Appendix D if I assume general k in the wave exponent. PhL 11.27.13

Here I started editing Appendix D changing the assumed z dependence from β_d to k. This was before I decided that you really do want to have $k = \beta_d$. But keep this around just in case. I don't know how far down I did this editing, I stopped somewhere before the end.

Appendix D : The General Electric Field Inside a Round Wire

In this Appendix we present a somewhat lengthy calculation of the electric field (and therefore the current) inside a round wire *without* assuming that such fields are symmetrical about the axis. This wire may be regarded as one conductor of a transmission line down which a wave is propagating.

The conclusions one can draw from the analytic solution given below are supportive of the general discussion elsewhere in this monograph:

1. The E_r and E_φ field components are *very small*. In fact, they are smaller than E_z by the factor (a/λ), where a = wire radius, and λ = wavelength of the wave on a transmission line containing the wire. This fraction is always assumed small in any analysis of a transmission line.

2. It is possible to maintain the condition $E_\varphi = 0$ on the wire surface. This condition is consistent with the idea that the wire surface is an electrical equipotential at any constant z.

3. We get an explicit formula for the *surface impedance*. We find that it is non-uniform around the boundary of the wire cross section, and that J_z is non-uniform inside the wire. This result is true even in the DC limit of a transmission line.

4. We suggest a method for computing the E fields in the low frequency limit of a transmission line consisting of round wires. First, solve the "electrostatic" problem illustrated in Chapter 6. Knowing the potential φ, compute the radial electric field at the surface of the conductors. From this compute the surface charge density $n(\varphi)$ as a function of azimuth around the wire. Then, as described below, compute the moments N_m (or η_m) of this charge distribution, and use the formulas below to find the electric field.

This is a highly technical appendix. The reader is invited to inspect the boxed results and comments in Section D.4.

D.1 The General Method and Solution for E_z

The starting point for the calculation is the Helmholtz equation (1.6.2) for the E field inside the wire,

$$(\nabla^2 + \beta^2) \mathbf{E}(r,\varphi, z, t) = 0 . \qquad (D.1.1)$$

Here we use cylindrical coordinates (r,φ,z), as appropriate for a straight round wire. The first step is to expose the assumed t and z dependence, and in doing so, define the field $\mathbf{E}(r,\varphi)$:

$$\mathbf{E}(r,\varphi,z,t) = e^{j(\omega t - kz)}\ \mathbf{E}(r,\varphi)\ . \qquad (D.1.2)$$

The form shown in (D.1.2) is a simple wave traveling down a transmission line in the +z direction. The wave has wavelength $\lambda = 2\pi/k$ where k is the wavenumber. We assume that one of the conductors of this transmission line is our round wire, while the other conductor is (other conductors are) unspecified.

One can regard (D.1.2) as an assumed variable-separated form for a solution, an "ansatz". Our second ansatz is that $E_\varphi(r=a,\varphi) = 0$ which means at any z = constant slice, the circular wire perimeter is an equipotential. If a consistent solution to the field equations can be found with these assumptions, they are justified *de facto*.

Convention Comment: Section 1.6 discusses the Fourier Transform (1.6.7) and (1.6.8) where $e^{+j\omega t}$ appears in the expansion formula (1.6.8). For $\mathbf{E}(\mathbf{x},\omega) = 2\pi\delta(\omega-\omega_1)$ one gets $\mathbf{E}(\mathbf{x},t) = e^{+j\omega_1 t}$ and then the form of a wave solution is $e^{+j(\omega t - kz)}$ with the + sign associated with ωt. In general EE people like to assume time dependence of the form $e^{+j\omega t}$ (and they like j in place of i for $\sqrt{-1}$). The Fourier Transform is of course valid with the other sign choice for the two exponentials, and for that other sign choice one would have $\mathbf{E}(\mathbf{x},\omega) = 2\pi\delta(\omega-\omega_1)$ => $\mathbf{E}(\mathbf{x},t) = e^{-j\omega_1 t}$ and one would think of a wave as $e^{-j(\omega t - kz)} = e^{+j(kz-\omega t)}$. This sign convention is common in many physics texts, but in this document we use the $e^{+j(\omega t - kz)}$ convention usually used in EE texts. It is just a convention choice and, as in (1.6.6), it just effects the sign of the imaginary physical field under consideration. If one thinks of the physical field under consideration as Re{ $\mathbf{E}(\mathbf{x},t)$}, the sign convention choice makes no difference at all.

(a) Partial Wave Expansions

The next step is to do a "partial wave expansion" (that is, a complex Fourier series expansion) of $\mathbf{E}(r,\varphi)$ in terms of "azimuthal harmonics" $e^{im\varphi}$, so that the variable φ is replaced with the partial wave index m:

$$\mathbf{E}(r,\varphi) = \sum_{m=-\infty}^{\infty} \mathbf{E}(r,m)\, e^{jm\varphi} \qquad \Leftrightarrow \qquad \mathbf{E}(r,m) = (1/2\pi) \int_{-\pi}^{\pi} d\varphi\ \mathbf{E}(r,\varphi)\, e^{-jm\varphi} \qquad (D.1.3)$$

Since the usual complex part has been extracted in (D.1.2), we assume that $\mathbf{E}(r,\varphi)$ is real so the right equation above says that $\mathbf{E}(r,-m) = \mathbf{E}(r,m)^*$. We can then re-express the above as,

$$\mathbf{E}(r,\varphi) = \mathbf{E}(r,0) + \sum_{m=1}^{\infty} [\ \mathbf{E}(r,m)e^{jm\varphi} + \mathbf{E}(r,-m)e^{-jm\varphi}] = \mathbf{E}(r,0) + \sum_{m=1}^{\infty} [\ \mathbf{E}(r,m)e^{jm\varphi} + [\mathbf{E}(r,m)e^{jm\varphi}]^*]$$

$$= \mathbf{E}(r,0) + 2 \sum_{m=1}^{\infty} \mathrm{Re}\{\ \mathbf{E}(r,m)e^{jm\varphi}\ \}$$

$$= \mathbf{E}(r,0) + 2 \sum_{m=1}^{\infty} [\ \mathrm{Re}(\mathbf{E}) + j\ \mathrm{Im}(\mathbf{E})]\ [\ \cos(m\varphi) + j\ \sin(m\varphi)]$$

$$= \mathbf{E}(r,0) + 2 \sum_{m=1}^{\infty} [\mathrm{Re}(\mathbf{E})\cos(m\varphi) - \mathrm{Im}(\mathbf{E})\ \sin(m\varphi)] + 2j \sum_{m=1}^{\infty} [\ \mathrm{Im}(\mathbf{E})\cos(m\varphi) + \mathrm{Re}(\mathbf{E})\ \sin(m\varphi)]$$

Since $\mathbf{E}(r,\varphi)$ is assumed real, the second sum must vanish for all r and φ, giving this final result,

$$\mathbf{E}(r,\varphi) = \mathbf{E}(r,0) + 2 \sum_{m=1}^{\infty} [\mathrm{Re}\{\mathbf{E}(r,m)\}\cos(m\varphi) - \mathrm{Im}\{\mathbf{E}(r,m)\} \sin(m\varphi)] \,. \tag{D.1.4}$$

In analogy with (D.1.2) and the discussion of Section 1.6 regarding complex functions, we define a complex surface (r=a) charge density n(φ,z,t) which has the following ansatz variable-separated form,

$$n(\varphi,z,t) = e^{j(\omega t - \beta_d z)}\, n(\varphi) \tag{D.1.5}$$

where n(φ) is real. We then expand n(φ) as in (D.1.3) and (D.1.4),

$$n(\varphi) = \sum_{m=-\infty}^{\infty} N_m\, e^{jm\varphi} \qquad \Leftrightarrow \qquad N_m = (1/2\pi) \int_{-\pi}^{\pi} d\varphi\, n(\varphi)\, e^{-jm\varphi} \,. \tag{D.1.6}$$

$$n(\varphi) = \mathbf{N_0} + 2 \sum_{m=1}^{\infty} [\mathrm{Re}\{N_m\}\cos(m\varphi) - \mathrm{Im}\{N_m\} \sin(m\varphi)] \tag{D.1.7}$$

Since n(φ) is real, $N_m = N_{-m}^*$. These N_m are the "moments" of the surface charge distribution, and their complex nature keeps track of cos(mφ) versus sin(mφ) components.

(b) The Vector Laplacian and the conversion of equations from φ space to m space

Given the following cylindrical-coordinates field components,

$$\mathbf{E}(r,\varphi,z,t) = E_r(r,\varphi,z,t)\hat{\mathbf{r}} + E_\varphi(r,\varphi,z,t)\hat{\boldsymbol{\varphi}} + E_z(r,\varphi,z,t)\hat{\mathbf{z}} \tag{D.1.8}$$

we may write out our ansatz wave form (D.1.2) and the Helmholtz equation (D.1.1) in more detail,

$$\begin{aligned}
E_r(r,\varphi,z,t) &= e^{j(\omega t - kz)}\, E_r(r,\varphi) \,. & [\nabla^2\mathbf{E}(r,\varphi,z,t)]_r &+ \beta^2 E_r(r,\varphi,z,t) = 0 \\
E_\varphi(r,\varphi,z,t) &= e^{j(\omega t - kz)}\, E_\varphi(r,\varphi) \,. & [\nabla^2\mathbf{E}(r,\varphi,z,t)]_\varphi &+ \beta^2 E_\varphi(r,\varphi,z,t) = 0 \\
E_z(r,\varphi,z,t) &= e^{j(\omega t - kz)}\, E_z(r,\varphi) \,. & [\nabla^2\mathbf{E}(r,\varphi,z,t)]_z &+ \beta^2 E_z(r,\varphi,z,t) = 0 \,.
\end{aligned} \tag{D.1.9}$$

In Cartesian coordinates, it happens that $[\nabla^2\mathbf{E}]_i = \nabla^2(E_i)$, but this is not generally true for curvilinear coordinates. In cylindrical coordinates, it is true for the z coordinate only. The operator ∇^2 when applied to a vector field is called the "vector Laplacian" and it is very different from the scalar Laplacian, so much so that some authors replace $[\nabla^2\mathbf{E}]$ by $[\text{✡}\mathbf{E}]$ which is defined in this manner

$$[\text{✡}\mathbf{E}] \equiv [\nabla^2\mathbf{E}] \equiv \mathrm{grad}(\mathrm{div}\ \mathbf{E}) - \mathrm{curl}\ (\mathrm{curl}\ \mathbf{E}) \tag{D.1.10}$$

whereas

$$\nabla^2\varphi \equiv \mathrm{div}(\mathrm{grad}\ \varphi) = \nabla\bullet(\nabla\varphi) \,. \tag{D.1.11}$$

It is the *vector* Laplacian that appears in our Helmholtz equation (D.1.1). For cylindrical coordinates it turns out that,

$$(\nabla^2\mathbf{E})_r = \nabla^2 E_r - (2/r^2)\,\partial_\varphi E_\varphi - (1/r^2)\,E_r$$
$$(\nabla^2\mathbf{E})_\varphi = \nabla^2 E_\varphi + (2/r^2)\,\partial_\varphi E_r - (1/r^2)\,E_\varphi$$
$$(\nabla^2\mathbf{E})_z = \nabla^2 E_z \qquad \text{(D.1.12)}$$

where

$$\nabla^2 = (1/r)\partial_r(r\partial_r) + (1/r^2)\partial_\varphi^2 + \partial_z^2 = \partial_r^2 + (1/r)\partial_r + (1/r^2)\partial_\varphi^2 + \partial_z^2\,. \qquad \text{(D.1.13)}$$

See for example Morse and Feshbach Vol I p 116, Moon and Spencer p 139, or do a web search on "vector Laplacian"; the author's Tensor Analysis document, Sections 13, 14 and 15, derives these results for arbitrary coordinate systems. Here is a summary of vector differential operators in cylindrical coordinates taken from Morse and Feshbach above, where the last line corresponds to the above discussion:

$$\text{grad}\ \psi = \mathbf{a}_r\frac{\partial\psi}{\partial r} + \mathbf{a}_\phi\frac{1}{r}\frac{\partial\psi}{\partial\phi} + \mathbf{a}_z\frac{\partial\psi}{\partial z}$$
$$\text{div}\ \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$
$$\text{curl}\ \mathbf{A} = \mathbf{a}_r\left(\frac{1}{r}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) + \mathbf{a}_\phi\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \mathbf{a}_z\left(\frac{1}{r}\frac{\partial}{\partial r}rA_\phi - \frac{1}{r}\frac{\partial A_r}{\partial\phi}\right)$$
$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$
$$\nabla^2\mathbf{A} = \mathbf{a}_r\left[\nabla^2 A_r - \frac{A_r}{r^2} - \frac{2}{r^2}\frac{\partial A_\phi}{\partial\phi}\right] + \mathbf{a}_\phi\left[\nabla^2 A_\phi - \frac{A_\phi}{r^2} + \frac{2}{r^2}\frac{\partial A_r}{\partial\phi}\right] + \mathbf{a}_z\nabla^2 A_z \qquad \text{(D.1.14)}$$

Using the ansatz form (D.1.2) and partial wave expansions of the form (D.1.3) or (D.1.6), it is a simple matter to convert an equation involving components like $E_i(r,\varphi,z,t)$ or $n(r,\varphi,z,t)$ to a simpler equation involving components like $E_i(r,m)$ and $n(r,m)$. We shall give an example, then state a simple set of rules that does the conversion automatically.

Example 1: The E_z Helmholtz Equation (D.1.9): $[\nabla^2\mathbf{E}]_z + \beta^2 E_z = 0$

Using (D.1.12) and (D.1.13), this may be written,

$$[\partial_r^2 + (1/r)\,\partial_r + (1/r^2)\,\partial_\varphi^2 + \partial_z^2 + \beta^2\ ]\ e^{j(\omega t - kz)}E_z(r,\varphi) = 0\,. \qquad (1)$$

Inserting the expansion (D.1.3) for $E_z(r,\varphi)$ and moving the m sum to the left gives

$$\sum_{m=-\infty}^{\infty} [\partial_r^2 + (1/r)\,\partial_r + (1/r^2)\,\partial_\varphi^2 + \partial_z^2 + k^2\ ]\, e^{j(\omega t - kz)} E_z(r,m)\, e^{jm\varphi} = 0\ . \qquad (2)$$

We can then make the obvious replacements $\partial_z = -jk$ and $\partial_\varphi = +jm$ to get,

$$\sum_{m=-\infty}^{\infty} \{\, [\partial_r^2 + (1/r)\,\partial_r - m^2\,(1/r^2) - k^2 + \beta^2\ ]\, e^{j(\omega t - \beta_d z)} E_z(r,m)\,\}\, e^{jm\varphi} = 0\ . \quad (3)$$

Due to the completeness of functions $e^{jm\varphi}$ on the interval $(-\pi.\pi)$, we conclude that $\{\ \} = 0$, or

$$[\partial_r^2 + (1/r)\,\partial_r - m^2\,(1/r^2) - k^2 + \beta^2\ ]\, e^{j(\omega t - kz)} E_z(r,m) \ = \ 0\ . \qquad (4)$$

In other words, one applies $\int_{-\pi}^{\pi} d\varphi\, e^{-jm'\varphi}$ to both sides of (3), uses the completeness property

$$\int_{-\pi}^{\pi} d\varphi\, e^{j(m-m')\varphi} = 2\pi\, \delta_{m,m'}\ , \qquad (5)$$

and then change m' to m to get (4).

Next, multiply both sides of (4) by $r^2\, e^{-j(\omega t - kz)}$ to get,

$$[r^2\partial_r^2 + r\,\partial_r - m^2 + r^2(\,\beta^2 - k^2)]\, E_z(r,m) \ = \ 0$$

We may then write these rules for converting equation (1) to equation (6)

Conversion Rules: (D.1.15)

$\partial_z \rightarrow -jk$
$\partial_\varphi \rightarrow +jm$
$\partial_t \rightarrow +j\omega$
$f(r,\varphi,z,t\ ) \rightarrow f(r,m)$

We can now practice with these rules to convert various equations of interest . Here a field with unstated arguments has the full arguments (r,φ,z,t).

Example 2: The E_r Helmholtz Equation (D.1.9): $[\nabla^2\mathbf{E}]_r + \beta^2\, E_r = 0$.

Using (D.1.12) we find,

$$\nabla^2(E_r)\ - (2/r^2)\,\partial_\varphi E_\varphi - (1/r^2)\,E_r + \beta^2 E_r \ = 0$$

$$[\partial_r^2 + (1/r)\partial_r + (1/r^2)\partial_\varphi^2 + \partial_z^2]\, E_r - (2/r^2)\,\partial_\varphi E_\varphi - (1/r^2)\,E_r + \beta^2 E_r \ = 0$$

$$[\partial_r^2 + (1/r)\partial_r + (1/r^2)\partial_\varphi^2 + \partial_z^2 - (1/r^2) + \beta^2]\, E_r - (2/r^2)\,\partial_\varphi E_\varphi = 0$$

Now apply the conversion rules to get

$$[\partial_r^2 + (1/r)\partial_r + (1/r^2)(-m^2) - k^2 - (1/r^2) + \beta^2]\, E_r(r,m) - (2/r^2)\, jm\, E_\varphi(r,m) = 0$$

Example 3: The E_φ Helmholtz Equation (D.1.9): $[\nabla^2 \mathbf{E}]_\varphi + \beta^2 E_\varphi = 0$.

Using (D.1.12) we find,

$$\nabla^2(E_\varphi) + (2/r^2)\, \partial_\varphi E_r - (1/r^2)\, E_\varphi + \beta^2 E_\varphi = 0$$

$$[\partial_r^2 + (1/r)\partial_r + (1/r^2)\partial_\varphi^2 + \partial_z^2]\, E_\varphi + (2/r^2)\, \partial_\varphi E_r - (1/r^2)\, E_\varphi + \beta^2 E_\varphi = 0$$

$$[\partial_r^2 + (1/r)\partial_r + (1/r^2)\partial_\varphi^2 + \partial_z^2 - (1/r^2) + \beta^2]\, E_\varphi + (2/r^2)\, \partial_\varphi E_r = 0$$

Now apply the conversion rules to get

$$[\partial_r^2 + (1/r)\partial_r + (1/r^2)(-m^2) + (-k^2) - (1/r^2) + \beta^2]\, E_\varphi(r,m) + (2/r^2)\, jm\, E_r(r,m) = 0$$

Example 4: The div $\mathbf{E} = 0$.

Using (D.1.14) we write div E = 0 as

$$\partial_r (r\, E_r) + \partial_\varphi E_\varphi + r\, \partial_z E_z = 0$$

Applying the conversion rules gives

$$\partial_r [r\, E_r(r,m)] + jmE_\varphi(r,m) + r\,(-jk)E_z(r,m) = 0$$

$$[1 + r\partial_r]\, E_r(r,m) + jmE_\varphi(r,m) + r\,(-jk)E_z(r,m) = 0$$

Summary of Examples:

$[\nabla^2 \mathbf{E}]_z + \beta^2 E_z = 0$:

$$[r^2\partial_r^2 + r\,\partial_r - m^2 + r^2(\beta^2 - k^2)]\, E_z(r,m) = 0 \qquad \text{(D.1.16)}$$

$[\nabla^2 \mathbf{E}]_r + \beta^2 E_r = 0$:

$$[r^2\partial_r^2 + r\partial_r - (m^2+1) + r^2(\beta^2-k^2)]\, E_r(r,m) - 2jm\, E_\varphi(r,m) = 0 \qquad \text{(D.1.17)}$$

$[\nabla^2 \mathbf{E}]_\varphi + \beta^2 E_\varphi = 0$:

$$[r^2\partial_r^2 + r\partial_r - (m^2+1) + r^2(\beta^2-k^2)]\, E_\varphi(r,m) + 2jmE_r(r,m) = 0 \qquad \text{(D.1.18)}$$

div $\mathbf{E} = 0$:

$$\partial_r [r\, E_r(r,m)] + jmE_\varphi(r,m) - jk\, r\, E_z(r,m) = 0 \qquad \text{(D.1.19)}$$

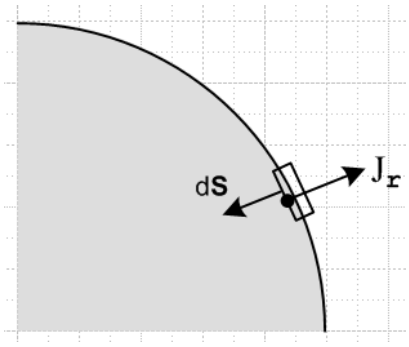
Comment: The *scalar* Helmholtz equation is fully separable in cylindrical coordinates (see M&S p 15) and the "atomic forms" or "harmonics" are of the generic form $e^{jm\varphi}$ x $e^{jk'z}$ x $J_m(jkr)$. In solving a problem involving such a scalar Helmholtz equation, one can expand the solution onto these harmonics with some coefficient for each term in the expansion, and then evaluate the expansion on some boundary to obtain those coefficients. We refer to this solution method as "the method of Smythian forms", admittedly an obscure phrase. In this situation, $e^{jm\varphi}$ is part of the "harmonic" functional form. In contrast, the *vector* Helmholtz equation is NOT separable in cylindrical coordinates (see M&S p 139), it is not even "R-separable", so there are no associated "harmonics" as there are with the scalar Helmholtz equation. Nevertheless, the functions $e^{jm\varphi}$ form a complete set for φ in (-π,π) and our expansion of each E_i onto these $e^{jm\varphi}$ is certainly allowed, even though these $e^{jm\varphi}$ are not part of any associated harmonics for the vector Helmholtz equation. As seen below, when we do this, we end up with ODE's whose solutions are Bessel functions of a strangely phased argument.

(c) The Charge Pumping Boundary Condition

The reason we are interested in surface charge n(φ) of (D.1.5) is that it acts as a driving source of the radial electric field in the wire. Recall the equation of continuity (1.1.8) converted to the ω domain

$$\text{div}J = -j\omega\rho \quad \Leftrightarrow \quad \int_S dS\bullet J = -j\omega \int_V dV\, \rho \ . \quad \text{// divergence theorem} \qquad \text{(D.1.20)}$$

When applied to a thin box of radial area dS straddling the wire surface,



one finds that $\int_S dS\bullet J = -J_r(r=a,\varphi)dS$ and $\int_V dV\, \rho = n(\varphi)\, dS$ so that

$$J_r(r=a,\varphi) = j\omega\, n(\varphi) \ . \qquad \text{(D.1.21)}$$

We assume that there is no current outside the wire to get this result. Since **J** = σ**E**, this is really a boundary condition on the radial electric field,

$$E_r(r=a,\varphi) = (j\omega/\sigma)\, n(\varphi) \ . \qquad \text{(D.1.22)}$$

We convert this to m-space using the conversion rules (D.1.15) to trivially obtain

$$E_r(r=a,m) = (j\omega/\sigma)\, N_m \quad . \qquad \text{(D.1.23)}$$

Thus, the radial electric field must have a certain value at the r=a boundary in each partial wave. And the value it must have is determined by the moment of the charge distribution.

By way of interpretation, the surface charge of a transmission line is "pumped" by the radial current in the wire. Since div**J** = 0 inside the wire, this radial current is accompanied by the usual longitudinal current one expects to find inside the conductors of a transmission line.

(d) The E_z Solution

As shown in (D.1.16), the Helmholtz equation for $E_z(r,m)$ is

$$[r^2\partial_r^2 + r\,\partial_r + (r^2\,\beta'^2 - m^2)]\, E_z(r,m) = 0 \qquad \text{(D.1.24)}$$

where

$$\beta'^2 = \beta^2 - k^2 \quad . \qquad \text{(D.1.25)}$$

Here I suddenly assume that $k = \beta_d$ for the first time. I might instead just assume that we are looking into long wavelength waves, so k (whatever it is) is very small, and then $\beta' \approx \beta$. But if this is not true, what is it that determines k?

In a conductor like copper, β is huge compared our parameter k which we assumed was small, so we could ignore the distinction between β and β'. Setting $x = \beta' r$ we find $\partial_r = \beta'\partial_x$ and then $r\partial_r = x\partial_x$ and so on so that (D.1.24) reads

$$[x^2\partial_x^2 + x\,\partial_x + (x^2 - m^2)]\, E_z(x/\beta',m) = 0\,. \qquad x = \beta' r \qquad \text{(D.1.26)}$$

This is Bessel's equation [Spiegel 24.1] and the solution subject to the condition that E_z be finite at r = 0 is $E_z(x/\beta',m) = C_{zm}\, J_m(x)$ or

$$E_z(r,m) = C_{zm}\, J_m(\beta' r) \qquad \text{(D.1.27)}$$

where C_{zm} is an arbitrary constant for each partial wave m.

Equation (D.1.27) is consistent with (2.1.22). In Section 2.1 we dealt only with the m=0 partial wave, which embodies the symmetrical part of the problem. Also, we assumed constant z behavior so k = 0 and the question of β versus β' never arose.

D.2 The Solutions for E_r and E_φ

(a) The E_r Solution

As shown in (D.1.17), the Helmholtz equation for $E_r(r,m)$ is

$$[r^2\partial_r^2 + r\partial_r - (m^2+1) + r^2\beta'^2]\, E_r(r,m) - 2jm\, E_\varphi(r,m) = 0 \qquad \text{(D.2.1)}$$

while the div **E** = 0 condition was stated in (D.1.19) as

$$[1 + r\partial_r]\, E_r(r,m) + jmE_\varphi(r,m) + r\,(-jk)E_z(r,m) = 0$$

or

$$-jmE_\varphi(r,m) = [1 + r\partial_r]\, E_r(r,m) - r\,(jk)E_z(r,m)\,. \qquad \text{(D.2.2)}$$

Inserting this into (D.2.1) gives

$$[r^2\partial_r^2 + r\partial_r - (m^2+1) + r^2\beta'^2]\, E_r(r,m) + [2 + 2r\partial_r]\, E_r(r,m) - 2r\,(jk)E_z(r,m) = 0$$

or

$$[r^2\partial_r^2 + 3r\partial_r + (1-m^2) + r^2\beta'^2)]\, E_r(r,m) = 2r\,(jk)E_z(r,m)\,. \qquad \text{(D.2.3)}$$

Inserting solution (D.1.27) for $E_z(r,m)$ this becomes

$$[r^2\partial_r^2 + 3r\partial_r + (1-m^2) + r^2\beta'^2)]\, E_r(r,m) = 2r\,(jk)\, C_{zm}\, J_m(\beta' r)$$

or

$$[r^2\partial_r^2 + 3r\partial_r + (1-m^2) + r^2\beta'^2)]\, E_r(r,m) = 2j\,(k/\beta')\, C_{zm}\, \beta' r\, J_m(\beta' r)$$

or

$$[r^2\partial_r^2 + 3r\partial_r + (1-m^2) + r^2\beta'^2)]\, E_r(r,m) = K_m\, \beta' r\, J_m(\beta' r) \qquad \text{(D.2.4)}$$

where

$$K_m \equiv 2j\,(k/\beta')\, C_{zm}\,. \qquad \text{(D.2.5)}$$

In order to get the left side of (D.2.4) into something recognizable, we define

$$E_r(r,m) = x^{-1}\, f_m(x) \qquad \text{(D.2.6)}$$

where x is a dimensionless radial variable which will play a major role in the following,

$$x \equiv \beta' r \qquad \text{and} \qquad x_a \equiv \beta' a\,. \qquad \text{(D.2.7)}$$

Then (D.2.4) becomes

$$[x^2\partial_x^2 + 3x\partial_x + (1-m^2) + x^2]\, \{x^{-1}\, f_m(x)\} = 2j\,(k/\beta')\, xC_{zm}\, J_m(x)$$

or

$$x\,[x^2\partial_x^2 + 3x\partial_x + (1-m^2) + x^2]\, \{x^{-1}\, f_m(x)\} = K\, x^2\, J_m(x)\,. \qquad \text{(D.2.8)}$$

Notice that parameter k has vanished!

Ever eager, Maple expands the left side of (D.2.7),

```
g := fm(x)/x:
L := x*(x^2*diff(g,x,x)+3*x*diff(g,x) + ((1-m^2) + x^2)*g):
     simplify(%): expand(%): collect(%,fm(x));
```

$$(-m^2+x^2)\,\mathrm{fm}(x)+\left(\frac{\partial^2}{\partial x^2}\mathrm{fm}(x)\right)x^2+\left(\frac{\partial}{\partial x}\mathrm{fm}(x)\right)x$$

so that (D.2.8) becomes

$$[\,x^2\,\partial_{x^2}\, + x\,\partial_x + (x^2-m^2)]\,f_m(x) = K_m\,x^2\,J_m(x)\,. \qquad \text{(D.2.9)}$$

The left side of (D.2.9) is the normal Bessel operator [Spiegel 24.1] , but the equation is also *driven* by a power times a Bessel function. The solution to the equation is the homogeneous solution of the Bessel equation plus the particular solution which is the response to the driving function on the right hand side.

The homogeneous solution is the usual linear combination of $J_m(x)$ and $Y_m(x)$, but we must reject $Y_m(x)$ since it blows up at x=0 and thereby causes the field E_r to be singular, which it cannot be, smack in the middle of a wire.

The particular solution is not very obvious and required some hunting to find. It is this

$$f_m(x)^{\mathbf{particular}} = (1/2)\,K_m\,[\,x\;J_{m+1}(x)\,]\quad . \qquad \text{(D.2.10)}$$

a Maple confirms, continuing the above code,

```
fm := (x) -> (1/2)*Km*x*BesselJ(m+1,x);
```

$$fm := x \to \frac{1}{2}\,Km\,x\,\mathrm{BesselJ}(m+1,x)$$

```
L: simplify(%);
```

$$x^2\,Km\,\mathrm{BesselJ}(m,x)$$

Therefore, we now have this full solution for $f_m(x)$

$$f_m(x) = f_m(x)^{\mathbf{particular}} + f_m(x)^{\mathbf{homogeneous}} = (1/2)\,K_m\,[\,x\;J_{m+1}(x)\,] + a_m\,J_m(x)$$

and then from (D.2.6) the full solution E_r ,

$$E_r(r,m) = a_m\;x^{-1}\,J_m(x) + \frac{K_m}{2}\;J_{m+1}(x)\quad . \qquad \text{(D.2.11)}$$

For each value of m, there are two as-yet undetermined constants, a_m and $(K_m/2)$. However, looking at (D.2.11), we see that, since $J_0(x) \approx 1$ for small x, we must have

$$a_0 = 0 \qquad \text{(D.2.12)}$$

to keep E_r finite at r = 0. We shall obtain expressions for a_m and $(K_m/2)$ below.

(b) The E_φ Solution

Recall (D.2.2) ,

$$jmE_\varphi(r,m) = -\partial_r[rE_r(r,m)] + r\,(jk)E_z(r,m)\,. \qquad (D.2.2)$$

We can then insert our known E_z and E_r to get E_φ

$$E_z(r,m) = C_{zm}\,J_m(x) \qquad (D.1.27)$$

$$E_r(r,m) = a_m\,x^{-1}\,J_m(x) + \frac{K_m}{2}\,J_{m+1}(x)\,. \qquad (D.2.11)$$

so (D.22) becomes the following

$$jmE_\varphi(r,m) = -\partial_r[r\{\,a_m\,x^{-1}\,J_m(x) + \frac{K_m}{2}\,J_{m+1}(x)\}] + r\,(jk)\,C_{zm}\,J_m(x)$$

$$jmE_\varphi(r,m) = -\partial_x[x\{\,a_m\,x^{-1}\,J_m(x) + \frac{K_m}{2}\,J_{m+1}(x)\}] + \frac{K_m}{2}\,x\,J_m(x) \qquad // \; K_m \equiv 2j\,(k/\beta')\,C_{zm}$$

$$jmE_\varphi(r,m) = -\partial_x[a_m\,J_m(x) + \frac{K_m}{2}\,x\,J_{m+1}(x)] + \frac{K_m}{2}\,x\,J_m(x)$$

$$jmE_\varphi(r,m) = -a_m\,J_m'(x) - \frac{K_m}{2}\,J_{m+1}(x) - \frac{K_m}{2}\,x\,J_{m+1}'(x) + \frac{K_m}{2}\,x\,J_m(x)$$

$$jmE_\varphi(r,m) = -a_m\,J_m'(x) + \frac{K_m}{2}\,[\,-J_{m+1}(x) - x\,J_{m+1}'(x) + x\,J_m(x)\,] \qquad (D.2.13)$$

At this point we invoke the recurrence relations AS 10.6.2

10.6.2

$$\mathscr{C}_\nu'(z) = \mathscr{C}_{\nu-1}(z) - (\nu/z)\,\mathscr{C}_\nu(z),$$
$$\mathscr{C}_\nu'(z) = -\mathscr{C}_{\nu+1}(z) + (\nu/z)\,\mathscr{C}_\nu(z).$$

to write

$$J_{m+1}' = J_m - (m+1)x^{-1}J_{m+1} \qquad \text{first relation with } \nu = m+1$$
$$J_m' = -J_{m+1} + (m/x)J_m \qquad \text{second relation with } \nu = m \qquad (D.2.14)$$

Insert these into (D.2.18) to get

$$jmE_\varphi(r,m) = -a_m\,J_m' + \frac{K_m}{2}\,[\,-x\,J_{m+1}' - J_{m+1} + xJ_m]$$

$$= -a_m\,\{-J_{m+1} + (m/x)J_m\} + \frac{K_m}{2}\,[\,-x\,\{J_m - (m+1)x^{-1}J_{m+1}\} - J_{m+1} + xJ_m]$$

$$= a_m\,J_{m+1} - a_m\,(m/x)J_m + \frac{K_m}{2}\,[\,-x\,J_m + (m+1)\,J_{m+1} - J_{m+1} + xJ_m]$$

$$= a_m J_{m+1} - a_m (m/x) J_m + \frac{K_m}{2} [\, m J_{m+1}]$$

$$= - a_m (m/x) J_m + \left(\frac{K_m}{2} m + a_m \right) J_{m+1}$$

Dividing by m then gives the final solution:

$$jE_\varphi(r,m) = - a_m x^{-1} J_m(x) + \left(\frac{K_m}{2} + \frac{a_m}{m} \right) J_{m+1}(x) \qquad x = \beta' r \quad . \tag{D.2.15}$$

What about $E_\varphi(r,m=0)$?

We have already noted that $a_0 = 0$. For m=0, (D.2.15) is invalid because we divided (D.2.13) by m. In fact, we really know nothing at all about the field $E_\varphi(r,m=0)$ from our solution method because E_φ is not even present in our starting equation (D.2.2) when m = 0,

$$-jmE_\varphi(r,m) = [1 + r\partial_r] E_r(r,m) - r(jk)E_z(r,m) \, . \tag{D.2.2}$$

However, we can use the E_φ Helmholtz equation (D.1.18) to learn more about $E_\varphi(r,m)$ with m = 0,

$$[r^2\partial_r^2 + r\partial_r - (m^2+1) + r^2\beta'^2] E_\varphi(r,m) + 2jmE_r(r,m) = 0 \tag{D.1.18}$$

With m = 0 we get

$$[r^2\partial_r^2 + r\partial_r - 1 + r^2\beta'^2] E_\varphi(r,0) = 0$$

or

$$[x^2\partial_x^2 + x\partial_x - 1 + x^2] E_\varphi(r,0) = 0 \qquad x = \beta' r \tag{D.2.16}$$

Since [..] is the ν = 1 Bessel operator and since E_φ must be finite at r = 0, we find that

$$E_\varphi(r,0) = C_{\varphi 0} J_1(\beta' r) \tag{D.2.17}$$

where $C_{\varphi 0}$ is some constant.

(c) Application of the Boundary Conditions

We have two boundary conditions to impose:

$$E_r(r=a,m) = (j\omega/\sigma) N_m \tag{D.2.18}$$

$$E_\varphi(r=a,m) = 0 \tag{D.2.19}$$

The first is the radial charge pumping condition shown in (D.1.23) above, while the second is our ansatz assumed earlier that any z = constant circle on the wire surface be an equipotential.

These two boundary conditions serve to determine the two constants a_m and K_m, though a bit of algebra is required. The first step is to use (D.2.11) for E_r and (D.2.15) for E_φ to write out the two boundary conditions as

$$a_m\ x_a^{-1}\ J_m(x_a) + \frac{K_m}{2}\ J_{m+1}(x_a)\ = (j\omega/\sigma)\ N_m \qquad (1)$$

$$-\ a_m\ x_a^{-1}\ J_m(x_a) + \left(\frac{K_m}{2}\ + \frac{a_m}{m}\right)\ J_{m+1}(x_a)\ = 0\ . \qquad (2)$$

Addition and subtraction of these equations gives two new equations,

$$\frac{K_m}{2}\ J_{m+1}(x_a)\ +\ \left(\frac{K_m}{2}\ + \frac{a_m}{m}\right)\ J_{m+1}(x_a)\ =\ (j\omega/\sigma)\ N_m \qquad (3)$$

$$2\ a_m\ x_a^{-1}\ J_m(x_a) - \frac{a_m}{m}\ J_{m+1}(x_a)\ = (j\omega/\sigma)\ N_m \qquad (4)$$

the second of which may be immediately solved for a_m

$$\frac{a_m}{m}\ = (j\omega/\sigma)\ N_m \frac{1}{J_{m-1}(x_a)}\ . \qquad (5)$$

Using the recursion relation $2m\ x^{-1}\ J_m = [J_{m+1} + J_{m-1}]$ and using (5) for a_m , equation (2) may be solved to get

$$\left(\frac{K_m}{2}\ + \frac{a_m}{m}\right)\ =\ (j\omega/2\sigma)\ N_m\ \left[\frac{1}{J_{m+1}(x_a)}\ + \frac{1}{J_{m-1}(x_a)}\right] \qquad (6)$$

Finally, subtracting (5) from (6) we find

$$\frac{K_m}{2}\ = (j\omega/2\sigma)\ N_m\ \left[\frac{1}{J_{m+1}(x_a)}\ - \frac{1}{J_{m-1}(x_a)}\right] \qquad (7)$$

For m = 0 we use (D.2.17) for E_φ and the two boundary conditions are then

$$\frac{K_0}{2}\ J_1(x_a)\ = (j\omega/\sigma)\ N_0 \qquad (D.2.20)$$

$$C_{\varphi 0}\ J_1(x_a)\ = 0\ \ \Rightarrow\ \ C_{\varphi 0} = 0 \qquad (D.2.21)$$

The K_0 result is compatible with (7) since $J_{-1} = -\ J_1$ and the (9) requires that $C_{\varphi 0} = 0$, barring some strange radius which happens to be a zero of the $J_1(x)$ function.

To summarize,

$$a_m = 2m(j\omega/2\sigma)\, N_m \frac{1}{J_{m-1}(x_a)} \qquad m\geq 0$$

$$\frac{K_m}{2} = (j\omega/2\sigma)\, N_m \left[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}\right] \qquad m\geq 0$$

$$\left(\frac{K_m}{2} + \frac{a_m}{m}\right) = (j\omega/2\sigma)\, N_m \left\{\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}\right\} \qquad m>0$$

$$C_{\varphi 0} = 0 \qquad m = 0 \qquad \text{(D.2.22)}$$

We have now completely solved the problem. To review, here are the three fields gathered together in one spot:

$$E_z(r,m) = -j(\beta'/k)\frac{K_m}{2} J_m(x) \qquad \textit{(D.1.27) and (D.2.5)}$$

$$E_r(r,m) = a_m\, x^{-1}\, J_m(x) + \frac{K_m}{2}\, J_{m+1}(x) \qquad \textit{(D.2.11)} \qquad \text{(D.2.23)}$$

$$jE_\varphi(r,m) = -\, a_m\, x^{-1}\, J_m(x) + \left(\frac{K_m}{2} + \frac{a_m}{m}\right) J_{m+1}(x) \qquad m > 0 \quad \textit{(D.2.15)}$$

$$jE_\varphi(r,0) = 0 \qquad m = 0 \quad \textit{(D.2.17)} \text{ and } C_{\varphi 0} = 0$$

D.3 What about the E_φ Helmholtz Equation ?

Except to find $E_\varphi(r,0)$ we have completely ignored the E_φ Helmholtz equation until now. It is reasonable to wonder whether the solution fields we have found above in fact solve this equation?

A related question is whether the three Helmholtz equations and div **E** = 0 are four independent equations, or is one of the three Helmholtz equations dependent? In Cartesian coordinates suppose we know that (implied sums on repeated indices)

$$(\partial_j\partial_j + \beta^2)\, E_1 = 0$$
$$(\partial_j\partial_j + \beta^2)\, E_2 = 0$$
$$\partial_i E_i = 0 \ . \qquad \text{(D.3.1)}$$

Can we show that $(\partial_j\partial_j + \beta^2)\, E_3 = 0$ so this third Helmholtz equation is dependent? If we apply the operator $(\partial_j\partial_j + \beta^2)$ to the last equation above we get

$$(\partial_j\partial_j + \beta^2)\, \partial_i E_i = 0$$

or

$$\partial_i\, (\partial_j\partial_j + \beta^2)\, E_i = 0$$

or

$$\partial_1 (\partial_j\partial_j + \beta^2)\, E_1 + \partial_2 (\partial_j\partial_j + \beta^2)\, E_2 + \partial_3 (\partial_j\partial_j + \beta^2)\, E_3 = 0$$

or

$$\partial_3 [(\partial_j\partial_j + \beta^2)\, E_3] = 0 \qquad \text{(D.3.2)}$$

This does not prove that $(\partial_j\partial_j + \beta^2)\, E_3 = 0$, but it certainly suggests it.

Perhaps $(\partial_j\partial_j + \beta^2)\, E_3 = f(x_1,x_2) \neq 0$, but other considerations might force f = 0, such as the nature of curvilinear coordinates and boundary conditions.

Rather than pursue this question further, we will now directly show that our solutions do in fact satisfy the E_φ Helmholtz equation. The E_φ Helmholtz equation (D.1.18) is this, expressed in terms of $x = \beta' r$,

$$[x^2\partial_x^2 + x\partial_x - (m^2+1) + x^2)]\, E_\varphi(r,m) = -\, 2jmE_r(r,m) = 0 \qquad \textit{(D.1.18)} \qquad \text{(D.3.3)}$$

and we found these solutions for E_r and E_φ ,

$$E_r(r,m) = a_m\, x^{-1}\, J_m(x) + \frac{K_m}{2}\, J_{m+1}(x) \ . \qquad \textit{(D.2.11)}$$

$$jE_\varphi(r,m) = -\, a_m\, x^{-1}\, J_m(x) + \left(\frac{K_m}{2} + \frac{a_m}{m} \right) J_{m+1}(x) \qquad x = \beta' r \ . \qquad \textit{(D.2.15)}$$

Treating (D.2.22) as LHS = RHS we then have

$$\text{LHS} = [x^2\partial_x^2 + x\partial_x - (m^2+1) + x^2]\, \{- a_m\, x^{-1}\, J_m(x) + \left(\frac{K_m}{2} + \frac{a_m}{m} \right) J_{m+1}(x)\, \}$$

$$\text{RHS} = 2m\{\, a_m\, x^{-1}\, J_m(x) + \frac{K_m}{2}\, J_{m+1}(x)\} \ . \qquad \text{(D.3.4)}$$

In order to have LHS = RHS, it must be true for arbitrary a_m and K_m . Thus, we want to show that each of the following equations is valid,

$$[x^2\partial_x^2 + x\partial_x - (m^2+1) + x^2]\, \left(\frac{K_m}{2}\, J_{m+1}(x) \right) = 2m\, \frac{K_m}{2}\, J_{m+1}(x)$$

$$[x^2\partial_x^2 + x\partial_x - (m^2+1) + x^2]\, \{- a_m\, x^{-1}\, J_m(x) + \frac{a_m}{m}\, J_{m+1}(x)\, \} = 2m\, a_m\, \{\, x^{-1}\, J_m(x)\, \}$$

which are the same as

$$[x^2\partial_x^2 + x\partial_x - (m^2+1) + x^2]\, (J_{m+1}(x)\,) = 2m\ J_{m+1}(x) \qquad \text{(D.3.5)}$$

$$[x^2\partial_x^2 + x\partial_x - (m^2+1) + x^2]\, \{- x^{-1}\, J_m(x) + \frac{1}{m}\, J_{m+1}(x)\, \} = 2m\, \{\, x^{-1}\, J_m(x)\, \} \ . \qquad \text{(D.3.6)}$$

Equation (D.3.5) may be written

$$[x^2\partial_x^2 + x\partial_x - (m^2+1) + x^2 - 2m]\, J_{m+1}(x) = 0$$

or

$[x^2\partial_x^2 + x\partial_x + x^2 - (m+1)^2]\, J_{m+1}(x) = 0$.

But this is Bessel's equation for $\nu = m+1$, thus (D.3.5) really is valid.

As for (D.3.6), we leave it to trusty Maple :

```
LHS := x^2*diff(f(x),x,x)+ x*diff(f(x),x) - (m^2+1)*f(x) + x^2*f(x);
```

$$LHS := x^2\left(\frac{\partial^2}{\partial x^2} f(x)\right) + x\left(\frac{\partial}{\partial x} f(x)\right) - (m^2+1)\, f(x) + x^2 f(x)$$

```
f := x -> -(1/x)*BesselJ(m,x) + (1/m)*BesselJ(m+1,x);
```

$$f := x \to -\frac{\mathrm{BesselJ}(m, x)}{x} + \frac{\mathrm{BesselJ}(m+1, x)}{m}$$

```
LHS := x^2*diff(f(x),x,x)+ x*diff(f(x),x) - (m^2+1)*f(x) + x^2*f(x): simplify(%);
```

$$2\,\frac{m\,\mathrm{BesselJ}(m, x)}{x}$$

Thus (D.3.6) is also valid

We conclude that the our round wire **E** field solutions satisfy the E_φ Helmholtz equation as well as the other two Helmholtz equations and the div **E** = 0 equation.

D.4 Statement of the Results

Although we have a complete solution to our problem, it is useful to normalize the results to some well defined absolute scale. The first step in doing this is to compute the total current I in the wire using $J_z = \sigma E_z$. The purpose here is relate the current I to the moment N_0 of the surface charge distribution. Many steps are required to ferret out this relationship:

$$I = \int_0^{2\pi} d\varphi \int_0^a r\, dr\, J_z(r,\varphi) = \int_0^{2\pi} d\varphi \int_0^a r\, dr \left\{ \sigma \sum_{m=-\infty}^{\infty} E_z(r,m)\, e^{jm\varphi} \right\} \quad \text{// (D.1.3)}$$

$$= \sigma \sum_{m=-\infty}^{\infty} \int_0^a r\, dr\, E_z(r,m) \int_0^{2\pi} d\varphi\, e^{jm\varphi} = 2\pi\sigma \int_0^a r\, dr\, E_z(r,0)$$

$$= 2\pi\sigma \int_0^a r\, dr \left\{ -j(\beta'/k)\frac{K_0}{2} J_0(x) \right\} \quad \text{// (D.2.21)} \quad x = \beta' r$$

$$= -j(\beta'/k)\, 2\pi\sigma \frac{K_0}{2} \int_0^a r\, dr\, J_0(x) \quad \text{// } x = \beta' r \text{ so } x\,dx = \beta'^2\, r\,dr$$

$$= -j(\beta' k)^{-1}\, 2\pi\sigma \frac{K_0}{2} \left[\int_0^{x_a} dx\, x\, J_0(x) \right] = -j(\beta'\beta_d)^{-1}\, 2\pi\sigma \frac{K_0}{2} \left[x_a\, J_1(x_a) \right]$$

$$= -j(\beta' k)^{-1}\, 2\pi\sigma \left\{ (j\omega/\sigma)\, N_0 \,/\, J_1(x_a) \right\} \left[x_a\, J_1(x_a) \right] \quad \text{// (D.2.20)}$$

$$= (\beta' k)^{-1}\, 2\pi\omega\, N_0\, x_a = (\beta' k)^{-1}\, 2\pi\omega\, N_0\, \beta' a$$

$= 2\pi\omega\,(a/k)\,N_0$

so that

$$N_0 = (k/2\pi\omega a)\,I \ . \tag{D.4.1}$$

At this point it is convenient to introduce the DC resistance per unit length of our wire,

$$R_{dc} = \frac{1}{\sigma\pi a^2} \tag{D.4.2}$$

along with a new symbol to indicate the *relative* surface charge moment,

$$\eta_m \equiv \frac{N_m}{N_0} \tag{D.4.3}$$

It follows from (D.4.1) that the normalization factor appearing in (D.2.22) may be written as

$$(j\omega/2\sigma)\,N_m = (j\omega/2\sigma)\,\frac{N_m}{N_0}\,N_0 = (j\omega/2\sigma)\,\eta_m\,[(k/2\pi\omega a)\,I\,] = (j/4)\,\eta_m\,(ak/\sigma\pi a^2)\,I$$

$$= (j/4)\,(ak)\,\eta_m\,I\,R_{dc} \ . \tag{D.4.5}$$

We may now construct the final form for our solutions from (D.2.23) and (D.2.22) for m > 0:

$$E_z(r,m) = -j(\beta'/k)\,\frac{K_m}{2}J_m(x) = -j(\beta'/k)\,(j\omega/2\sigma)\,N_m\,[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}]\,J_m(x)$$

$$= -j(\beta'/k)\,[(j/4)\,(ak)\,\eta_m\,I\,R_{dc}]\,[\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}]\,J_m(x)$$

$$= (1/4)\,\eta_m\,I\,R_{dc}\,[\frac{x_a J_m(x)}{J_{m+1}(x_a)} - \frac{x_a J_m(x)}{J_{m-1}(x_a)}]$$

$$E_r(r,m) = a_m\,x^{-1}\,J_m(x) + \frac{K_m}{2}\,J_{m+1}(x)$$

$$= [(j\omega/2\sigma)\,N_m]\,\{\ 2m\,\frac{1}{J_{m-1}(x_a)}\,x^{-1}\,J_m(x) + [\frac{1}{J_{m+1}(x_a)} - \frac{1}{J_{m-1}(x_a)}]\,J_{m+1}(x)\ \}$$

$$= (j/4)\,(ak)\,\eta_m\,I\,R_{dc}\ \{\frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m+1}(x)}{J_{m-1}(x_a)}\}$$

$$jE_\varphi(r,m) = -a_m\,x^{-1}\,J_m(x) + (\frac{K_m}{2} + \frac{a_m}{m})\,J_{m+1}(x)$$

$$= [(j\omega/2\sigma)\, N_m] \{ - 2m\, x^{-1} \frac{1}{J_{m-1}(x_a)} + [\frac{1}{J_{m+1}(x_a)} + \frac{1}{J_{m-1}(x_a)}]\, J_{m+1}(x)$$

$$= (j/4)\,(ak)\,\eta_m\, I\, R_{dc} \{ - \frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + [\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m+1}(x)}{J_{m-1}(x_a)}] \}$$

We summarize these results in a box. The m=0 results are all obtainable from the general expressions with m = 0 using $J_{-1} = -J_1$:

E Fields Inside a Round Wire

$x = \beta' r \qquad \beta'^2 = \beta^2 - k^2 \approx \beta^2 \qquad \beta = \omega\sqrt{\mu\xi} \qquad \xi = [\varepsilon + \sigma/(j\omega)] \approx \sigma/(j\omega)$ conductor

$x_a = \beta' a$ k = assumed wavenumber for wave solution

$\mathbf{E}(r,\varphi,z,t) = e^{j(\omega t - kz)}\,\mathbf{E}(r,\varphi)$ *(D.1.2)* $\mathbf{E} = E_r\hat{\mathbf{r}} + E_\varphi\hat{\boldsymbol{\varphi}} + E_z\hat{\mathbf{z}}$ (for any arguments)

$$\mathbf{E}(r,\varphi) = \mathbf{E}(r,0) + 2\sum_{m=1}^{\infty}[\mathrm{Re}\{\mathbf{E}(r,m)\}\cos(m\varphi) - \mathrm{Im}\{\mathbf{E}(r,m)\}\sin(m\varphi)] = \text{real} \qquad (D.1.4)$$

where:

$$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\left[\frac{x_a J_m(x)}{J_{m+1}(x_a)} - \frac{x_a J_m(x)}{J_{m-1}(x_a)}\right] \qquad a = \text{radius} \qquad \eta_m \equiv \frac{N_m}{N_0}$$

$$E_z(r,0) = (1/2)\, I\, R_{dc}\left[\frac{x_a\, J_0(x)}{J_1(x_a)}\right] \qquad // = (2.1.22) \text{ for } k = 0 \qquad \eta_0 = 1$$

$$E_r(r,m) = (j/4)\,(ak)\,\eta_m\, I\, R_{dc}\left[\frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m+1}(x)}{J_{m-1}(x_a)}\right]$$

$$E_r(r,0) = (j/2)\,(ak)\, I\, R_{dc}\left[\frac{J_1(x)}{J_1(x_a)}\right]$$

$$E_\varphi(r,m) = (1/4)\,(ak)\,\eta_m\, I\, R_{dc}\left[-\frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} + \frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m+1}(x)}{J_{m-1}(x_a)}\right]$$

$$E_\varphi(r,0) = 0$$

(D.4.6)

ok to here and holding

Observations about the solution:

(1) We looked for a wave solution inside a round wire in which phase fronts propagate down the wire (z direction) with angular frequency ω and wavelength λ = 2π/k, as indicated by (D.1.2) in the box. We found the solution shown in the box. It satisfies all three components of the Helmholtz equation (D.1.1) as well as the div **E** = 0 equation (no charge inside conductor).

(2) For a good conductor and a good dielectric, one has $\xi \approx \sigma/(j\omega)$ and $\xi_d \approx \varepsilon_d$ as shown in the box. These are the complex dielectric "constants". The corresponding wavenumbers are then

$$\beta = \omega\sqrt{\mu\xi} \approx \omega\sqrt{\mu\,\sigma/(j\omega)} = e^{j3\pi/4}\sqrt{\omega\mu\sigma} = e^{j3\pi/4}(\sqrt{2}/\delta) \qquad (2.1.18),\ (2.1.21)$$

$$\beta_d = \omega\sqrt{\mu_d\xi_d} \approx \omega\sqrt{\mu_d\varepsilon_d} = \omega / v_d \qquad v_d = \text{speed of light in the dielectric} \qquad (D.4.7)$$

Thus, in our wave solution (D.1.2), the phase fronts propagate down the *inside* of the wire at v_d, the speed of light in the dielectric *outside* the wire. Although we have been silent about the fields outside the wire, it seems reasonable to presume there is a wave outside also moving down the wire at v_d . See below.

(3) The fact that the wavenumber β inside the conductor is complex indicates the presence of absorption of energy in the conductor due to σ which one could calculate in detail.

(4) In our axially symmetric round wire analysis of Section 2.1 we assumed *no* z variation of the fields, which means wave number $\beta_d = 0$ so β' = β. In this case the box above shows $E_r(r,m) = 0$ and $E_\varphi(r,m) = 0$ for all $m \geq 0$. Our symmetry assumptions in Section 2.1 limited the analysis there to m = 0 and the $E_z(r,0)$ shown in the box above agrees with result (2.1.22).

(5) The components E_φ and E_r are *smaller* than E_z by factor ($\beta_d a$). This is the dimensionless smallness parameter which defines the "transmission line limit", see Chapter 4.

(6) If there exist moments N_m of the charge distribution on the wire with m > 1, then the corresponding $\eta_m \neq 0$ and it is clear that $E_z(r,\varphi)$ and hence $J_z(r,\varphi)$ is *non-uniform* around the surface of the wire. That is, these fields vary with φ as cos(mφ). The non-uniformity is not "small" but has the full strength of η_m. Of course we only expect to get significant moments of charge density n(φ) when conductors are "fat and close". It seems likely in this case that the largest contribution will come from m=1.

(7) The surface impedance from (C.2.1) is just $Z_s(\varphi) = E_z(r=a,\varphi)/I$. Using the boxed result for $E_z(r,\varphi)$ we quickly find that

$$Z_s(\varphi) = (1/2)\, R_{dc} \left\{ \frac{x_a\, J_0(x_a)}{J_1(x_a)} + \sum_{m=1}^{\infty} \left[\frac{x_a J_m(x_a)}{J_{m+1}(x_a)} - \frac{x_a J_m(x_a)}{J_{m-1}(x_a)} \right] \mathrm{Re}[\eta_m e^{jm\varphi}] \right\}$$

where (D.4.8)

$$\mathrm{Re}[\eta_m e^{jm\varphi}] = \mathrm{Re}(\eta_m)\cos(m\varphi) - \mathrm{Im}(\eta_m)\sin(m\varphi).$$

Thus we see the expected non-uniformity of $Z_z(\varphi)$ around the perimeter of the wire cross section for the m > 0 components.

D.5 The Low Frequency Limit

If we assume that ω is small enough that $|x_a| = |\beta'|a << 1$, we can greatly simplify the above results. For a good conductor, $\beta' \approx \beta = e^{j3\pi/4}\,(\sqrt{2}/\delta)$ so $|\beta'|a << 1 \Rightarrow (\sqrt{2}\, a/\delta) << 1$ or $(a/\delta) << 1$, which means the skin depth is much larger than the radius of the wire. In terms of frequency ω this condition is

$$\sqrt{\omega\mu\sigma}\; a << 1 \qquad \Rightarrow \qquad \omega << \frac{1}{\mu\sigma a^2} \; .$$

Using our ongoing example, for the Belden 8281 coax center conductor with a = .394 mm, $\mu = \mu_0 = 4\pi \times 10^{-7}$ henry/m, and $\sigma = 5.81 \times 10^7$ mho/m (copper), the limit requires that

$$\omega << \frac{10^6}{4\pi * 5.81 * (.394)^2} = 88 \text{ KHz}.$$

All limits come from the leading small-x term of $J_n(x)$ which is

$$J_m(x) \approx x^m/(2^m m!) \quad m \geq 0 \qquad \text{// Spiegel (24.2)} \qquad \text{(D.5.1)}$$

To avoid errors, we have Maple compute the limits of the various terms appearing in box ***.

```
A := xa*J(m,x)/J(m+1,xa): simplify(%);
```

$$\frac{xa\,J(m,x)}{J(m+1,xa)}$$

```
B := xa*J(m,x)/J(m-1,xa): simplify(%);
```

$$\frac{xa\,J(m,x)}{J(m-1,xa)}$$

```
C := (2*m/x)*J(m,x)/J(m-1,xa): simplify(%);
```

$$2\frac{m\,J(m,x)}{x\,J(m-1,xa)}$$

```
D_ := J(m+1,x)/J(m+1,xa): simplify(%);
```

$$\frac{J(m+1,x)}{J(m+1,xa)}$$

```
E := J(m+1,x)/J(m-1,xa): simplify(%);
```

$$\frac{J(m+1,x)}{J(m-1,xa)}$$

```
F := xa*J(0,x)/J(1,xa): simplify(%);
```

$$\frac{xa\,J(0,x)}{J(1,xa)}$$

```
G := J(1,x)/J(1,xa): simplify(%);
```

$$\frac{J(1,x)}{J(1,xa)}$$

```
J := (m,x) -> x^m/(2^m*m!);
```

$$J := (m,x) \to \frac{x^m}{2^m\,m!}$$

```
A := xa*J(m,x)/J(m+1,xa): simplify(%);
```

$$2\,xa^{(-m)}\,x^m\,(m+1)$$

```
B := xa*J(m,x)/J(m-1,xa): simplify(%);
```

$$\frac{1}{2}\frac{xa^{(2-m)}\,x^m}{m}$$

```
C := (2*m/x)*J(m,x)/J(m-1,xa): simplify(%);
```

$$x^{(m-1)}\,xa^{(-m+1)}$$

```
D_ := J(m+1,x)/J(m+1,xa): simplify(%);
```

$$x^{(m+1)}\,xa^{(-m-1)}$$

```
E := J(m+1,x)/J(m-1,xa): simplify(%);
```

$$\frac{1}{4}\frac{x^{(m+1)}\,xa^{(-m+1)}}{m\,(m+1)}$$

```
F := xa*J(0,x)/J(1,xa): simplify(%);
```

$$2$$

```
G := J(1,x)/J(1,xa): simplify(%);
```

$$\frac{x}{xa}$$

The left column is for visual check, the right column then provides the limits. Our limit is $x_a << 1$ and since $x \leq x_a$ we also have $x << 1$, but the ratio $(x/x_a) = (r/a)$ is merely ≤ 1 and not $<< 1$. From the column on the right above we read off the limits:

$$A = \frac{x_a J_m(x)}{J_{m+1}(x_a)} \approx 2(m+1)(x/x_a)^m$$

$$B = \frac{x_a J_m(x)}{J_{m-1}(x_a)} \approx (1/2m)(x/x_a)^m\, x_a^2 \approx 0 \qquad \text{since } x_a << 1$$

$$C = \frac{2mx^{-1}J_m(x)}{J_{m-1}(x_a)} \approx (x/x_a)^{m-1}$$

$$D = \frac{J_{m+1}(x)}{J_{m+1}(x_a)} \approx (x/x_a)^{m+1}$$

$$E = \frac{J_{m+1}(x)}{J_{m-1}(x_a)} \approx \frac{1}{4m(m+1)}\,(x/x_a)^{m+1}\, x_a^2 \approx 0$$

$$F = \frac{x_a\ J_0(x)}{J_1(x_a)} \qquad \approx\ 2$$

$$G = \frac{J_1(x)}{J_1(x_a)} \qquad \approx\ (x/x_a) \qquad \text{(D.5.2)}$$

In terms of the letters A through G the E(r,m) fields from box *** are given by

$$E_z(r,m) = (1/4)\ \eta_m\ I\ R_{dc}\ [A - B]$$
$$E_z(r,0) = (1/2)\ I\ R_{dc}\ [F]$$

$$E_r(r,m) = (j/4)\ (a\beta_d)\ \eta_m\ I\ R_{dc}\ [C + D - E]$$
$$E_r(r,0) = (j/2)\ (a\beta_d)\ I\ R_{dc}\ [G]$$

$$E_\varphi(r,m) = (1/4)\ (a\beta_d)\ \eta_m\ I\ R_{dc}\ [-C + D + E]$$
$$E_\varphi(r,0) = 0 \qquad \text{(D.5.3)}$$

Installing the limits and setting $(x/x_a) = (r/a)$ we find

E Fields Inside a Round Wire (low frequency limit)

$$E_z(r,m) = (1/2)\ \eta_m\ I\ R_{dc}\ (m+1)(r/a)^m$$
$$E_z(r,0) = I\ R_{dc}$$

$$E_r(r,m) = (j/4)\ (a\beta_d)\ \eta_m\ I\ R_{dc}\ [(r/a)^{m-1} + (r/a)^{m+1}]$$
$$E_r(r,0) = (j/2)\ (a\beta_d)\ I\ R_{dc}\ (r/a)$$

$$E_\varphi(r,m) = (1/4)\ (a\beta_d)\ \eta_m\ I\ R_{dc}\ [-(r/a)^{m-1} + (r/a)^{m+1}]$$
$$E_\varphi(r,0) = 0 \qquad \text{(D.5.4)}$$

The equation $E_z(r,0) = I\ R_{dc}$ states that in this low frequency limit the wire acts as a resistor having the expected resistance per unit length $R_{dc} = 1/(\sigma\pi a^2)$ as in (D.4.2). In the transmission line limit $a\beta_d << 1$ of interest to us, we see that there is always a small radial current which pumps the changing surface charge. In the m = 0 this radial current is linear in r, peaking at the surface.

In fact, if you integrate the radial current over the surface of a long piece of wire of length $\lambda/2$, taking into account the z dependence, you get 2I.

[I think the above has no basis in fact! At DC, we have $\omega = 0$ and $E_r(a,0) = (j/2)\ (a\beta_d)\ I\ R_{dc}$. But $\beta_d = \omega\sqrt{\mu_d\xi_d} \approx \omega\sqrt{\mu_d\varepsilon_d} = 0$ at DC, so the integral of the radial current is 0. For m = 1, the radial current depends on η_1 which we don't know, and then $E_r(a,1) = (j/4)\ (a\beta_d)\ \eta_m\ I\ R_{dc}(2)$ and we have no conclusion at all. Yes it is *proportional* to I. I don't know where this strange claim came from! Throwing in the sinφ angle and integrating over half a wavelength is not going to rescue this weird claim.]

See Section 3.7, Figure 2 for a drawing of the current we have just computed.

continue here after reading Chapter 3.

D.6 What about the E fields *outside* the round wire?

This is an Exercise for the Reader, with some preliminary discussion given below.

The Helmholtz equation (D.1.1) outside the wire now contains β_d instead of β. The assumed wave solution form is still (D.1.2). In the three component Helmholtz equations this means that

$\beta' = \beta_d^2 - \beta_d^2 = 0$ so that:

$[\nabla^2 \mathbf{E}]_z + \beta^2 E_z = 0$:

$[r^2\partial_r^2 + r\,\partial_r - m^2]\, E_z(r,m) = 0$ *(D.1.16)*$_o$

$[\nabla^2 \mathbf{E}]_r + \beta^2 E_r = 0$:

$[r^2\partial_r^2 + r\partial_r - (m^2+1)]\, E_r(r,m) - 2jm\, E_\varphi(r,m) = 0$ *(D.1.17)*$_o$

$[\nabla^2 \mathbf{E}]_\varphi + \beta^2 E_\varphi = 0$:

$[r^2\partial_r^2 + r\partial_r - (m^2+1)]\, E_\varphi(r,m) + 2jmE_r(r,m) = 0$ *(D.1.18)*$_o$

div $\mathbf{E} = 0$:

$\partial_r\,[r\,E_r(r,m)] + jmE_\varphi(r,m) - j\beta_d\, r\, E_z(r,m) = 0$ *(D.1.19)*$_o$

The differential operators appearing in the above equations are no longer Bessel-style operators, they are Euler-style operators. Euler ODE's have the general form $[\, r^2\partial_r^2 + a\, r\, \partial_r + b]\, f(r) = 0$, and the solutions have this form from p 45 of Polyanin's excellent ODE compendium,

118. $x^2 y''_{xx} + axy'_x + by = 0$.

The Euler equation.

Solution:

$$y = \begin{cases} |x|^{\frac{1-a}{2}} \left(C_1 |x|^{\mu} + C_2 |x|^{-\mu}\right) & \text{if } (1-a)^2 > 4b, \\ |x|^{\frac{1-a}{2}} \left(C_1 + C_2 \ln|x|\right) & \text{if } (1-a)^2 = 4b, \\ |x|^{\frac{1-a}{2}} \left[C_1 \sin(\mu \ln|x|) + C_2 \cos(\mu \ln|x|)\right] & \text{if } (1-a)^2 < 4b, \end{cases}$$

where $\mu = \frac{1}{2}|(1-a)^2 - 4b|^{1/2}$.

For $E_z(r,m)$ the equation *(D.1.17)*$_o$ shown just above is in fact an Euler equation which has $a = 1$ and $b = -m^2$ so $\mu = m$ and the solution forms are these ($r \geq a$ outside the wire),

$$E_z(r,m) = A_{zm} r^m + B_{zm} r^{-m} \qquad m > 0$$
$$E_z(r,0) = C_z \ln(r) + D_z \qquad m = 0 \quad .$$

This E_z Helmholtz equation is the same as the radial equation one obtains when solving the 2D Laplace equation $\nabla^2 u = 0$ in separated polar coordinates, see for example Stakgold Vol II p 92 (6.7).

It certainly seems reasonable that $A_{zm} = 0$ to avoid a dramatic blowup at $r = \infty$, but the $m = 0$ coefficients are less obvious, since we have already seen logarithmic divergences associated with infinitely long wires.

One approach is to mimic the method used for the inside solution, making use of the div **E** = 0 equation shown above, and thereby coming up with a full set of E field solutions. Along the way one must determine appropriate boundary conditions at r = a so the outside solutions are consistent with the inside solutions. We can at least see that for $m > 0$ $E_z(r,m) \sim (r/a)^{-m}$ outside the round wire.

D.7 What about φ, A and B ?

Inside the conductor, we know that potential φ satisfies (D.1.1), and we know that φ at r=a does not depend on angle φ. This means that φ is similar to our E_z solution in the m=0 partial wave, and vanishes for all higher partial waves. Thus we write

$$\varphi(r,m) = \delta_{m,0}\ \varphi_0\ J_0(x)/J_0(x_a) \qquad x = \beta' r \qquad \text{(D.7.1)}$$

where φ_0 is the value of the potential on the surface at r=a. Since we know **E**(r,m) from Section 3 (6), we can solve for the vector potential **A** as follows:

$$-j\omega \mathbf{A} = \mathbf{E} + \text{grad}\ \varphi \qquad \text{(D.7.2)}$$

Converted to partial waves, this says

$$-j\omega A_r(r,m) = E_r(r,m) + \delta_{m,0}\ \partial_r \varphi(r,0)$$

$$-j\omega A_\varphi(r,m) = E_\varphi(r,m) - (1/r)\, jm\ \delta_{m,0}\ \varphi(r,0) = E_\varphi(r,m)$$

$$-j\omega A_z(r,m) = E_z(r,m) - j\beta_d\ \delta_{m,0}\ \varphi(r,0) \qquad \text{(D.7.3)}$$

Thus, for $m \neq 0$ we have $-j\omega \mathbf{A} = \mathbf{E}$. For m = 0 there are extra pieces as shown for $A_r(r,0)$ and $A_z(r,0)$.

Since **A**(r,m) is known, **A**(r,φ,z) is also known, and then so too is **B** = curl **A**. This curl can also be performed in each partial wave if desired by replacing $\partial_\varphi = -jm$ and $\partial_z = -j\beta_d$ as usual.

Since $-j\omega \mathbf{A} = \mathbf{E}$ in the higher partial waves, we may conclude that the transverse components of **A** are small compared to the longitudinal component, just as is the case for **E**, see (D.3.6). For m=0 we know

that $A_{\varphi} = E_{\varphi} = 0$, but A_r is a combination of E_r and $\partial_r \varphi$ which is *not* small compared to A_z, due to the $\partial_r \varphi$ contribution.

Thus, we arrive at the conclusion that, although A_r is negligible in the dielectric, it cannot be ignored inside the conductor. This explains, incidentally, the problem one encounters with the gauge condition (1.5.5) div$\mathbf{A} = -j(\beta^2/\omega)\varphi$. Since φ is continuous at the boundary, and β^2 takes a jump of many orders of magnitude (from β_d to β'), something on the left side must change violently. But A_z is also continuous. It is the m=0 A_r inside the conductor that takes up the slack. In fact, taking the difference inside minus outside we conclude that

$$\partial_r A_r\,(r=a-\varepsilon) = -j(\beta'^2/\omega)\varphi_0 \qquad \text{(D.7.4)}$$

from which we can determine φ_0The potential A_r is discontinuous at r=a.