E dot B = 0 text for lines doc PhL 1.28.14

This is the third version of this doc in which I attempt to come up with a lower bound for ω above which one will have **E** • **B** = 0 in a transmission line. I could never get happy with my estimates. At one point below I get a result that the bound is about f = 1 Hz, but who knows. I gave up and left it as a Reader Exercise! I really have no idea what that lower limit is. One could compute it perhaps for the Chapter 6 worked out example, but much work would be required. I added that as part of the Exercise.

Appendix M: Estimate of lowest frequency for which E•B = 0 at the conductor surface.

This is a surprisingly complicated estimate. We shall assume rectangular conductors with cross section as shown in Fig C.*, where we know that **E•H** ≠ 0 at ω = 0. Our interest will be the nature of the perpendicularity of the fields just outside the conductor surface in the cross section picture.

(a) Estimate of H_z / H_φ

We assume our conductors are widely spaced so we can crudely use certain m = 0 results from Appendix D (even though Appendix D is for a round wire). The main result of interest from Appendix D comes from (D.6.1) which estimates the ratio of the longitudinal H field to the transverse H field. We assume $\mu_d = \mu_c = \mu$ so that H is continuous at the conductor boundary, so the (D.6.1) results then apply just outside the surface. The estimate implied by (D.6.1) is this:

$$| H_z / H_\varphi | \sim | \beta_d/\beta_c | .$$

This ratio appears in the form (β_d/β') but $\beta' \approx \beta = \beta_c$ for the conductor. We assume the conductor is copper with a large σ_c, and for the dielectric we assume $\sigma_d = 0$ and $\varepsilon = \varepsilon_0$. Also, $\mu_d = \mu_c = \mu_0$. Then from (1.5.1),

$$\beta_c^2 = -j\omega\mu_0\sigma_c$$

$$\beta_d^2 = \omega^2\mu_0\varepsilon_0$$

$$\Rightarrow | \beta_d/\beta_c |^2 = \frac{\omega^2\mu_0\varepsilon_0}{-j\omega\mu_0\sigma_c} = \omega\,(\varepsilon_0/\sigma_c) \approx \omega\,(10^{-11}/10^{8}) \approx \omega\ 10^{-18}$$

$$\Rightarrow | H_z / H_\varphi | \approx | \beta_d/\beta_c | \approx \sqrt{\omega}\ 10^{-9} \qquad \text{// SI units}$$

Certainly for any reasonable frequency this ratio is very small. It seems likely this ratio estimate is valid for any non-singular point on the cross section of a conductor of any shape.

(b) Estimate of E_r/H_φ

For a round wire and m = 0, Appendix D gives the following surface charge density [(D.1.5a) and (D.2.31)]

$$n(\varphi) = N_0 = (\beta_d/2\pi\omega a)\, I$$

The transverse electric field at the conductor surface is then, from (1.1.47),

$$E_r = n/\varepsilon_0 = (\beta_d/2\pi\omega a\varepsilon_0)\, I \ . \qquad \text{dim OK}$$

Since our assumed transmission line has widely spaced conductors, the m = 0 estimate is reasonable. For non-round conductors (at non-singular points on the periphery) we use this same formula, with the understanding that a is some transverse dimension of the conductor. Inserting ** for β_d we get

$$E_r^2 \approx (\beta_d/2\pi\omega a\varepsilon_0)^2\, I^2 = \beta_d^2\, (I/2\pi\omega a\varepsilon_0)^2 = \omega^2\mu_0\varepsilon_0\ I^2/\,(4\pi^2\omega^2 a^2\varepsilon_0^2) = (\mu_0/\varepsilon_0)\, I^2/\,(2\pi a)^2$$

so

$$E_r \approx I/(2\pi a)\,\sqrt{\mu_0/\varepsilon_0} \approx (377/2\pi)\ I/a \approx 60\Omega\ I/a \qquad \text{dim OK}$$

For a round wire we also know from Ampere's law the value of H_φ at the wire surface,

$$2\pi a\, H_\varphi = I \qquad \Rightarrow \qquad H_\varphi = I/(2\pi a)$$

The ratio then of E_r to H_φ is given by

$$E_r/\, H_\varphi = \frac{60\Omega\ I/a}{I/(2\pi a)} = 2\pi\ 60\Omega = 377\Omega$$

which replicates the plane-wave result that E/H = the impedance of free space.

(c) Examination of E • H

Let us review our proof from (***) that **E** • **H** = 0. First, Maxwell's curl H equation (1.1.1) says, using both **D** = ε**E** and **J** = σ**E**,

$$\text{curl}\ \mathbf{H} = (j\omega\mathbf{D} + \mathbf{J}) = (j\omega\varepsilon\mathbf{E} + \sigma\mathbf{E}) = (j\omega\varepsilon + \sigma)\mathbf{E}$$

Then

$$\mathbf{E} \bullet \mathbf{H} = (j\omega\varepsilon + \sigma)^{-1}\ \text{curl}\ \mathbf{H} \bullet \mathbf{H} \ .$$

In cylindrical coordinates we have

$$\mathbf{H} = \hat{\mathbf{r}}\ H_r + \hat{\boldsymbol{\theta}}\, H_\theta + \hat{\mathbf{z}}\, H_z$$

$$\text{curl}\ \mathbf{H} = \hat{\mathbf{r}}\,[\, r^{-1}\partial_\theta H_z - \partial_z H_\theta] + \hat{\boldsymbol{\theta}}\,[\partial_z H_r - \partial_r H_z] + \hat{\mathbf{z}}\,[\, r^{-1}\partial_r(rH_\theta) - r^{-1}\partial_\theta H_r\,]$$

so

$$\text{curl } \mathbf{H} \bullet \mathbf{H} = H_r [\, r^{-1}\partial_\theta H_z - \partial_z H_\theta] + H_\theta[\partial_z H_r - \partial_r H_z] + H_z [\, r^{-1}\partial_r(rH_\theta) - r^{-1}\partial_\theta H_r \,]$$

If we set $H_z \equiv 0$ and $H_r \equiv 0$ with the argument that H_θ is so large, this becomes

$$\text{curl } \mathbf{H} \bullet \mathbf{H} = 0 [\, r^{-1}\partial_\theta 0 - \partial_z H_\theta] + H_\theta[\partial_z 0 - \partial_r 0] + 0 [\, r^{-1}\partial_r(rH_\theta) - r^{-1}\partial_\theta 0 \,] = 0$$

and from this we concluded that $\mathbf{E} \bullet \mathbf{H} = 0$.

But in reality we don't have H_z and H_r both vanishing, so we then back up to write

$$|\mathbf{E}|\,|\mathbf{H}| \cos\alpha$$

$$= (j\omega\varepsilon + \sigma)^{-1} \{ H_r [\, r^{-1}\partial_\theta H_z - \partial_z H_\theta] + H_\theta[\partial_z H_r - \partial_r H_z] + H_z [\, r^{-1}\partial_r(rH_\theta) - r^{-1}\partial_\theta H_r \,] \}$$

Maybe better to do this with the other curl equation to get'

$$|\mathbf{E}|\,|\mathbf{B}| \cos\alpha$$

$$= (\omega\varepsilon)^{-1} | E_r [\, r^{-1}\partial_\theta E_z - \partial_z E_\theta] + E_\theta[\partial_z E_r - \partial_r E_z] + E_z [\, r^{-1}\partial_r(rE_\theta) - r^{-1}\partial_\theta E_r \,] |$$

Since we are just above the conductor surface, using the local cylindrical coordinate system of Fig 3.3 we argue that $E_\theta = 0$ because the surface is an equipotential and we are just above it. So,

$$|\mathbf{E}|\,|\mathbf{B}| \cos\alpha$$

$$= (\omega\varepsilon)^{-1} | E_r [\, r^{-1}\partial_\theta E_z] + E_z [- r^{-1}\partial_\theta E_r \,] |$$

$$|\mathbf{E}|\,|\mathbf{B}| \cos\alpha = (\omega\varepsilon)^{-1} | E_r [- \partial_z E_\theta] + E_\theta[\partial_z E_r - \partial_r E_z] + E_z [- r^{-1}\partial_r(rE_\theta) \,] |$$

Next, we replace ∂_z derivatives by $j\beta_d$ based on the travelling wave on the line, so

$$|\mathbf{E}|\,|\mathbf{B}| \cos\alpha = (\omega\varepsilon)^{-1} | E_r [- j\beta_d E_\theta] + E_\theta[j\beta_d E_r - \partial_r E_z] + E_z [\, -\partial_r E_\theta - r^{-1}E_\theta \,] |$$

Now we make a wild guess that we can replace ∂_r by (1/a) where a is a generic transverse dimension of the conductor, and we also set r = a, so then

$$|\mathbf{E}|\,|\mathbf{B}| \cos\alpha = (\omega\varepsilon)^{-1} | E_r [- j\beta_d E_\theta] + E_\theta[j\beta_d E_r - (1/a)E_z] + E_z [-(1/a)E_\theta - a^{-1}E_\theta \,] |$$

$$= (\omega\varepsilon)^{-1} | -j\beta_d E_r E_\theta + j\beta_d E_\theta E_r - (1/a)E_z E_\theta - (2/a)E_z E_\theta |$$

$$= (\omega\varepsilon)^{-1} | (3/a)E_z E_\theta |$$

Notice that $E_z E_\theta$ is doubly small here! We then get

$$\cos\alpha \approx (\omega\varepsilon)^{-1} \,|\, (3/a) E_z E_\theta \,|\, / \,(E_r B_\theta)$$

Above we found that

$$E_r \approx 60\Omega\, I/a$$
$$H_\theta \approx I/(2\pi a) \qquad B_\theta \approx \mu_0 I/(2\pi a)$$

How can you have E_θ ???

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Next we stare at (D.6.1) to conclude that, since E_z and E_φ are continuous at the boundary, being tangential E fields, we can neglect E_φ (= E_θ) relative to E_z since $E_\varphi/E_z \approx (\beta_d/\beta_c)$. Then

$$|\mathbf{E}|\,|\mathbf{B}|\cos\alpha = (-j\omega\varepsilon)^{-1}\,\{\, E_r\,[-j\beta_d E_\theta] + E_\theta[j\beta_d E_r - \partial_r E_z] + E_z\,[\partial_r E_\theta + r^{-1}E_\theta\,]\,\}$$

From Appendix D

since maybe I know a little more about the E fields. Then very roughly keep just the E_r term,

$$|\mathbf{E}|\,|\mathbf{B}|\cos\alpha \approx (-j\omega\varepsilon)^{-1}\,\{\, E_r\,[\, r^{-1}\partial_\theta E_z - \partial_z E_\theta]$$

Since the conductor surface is an equipotential surface and we are just above it, we set $\partial_\theta E_z = 0$. And from the wave nature of the fields $\partial_z E_\theta = j\beta_d E_\theta$ so then we have

$$|\mathbf{E}|\,|\mathbf{B}|\cos\alpha \approx (-j\omega\varepsilon)^{-1}\,\{\, E_r\,[-j\beta_d E_\theta]\,\}$$

The next required ingredient is an estimate for the ratio of the transverse E field to the transverse H field. For this estimate we

We assume a wide spacing of the conductors so that we can crudely use our m = 0 field results of Appendix D to es H_z/H_φ

We shall now try to estimate a lower limit for ω such that we expect **E•H** = 0. There is probably a simple way to do this, but we proceed as best we can. For a ballpark estimate, let us assume that σ = 0 in the dielectric, so the neglected terms in (3.7.3) are these

$$\cos\alpha\,|^{\mathbf{neglected}} = \{ (\partial_x H_y - \partial_y H_x)H_z + (\partial_y H_z)H_x + (- \partial_x H_z)H_y \} \;/\; \{(j\omega\varepsilon)\,|\mathbf{E}|\,|\mathbf{H}|\} \;.$$

Although the numerator terms involving H_z are small, their effect is amplified by dividing by ω when ω is small. A reasonable condition for the applicability of our claim above that **E•H** = 0 would be this

$$|\cos\alpha\,|^{\mathbf{neglected}}| \ll 1$$

or

$$|(\partial_x H_y - \partial_y H_x)H_z + (\partial_y H_z)H_x + (- \partial_x H_z)H_y| \;/\; \{(\omega\varepsilon)\,|\mathbf{E}|\,|\mathbf{H}|\} \ll 1 \;.$$

As part of our ballpark estimate, we let H_x "represent" the transverse action (probably cylindrical coordinates would be better here for which $H_x \to H_\varphi$) and set $H_y = 0$ to get

$$|(- \partial_y H_x)H_z + (\partial_y H_z)H_x| \;/\; \{(\omega\varepsilon)\,|\mathbf{E}|\,|\mathbf{H}|\} \ll 1$$

or

$$|H_z^2\, \partial_y(H_x/H_z)| \;/\; \{(\omega\varepsilon)\,|\mathbf{E}|\,|\mathbf{H}|\} \ll 1$$

We next assume that $\partial_y(H_x/H_z) \sim (H_x/H_z)/D$ where D is some transverse dimension of the transmission line in question. Then

$$|H_z^2\,(H_x/H_z)/D| \;/\; \{(\omega\varepsilon)\,|\mathbf{E}|\,|\mathbf{H}|\} \ll 1$$

or

$$|H_z\,H_x\,/D| \;/\; \{(\omega\varepsilon)\,|\mathbf{E}|\,|\mathbf{H}|\} \ll 1$$

But since $|\mathbf{H}| \approx |H_x|$ in our ballpark scenario, this says

$$|H_z/D| \;/\; \{(\omega\varepsilon)\,|\mathbf{E}|\} \ll 1 \;.$$

From the Appendix D study of a round wire, using the m = 0 results shown in (D.6.1) we find that

$$|H_z / H_x| \sim |H_z / H_\varphi| = |B_z / B_\varphi| \sim |\beta/\beta_c|$$

where β_c is for the conductor and β for the dielectric. Although this result applies to the **H** field inside the round wire, if $\mu = \mu_c$ then **H** is continuous at the boundary, so the result then applies for **H** just outside the wire as well. From (1.5.1) this ratio is roughly

$$|\beta/\beta_c|^2 = |\omega^2\mu\varepsilon / (-j\omega\mu_c\sigma_c)| \approx (\varepsilon/\sigma_c)\,\omega \qquad \mu \approx \mu_c$$

Then, replacing H_x by H_φ which is the main transverse field in cylindrical coordinates,

$$| H_z|^2 \approx | H_\varphi|^2 \, | \beta/\beta_c |^2 \;\; = \;\; | H_\varphi|^2 \, (\varepsilon/\sigma_c)\, \omega$$

Squaring our condition *** gives

$$| H_z |^2 \; << D^2(\omega\varepsilon)^2 \, |\mathbf{E}|^2$$

or

$$| H_\varphi|^2 \, (\varepsilon/\sigma_c)\, \omega \; << D^2\varepsilon^2\omega^2 \, |\mathbf{E}|^2$$

or

$$| H_\varphi|^2 \, (1/\sigma_c) \; << D^2\varepsilon \; \omega \, |\mathbf{E}|^2$$

or

$$\omega >> (\, |H_\varphi|/ \, |\mathbf{E}|)^2 \, (1/D^2\sigma_c\varepsilon_0) \qquad\qquad \varepsilon \; \approx \varepsilon_0$$

The next task is to estimate the ratio $|H_\varphi|/ \, |\mathbf{E}|$ for a transmission line. Roughly we can estimate

$$|\mathbf{E}| \;\; = V(z)/D$$

where V(z) is the potential difference of Chapter 4 and D is a transverse dimension of the line. For a round conductor of radius a, we know that just outside the conductor surface

$$2\pi a \, H_\varphi \; \approx \; i(z)$$

where i(z) is the conductor current. Then:

$$|H_\varphi|/ \, |\mathbf{E}| \approx \; (1/2\pi a) \, i(z) \, / \, [V(z)/D] \;\; = (1/2\pi)(D/a) \, i(z)/V(z)$$

But V(z)/i(z) is Z_0 for the transmission line. Even for a 60 Hz power transmission line, one is in the "high frequency limit" in terms of the formula (4.4.16)

$$Z_0 \; \approx \; (K \, / \sqrt{\varepsilon_{rel}} \,) \, 30\Omega \qquad\qquad \varepsilon_{rel} \equiv \varepsilon/\varepsilon_0 \;\; . \qquad\qquad \textit{(4.4.16)}$$

The reason is that for such a transmission line, G is extremely small and ωL dominates R which is very low as well in order to reduce ohmic losses in the line. Setting $\varepsilon_{rel} \approx 1$ we then have

$$|H_\varphi|/ \, |\mathbf{E}| \; \approx \; (1/2\pi)(D/a) \, i(z)/V(z) \; \approx \;\; (1/2\pi)(D/a) \, / \, Z_0 \; \approx (1/2\pi)(D/a)(1/K)(1/30\Omega)$$

The dimensions on the left are (amps/m) / (volts/m) = (amps/volts) = 1/ohms, in agreement with the right where only the last factor has dimensions. Then ** becomes

$$\omega >> (\, |H_\varphi|/ \, |\mathbf{E}|)^2 \, (1/D^2\sigma_c\varepsilon_0) \;\; = (1/2\pi)^2(D/a) \, ^2(1/K)^2(1/30\Omega)^2 \, / \, (D^2\sigma_c\varepsilon_0 \,)$$

or

$$\omega >> (1/2\pi a \; K \; 30\Omega)^2 \, / \, (\sigma_c\varepsilon_0 \,)$$

Another dimension check for the right side [looking as usual at (1.1.28) and (1.1.29)],

$$\dim(\text{RHS}) = m^{-2}\ \text{ohm}^{-2} \,/\, [\text{ohm}^{-1}\, m^{-1} * \text{farad}\, m^{-1}] = \text{ohm}^{-1}/\text{farad} = \sec^{-1}$$

Then

$$\omega >> 1/(60\pi)^2\ 1/K^2\ (1/a)^2 \,/\, (\sigma_c \varepsilon_0)$$

Using $\varepsilon_0 \sim 10^{-11}$ and $\sigma_c \sim 10^8$ we have

$$\omega >> 1000/(60\pi)^2 * \ 1/K^2\ (1/a)^2$$

For a typical transmission line with a = 1/2 cm and wire separation 1 m, we find from *** that

$$K = 4 \ln(b/a) = 4 \ln(1\text{m}/\ 0.5*10^{-2}\text{m}) = 4 \ln(20) = 12$$

which corresponds to $Z_0 = K\ 30\Omega = 360\Omega$ (normally 400-600 Ω for a power line). Then

$$\omega >> .03 \quad 1/K^2\ (1/a)^2 = .03\ (1/144)\ (200)^2 = 8$$

Then $2\pi f = 8$ says $f \sim 1$ Hz !!

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I am getting nowhere with all this work, so I think I will bail out and just make a comment.

Here then is a rewrite of Fact 4:

Fact 4: In a cross sectional sketch of a transmission line, the **E** and **B** field lines are perpendicular at every point in the dielectric.

Proof: From Maxwell's curl E equation (1.1.2) in the ω domain we have

$$\text{curl}\ \mathbf{E} = -j\omega \mathbf{B} \ .$$

Then

$$\mathbf{B} \bullet \mathbf{E} = (-j\omega)^{-1}\ \mathbf{curl\ E} \bullet \mathbf{E}$$

$$= (-j\omega)^{-1} [\ (\partial_x E_y - \partial_y E_x)E_z + (\partial_y E_z - \partial_z E_y)E_x + (\partial_z E_x - \partial_x E_z)E_y\ ] .$$

To the extent that $E_z << E_x, E_y$ we set $E_z \approx 0$ to get

$$\mathbf{B} \bullet \mathbf{E} = (-j\omega)^{-1} [- (\partial_z E_y)E_x + (\partial_z E_x)E_y\ ] + (-j\omega)^{-1} [\text{ correction terms }]$$

where "correction terms" accounts for the fact that the dismissed terms are not exactly zero. Then

$$\mathbf{B} \bullet \mathbf{E} = (-j\omega)^{-1} [E_y^2 \, \partial_z(E_x/E_y)] + (-j\omega)^{-1} [\text{ correction terms }] \qquad (3.7.1)$$

However, we argued in Fact 3 that the shape of fields does not vary with z. Thus, the ratio of two components like E_x/E_y cannot vary with z. Thus $\partial_z(E_x/E_y) = 0$ so

$$\mathbf{B} \bullet \mathbf{E} = (-j\omega)^{-1} [\text{ correction terms }] \ .$$

Letting α be the angle between **B** and **E**, we can write

$$|\mathbf{B}| \, |\mathbf{E}| \cos\alpha = (\omega)^{-1} | \text{ correction terms } |$$

so then

$$\cos\alpha = (\omega)^{-1} | \text{ correction terms } | / (|\mathbf{B}| \, |\mathbf{E}|) \ .$$

Since the correction terms involve E_z , they are small compared to $(|\mathbf{B}| \, |\mathbf{E}|)$ and we then conclude that $\cos\alpha \approx 0$ and $\alpha \approx \pi/2$ and then $\mathbf{B} \bullet \mathbf{E} = 0$.

Caveat: However, for sufficiently small ω the right side of ** blows up and our conclusion is no longer valid. A condition for validity would then be

$$\cos\alpha << 1$$

or

$$(\omega)^{-1} | \text{ correction terms } | / (|\mathbf{B}| \, |\mathbf{E}|) << 1$$

or

$$\omega >> | \text{ correction terms } | / (|\mathbf{B}| \, |\mathbf{E}|)$$

In particular, this condition is violated when ω = 0. This then explains why one does not see $\mathbf{B} \bullet \mathbf{E} = 0$ at the surface of rectangular conductor operating at low frequency as shown in Fig C.2.

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older stuff:

Recall these two equations from earlier sections

$$\text{curl } \mathbf{B} = \mu \ (j\omega\varepsilon \, \mathbf{E} + \mathbf{J}) = \mu \ (j\omega\varepsilon \, \mathbf{E} + \sigma\mathbf{E}) = \mu(j\omega\varepsilon + \sigma)\mathbf{E} \qquad \textit{(2.2.1)}$$

$$\beta^2 = \omega^2 \mu \, [\, \varepsilon + \sigma/j\omega] = \omega^2 \mu(\, j\omega\varepsilon + \sigma) \, /j\omega = \omega\mu(\, j\omega\varepsilon + \sigma) \, /j \quad . \qquad \textit{(1.5.1)}$$

If follows from these equations that,

curl $\mathbf{B} = j(\beta^2/\omega)\,\mathbf{E} \equiv C\,\mathbf{E}$. (3.7.1)

To show that the **E** and **B** fields are perpendicular, we will show that $\mathbf{E}\bullet\mathbf{B} = 0$. We have from (3.7.1),

$$C\,\mathbf{E}\bullet\mathbf{B} = \text{curl}\,\mathbf{B}\bullet\mathbf{B} = (\partial_x B_y - \partial_y B_x)B_z + (\partial_y B_z - \partial_z B_y)B_x + (\partial_z B_x - \partial_x B_z)B_y .$$

Since $B_z \approx 0$ (see estimate in previous section), we are left with only two terms

$$C\,\mathbf{E}\bullet\mathbf{B} = B_y(\partial_z B_x) - B_x(\partial_z B_y) = B_y^2\,\partial_z\,(B_x/B_y) .$$

However, we argued in Fact 3 that the shape of fields does not vary with z. Thus, the ratio of two components like B_x/B_y cannot vary with z. Thus, $\mathbf{E}\bullet\mathbf{B} = 0$ so the **E** and **B** lines are perpendicular everywhere in the dielectric. (We also assume $E_z \approx 0$ since it is small in the dielectric).