$$

so this really is looking like my model for the Chapter 2 round wire -- center of a transmission line. It cannot operate without having this surface charge! This charge density is controlled by I and the exterior region quantity $\sqrt{\mu_d\xi_d}$. The total charge per length on the round wire is

$$Q = \sqrt{\mu_d\xi_d}\ I = I/v_d \qquad \text{Coul/m} = \text{Coul/sec} * \text{sec/m} \qquad \text{check}$$

So that is an interesting fact all in itself. $Q = I/v_d$.

In Chapter 2 I ignored this charge, and I also ignored the E_r field and we see it is very small since $(a\beta_d) << (a\beta')$ or $\beta_d << \beta'$ which is certainly true. [do not need to invoke "transmission line limit"]

Can I relate this to capacitance? I know $C = 2\pi\varepsilon_d \ln(b/a)$ per unit length for a coax center round wire. Suppose the inner conductor has $n = \sqrt{\mu_d\xi_d}\ I/(2\pi a)$ charge density which is then $Q = \sqrt{\mu_d\xi_d}\ I$ per unit length of wire. Then $Q = CV$ so $\Delta V = Q/C = \sqrt{\mu_d\xi_d}\ I / 2\pi\varepsilon_d \ln(b/a)$ and all I can say is that the ratio of the radii tells you ΔV, so no great realization here.

This charge Q is of course going to affect the exterior E_r field for my coax problem.

What about $E_\varphi(r,0) = C_{\varphi 0} J_1(\beta' r)$? I see easily from (D.1.18) why the E_φ field **decouples** from the other fields, and that is why it has a separate constant. If $C_{\varphi 0} > 0$, the is an electromotive force around the wire circumference which I guess exists just under the surface and it would drive a J_φ current. This is basically an independent torsional wave solution that the round wire can support. Even if I = 0 and the other fields went away, you could still have this torsion wave going down the wire. One way maybe to drive such a wave would be to put a little axial cut in the wire at one end and somehow apply and AC voltage to it. Maybe you apply a B_z magnetic AC field at the end and the Faraday's law might drive it.

I hunted for torsion wave and azimuthal wave on line, the latter appears but not in my context.

Now what about the m> 0 modes?

$$E_z(r,m) = C_{zm} J_m(\beta' r) = K_m (\beta'/\beta_d)(1/2j) J_m(\beta' r) \qquad \text{(D.1.27)}$$

$$E_r(r,m) = a_m x^{-1} J_m(x) + \frac{K_m}{2} J_{m+1}(x) \ . \qquad \text{(D.2.11)}$$

$$jE_\varphi(r,m) = - a_m x^{-1} J_m(x) + \left(\frac{K_m}{2} + \frac{a_m}{m} \right) J_{m+1}(x) \qquad x = \beta' r \ . \qquad \text{(D.2.15)}$$

Now for general m, **the E_φ(r,m) field is coupled** to the other fields and you don't get that torsion wave freedom you had with m = 0. The E_φ field just "comes out" as you see it here. All three components are now interrelated.

9. Am I done with the Meaning for Chapter 2?

Let's try to spin an answer. The round wire is the center conductor of a coaxial cable, and we are only interested in the fields inside this round wire. We *assume* this coax cable is a transmission line which carries a wave having $e^{j(\omega t - \beta_d z)}$ time and z dependence. This is our ansatz. For a fat coax cable if we are far from the round wire, we can imagine a wave with $e^{j(\omega t - \beta_d z)}$ where β_d is the dielectric's wave number. This is the nature of a plane longitudinal wave through the dielectric. At the round wire surface the dielectric wave has this $e^{j(\omega t - \beta_d z)}$ dependence and through that boundary it "rubs off" on what goes on inside the round wire. That is our ansatz.

10. Decision: The purpose of the current document was to "find a meaning for Chapter 2", and that task has now been carried out. Only the m = 0 part of the Appendix D analysis matters. I just added more text at the front of Chapter 2 to put the round wire into its proper "coaxial cable wave context". This was really my Big Problem and I think it is now solved, so I will close this doc down.

The issue of this a_m and K_m coefficients I will put off for a future day. You cannot solve all problems at once or you get nowhere fast!