Maxwell's Equations and Monochrome Fields PhL 9.26.13
("round wire ponderings.doc")

Three good "comments" all of which are correct.

Comment 1. Consider the field wave equations in the general case inside either a cylinder of dielectric or a cylinder of copper. Inside the cylinder in either case there is no charge and no applied sources, so

$$(\nabla^2 - \mu\varepsilon\, \partial_t^2 - \mu\sigma\, \partial_t)\, \mathbf{H} = 0 \qquad (1.2.1)$$
$$(\nabla^2 - \mu\varepsilon\, \partial_t^2 - \mu\sigma\, \partial_t)\, \mathbf{E} = 0 \qquad (1.2.2) \qquad \text{ok}$$

In the dielectric cylinder we have $\sigma = 0$ and we get our usual wave equations

$$(\nabla^2 - \mu\varepsilon\, \partial_t^2)\, \mathbf{H} = 0$$
$$(\nabla^2 - \mu\varepsilon\, \partial_t^2)\, \mathbf{E} = 0 \qquad \text{ok}$$

These equations have solutions of the form $\exp[i(\mathbf{k}\bullet\mathbf{x}-\omega t]$ where, because $v = \lambda/T = 2\pi/k * f = \omega/k$, we have $k(\omega) = \omega/v$ where $1/v^2 = \mu\varepsilon$. In this situation, I think it is possible to have solutions where both H and E are monochrome (that is, having simple $e^{-i\omega t}$ time dependence). [correct]

In the copper cylinder the last term is present and we know that the solutions are "damped waves". In this case, if you plot such a wave, you find that it does not have simple monochrome behavior. In this situation, the assumption of monochrome time dependence might be an error. [no driving force]

However, as will be seen in the next comment, for our copper cylinder we are really going to have this situation,

$$(\nabla^2 - \mu\varepsilon\, \partial_t^2 - \mu\sigma\, \partial_t)\, \mathbf{H} = -\text{curl}\, \mathbf{J_a} \qquad (1.2.1)$$
$$(\nabla^2 - \mu\varepsilon\, \partial_t^2 - \mu\sigma\, \partial_t)\, \mathbf{E} = \mu \frac{\partial \mathbf{J_a}}{\partial t} \qquad (1.2.2)$$

where there *is* a certain J_a and corresponding E_a which exists at the surface and is a prescribed functional value or "boundary condition" for our problem. So then we really have a *driven* damped wave equation and in that case we do expect to have monochrome time dependence of the solution if the driving term is monochrome. We might expect however that the phase of the solution is a function of ω. [all correct]

Just as a reminder, consider the elementary problem of a 1D driven **damped harmonic oscillator**. This problem appears starting at page 211 in my blue Berkeley series mechanics book. The solution for a driving term of the form $\sin(\omega t)$ is this

$$x(t) = x(0)\, \sin[\omega t + \varphi(\omega)] \qquad \tan\varphi(\omega) = -(\omega/\tau)/(\omega_0^2 - \omega^2) \qquad [\text{ an excellent example }]$$

where τ is from the damping term and ω_0 is the natural frequency of the DHO. Here you see a monochrome solution which has a phase which is a function of ω ! Now it is true that the differential

equation for the ω-driven DHO (with ω_0 and τ) is not the same as the damped wave equation shown above. The DHO for example has variable position x(t), not field **E**(x,t), but the concept of a monochrome response with an ω-dependent phase is common I think to both systems.

Comment 2. I think my ω-space copper cylinder solution is accurate to the extent that $\varepsilon\omega << \sigma$. One could do the Fourier Transform to find out what the time domain solution looks like, and in particular, what its time dependence looks like. For example, in the notation I used in lines, we have

$$E(r) = E(a)\,[J_0(\beta r) / J_0(\beta a)] \qquad \beta = e^{j3\pi/4}\sqrt{\omega\mu\sigma}$$

or

$$E(r) = E(a)\,\frac{\mathrm{ber}_0[\sqrt{2}(r/\delta)] + j\,\mathrm{bei}_0[\sqrt{2}(r/\delta)]}{\mathrm{ber}_0[\sqrt{2}(a/\delta)] + j\,\mathrm{bei}_0[\sqrt{2}(a/\delta)]} \quad . \qquad z = \sqrt{2}\,(r/\delta) = r\sqrt{\omega\mu\sigma} \qquad (2.2.2)$$

or

$$E(r,\omega) = E(a,\omega)\,\frac{\mathrm{ber}_0[r\sqrt{\omega\mu\sigma}\,] + j\,\mathrm{bei}_0[r\sqrt{\omega\mu\sigma}\,]}{\mathrm{ber}_0[a\sqrt{\omega\mu\sigma}\,] + j\,\mathrm{bei}_0[a\sqrt{\omega\mu\sigma}\,]}$$

The time domain behavior would be the FT of this thing, or

$$E(r,t) = (1/2\pi)\int_{-\infty}^{\infty} d\omega\, e^{j\omega t}\, E(r,\omega)$$

$$= (1/2\pi)\int_{-\infty}^{\infty} d\omega\, e^{j\omega t}\, E(a,\omega)\,\frac{\mathrm{ber}_0[r\sqrt{\omega\mu\sigma}\,] + j\,\mathrm{bei}_0[r\sqrt{\omega\mu\sigma}\,]}{\mathrm{ber}_0[a\sqrt{\omega\mu\sigma}\,] + j\,\mathrm{bei}_0[a\sqrt{\omega\mu\sigma}\,]}$$

I guess at this point we should impose a boundary condition at the surface r = a. Suppose at the surface we have a driving system all along our piece of cylinder which applies $E(a,t) = E_0\, e^{j\omega_1 t}$ in our complex notational world. Then

$$E(a,\omega) = \int_{-\infty}^{\infty} dt\; e^{-i\omega t}\, E(a,t) = E_0\, 2\pi\delta(\omega-\omega_1)$$

and then the above tells us that

$$E(r,t) = E_0 e^{j\omega_1 t}\,\frac{\mathrm{ber}_0[r\sqrt{\omega_1\mu\sigma}\,] + j\,\mathrm{bei}_0[r\sqrt{\omega_1\mu\sigma}\,]}{\mathrm{ber}_0[a\sqrt{\omega_1\mu\sigma}\,] + j\,\mathrm{bei}_0[a\sqrt{\omega_1\mu\sigma}\,]}$$

If we now say that the driving frequency is ω instead of ω_1, our solution is

$$E(r,t) = E_0 e^{i\omega t}\,\frac{\mathrm{ber}_0[r\sqrt{\omega\mu\sigma}\,] + j\,\mathrm{bei}_0[r\sqrt{\omega\mu\sigma}\,]}{\mathrm{ber}_0[a\sqrt{\omega\mu\sigma}\,] + j\,\mathrm{bei}_0[a\sqrt{\omega\mu\sigma}\,]} \quad . \qquad \text{[all seems ok]}$$

Well, if I have not screwed up, this says that E(r,t) really is sinusoidal in time at all values of r inside the cylinder. The phase of E(r,t) is going to be a function of ω, just as it is in the damped harmonic oscillator problem (here E(r,t) is in fact a function of $r\sqrt{\omega}$). So the nature of the solution above is this

$$E(r,t) = e^{j\omega t}\, e^{jf(\omega,r,a)}\; g(\omega,r,a)$$

where f and g are real functions.

So once again, here is the ω-domain and time-domain solution for the E field in the cylinder with the monochrome driving E field boundary condition:

$$E(r,\omega) = E_0 2\pi\delta(\omega-\omega_1)\,\frac{\mathrm{ber}_0[r\sqrt{\omega\mu\sigma}\,] + j\,\mathrm{bei}_0[r\sqrt{\omega\mu\sigma}\,]}{\mathrm{ber}_0[a\sqrt{\omega\mu\sigma}\,] + j\,\mathrm{bei}_0[a\sqrt{\omega\mu\sigma}\,]} \qquad E(a,\omega) = E_0 2\pi\delta(\omega-\omega_1)$$

$$E(r,t) = E_0 e^{i\omega t}\frac{\mathrm{ber}_0[r\sqrt{\omega\mu\sigma}\,] + j\,\mathrm{bei}_0[r\sqrt{\omega\mu\sigma}\,]}{\mathrm{ber}_0[a\sqrt{\omega\mu\sigma}\,] + j\,\mathrm{bei}_0[a\sqrt{\omega\mu\sigma}\,]} \qquad E(a,t) = E_0 e^{i\omega t} \qquad \text{[all seems ok]}$$

Might be good to include some of the above results in Section 2 somewhere.

Comment 3. For the cylinder we have seen above that the solution E(r,ω) which I call just E(r) has a phase which varies with ω, and I am completely simpatico with that notion. What can be said about the phase of the B(r) field? We have these two curl equations for the round wire problem which are

$$\frac{\partial\,[\,rB(r)]}{\partial r} = (\mu\sigma)\,[\,r\,E(r)\,] \qquad\qquad \frac{\partial E(r)}{\partial r} = j\omega B(r)$$

where E(r) is the E(r,ω) above and similarly for B(r). Both of these fields contain the factor $2\pi\delta(\omega-\omega_1)$ we suspect. I must now cure my mental block of yesterday concerning the second equation.

If you had an equation which said E(r) = jωB(r), you would be correct in saying that there is a constant phase shift of 90^0 between E(r,ω) and B(r,ω). But that is not true if the equation is the equation shown above. Let's try to come up with a trivial example unrelated to the round wire problem. Suppose,

$$E(r) = \exp(\sqrt{j}\,r\omega)$$

so

$$\frac{\partial E(r)}{\partial r} = \sqrt{j}\;\omega\,\exp(\sqrt{j}\,r\omega) = j\omega\,[\exp(\sqrt{j}\,r\omega)/\sqrt{j}\,] = j\omega B(r) \qquad B(r) = \exp(\sqrt{j}\,r\omega)\,/\,\sqrt{j}$$

So here we have functions E(r) and B(r) which satisfy $\partial_r E(r) = j\omega B(r)$. The field ratio is

$$E(r)/B(r) = \sqrt{j} = e^{j\pi/4}$$

In this simple example, the phase of the field ratio is 45^0 and not 90^0, so this is a fine counterexample. [good]

Side note: $\exp(j^{\alpha}x) = \exp[e^{j\pi\alpha/2}x] = \exp[x\cos(\pi\alpha/2)+jx\sin(\pi\alpha/2)]$

$$= e^{x\cos(\pi\alpha/2)}\, e^{jx\sin(\pi\alpha/2)}$$

and therefore

$$\text{mag}[\exp(j^{\alpha}x)] = e^{x\cos(\pi\alpha/2)}$$

$$\arg[\exp(j^{\alpha}x)] = x\sin(\pi\alpha/2) .$$

In our little example above, $\alpha = 1/2$ and $x = r\omega$ so we have

$$\text{mag}[\exp(\sqrt{j}\, r\omega)] = e^{r\omega\cos(\pi/4)} = \exp(r\omega\sqrt{2})$$

$$\arg[\exp(\sqrt{j}\, r\omega)] = r\omega\sin(\pi/4) = r\omega\sqrt{2}$$

So like any other complex function, our little $E(r) = \exp(\sqrt{j}\, r\omega)$ has a magnitude and phase.

Let's look one more time at this little equation:

$$\frac{\partial E(r,\omega)}{\partial r} = j\omega B(r,\omega)$$

Having looked a bit at **the PDE world,** I now know the most general form for a solution is

$$E(r,\omega) = j\omega \int^{r} dr\, B(r,\omega) + Q(\omega)$$

In our little example we had $B(r,\omega) = \exp(\sqrt{j}\, r\omega)/\sqrt{j}$, so $\int^{r} dr\, B(r,\omega) = 1/(\omega\sqrt{j}\,\sqrt{j})\exp(\sqrt{j}\, r\omega)$ which is then $1/(j\omega)\exp(\sqrt{j}\, r\omega)$ and with $Q(\omega) = 0$ we find $E(r,\omega) = \exp(\sqrt{j}\, r\omega)$ as per the example. Looking at this general form, then, certainly with some $Q(\omega)$ present, the phase of $E(r,\omega)$ is out of control and in general is some mess. But even with $Q(\omega) = 0$, assuming $B(r,\omega)$ is complex, then the phase of $\int^{r} dr\, B(r,\omega)$ could be totally different from the phase of $B(r,\omega)$, as in our simple example. So I am just confirming once again that the ODE above does not imply that phase(E/B) = j !! [all correct]

I am now going to reread my confused "complex E and B" notes and add comments in red. [done]