3.4 Size of E fields in conductor and dielectric; conservation of total current at a boundary

We know from (1.1.48) that the following E field condition applies at a boundary between two media, where n refers to the normal component,

$$\xi_1 E_{n1} = \xi_2 E_{n2} \qquad \text{// frequency domain} \qquad \textit{(1.1.48)}$$

or

$$(\varepsilon_1 + \sigma_1/j\omega)\, E_{n1} = (\varepsilon_2 + \sigma_2/j\omega)\, E_{n2} \qquad (3.4.1)$$

or

$$\frac{E_{n1}}{E_{n2}} = \frac{\xi_2}{\xi_1} = \frac{\varepsilon_2 + \sigma_2/j\omega}{\varepsilon_1 + \sigma_1/j\omega} = \frac{j\omega\varepsilon_2 + \sigma_2}{j\omega\varepsilon_1 + \sigma_1} \quad . \qquad (3.4.2)$$

Since most dielectrics have a small imaginary part for ε_1, we must interpret σ_1 as being the σ_{eff} shown in (3.3.4).

$$\sigma_1 \approx (\sigma_{1DC} + \omega\varepsilon_1 \tan_L) \, . \qquad (3.4.3)$$

Let 1 = dielectric = polyethylene and 2 = conductor = copper with these assumed parameters

$$\varepsilon_1 = 2.3\,\varepsilon_0 \qquad \varepsilon_2 = \varepsilon_0 \qquad \varepsilon_0 = 8.85 \times 10^{-12}$$
$$\sigma_{1DC} = 10^{-15} \qquad \sigma_2 = 5.81 \times 10^{7} \quad . \qquad \tan_L = .0002 \qquad (3.4.4)$$

Then

$$\text{rat} \equiv \frac{E_{n1}}{E_{n2}} = \frac{j\omega\varepsilon_2 + \sigma_2}{j\omega\varepsilon_1 + \sigma_1} = \frac{j2\pi f\varepsilon_2 + \sigma_2}{j2\pi\varepsilon_1 + \sigma_1} \qquad (3.4.5)$$

```
pi := evalf(Pi):
e1 := 2.3*e0:  e2 := e0: e0 := 8.85e-12:
s1dc := 1e-15:  s2 := 5.81e7: tanL := .0002:
s1 := s1dc + 2*pi*f*e1*tanL:
rat := (j*2*pi*f*e2+s2)/(j*2*pi*f*e1+s1): collect(%,f);
```

$$\frac{.5560618998\ 10^{-10} jf + .581\ 10^{8}}{(.2557884738\ 10^{-13} + .1278942369\ 10^{-9} j) f + .1\ 10^{-14}}$$

Looking at this ratio, for f in the range (10 Hz, 10^{16} Hz) we can approximate the ratio as

$$\text{rat} = \frac{E_{n1}}{E_{n2}} = \frac{\sigma_2}{j2\pi f\varepsilon_1} \qquad \left|\frac{E_{n1}}{E_{n2}}\right| \approx \frac{\sigma_2}{2\pi f\varepsilon_1} = .45 \times 10^{18} / f \quad . \qquad (3.4.6)$$

Then for the "practical" range (10 Hz, 500 GHz) this ratio varies from ~10^{17} to 10^{6}. We conclude:

Fact 1: At a boundary between a good dielectric and a good conductor, the normal E field is at least 1 million times larger in the dielectric than it is in the conductor for frequencies under 500 GHz. (3.4.7)

Fact 2: This large jump in E_n at the boundary must be supported by a significant surface charge density n_s on the boundary since, according to (1.1.47), $n_s = \varepsilon_1 E_{n1} - \varepsilon_2 E_{n2} \approx \varepsilon_1 E_{n1}$. (3.4.8)

Using $J_i = \sigma_i E_i$ we can restate the above facts in terms of current densities,

$$(j\omega\varepsilon_1 + \sigma_1)\, E_{n1} = (j\omega\varepsilon_2 + \sigma_2)\, E_{n2} \,. \qquad \textit{(3.4.1)}$$

$$(j\omega\varepsilon_1 + \sigma_1)\, (J_{n1}/\sigma_1) = (j\omega\varepsilon_2 + \sigma_2)\, (J_{n2}/\sigma_2) \,.$$

$$[1 + j\omega(\varepsilon_1/\sigma_1)]\, J_{n1} = [1 + j\omega(\varepsilon_2/\sigma_2)]\, J_{n2} \qquad (3.4.9)$$

$$\left[\, 1 + \frac{j\omega\varepsilon_1}{\sigma_{1DC} + \omega\varepsilon_1 \tan_L} \,\right] J_{n1} = [1 + j\omega(\varepsilon_2/\sigma_2)]\, J_{n2} \qquad \text{// using (3.4.3)}$$

$$\left[\, 1 + \frac{j2\pi f\varepsilon_1}{\sigma_{1DC} + 2\pi f\varepsilon_1 \tan_L} \,\right] J_{n1} = [1 + j2\pi f(\varepsilon_2/\sigma_2)]\, J_{n2} \qquad \text{// } \omega = 2\pi f \qquad (3.4.10)$$

For $f >> \sigma_{1DC}/(2\pi\varepsilon_1 \tan_L) = .04$ Hz, one can ignore σ_{1DC} on the left to get

$$\underset{\text{cond}}{1} + \underset{\text{disp}}{j/\tan_L}\,]\, J_{n1} = [\underset{\text{cond}}{1} + \underset{\text{disp}}{j2\pi f\,(\varepsilon_2/\sigma_2)}]\, J_{n2} \qquad f >> \sigma_{1DC}/(2\pi\varepsilon_1\tan_L) \qquad (3.4.11)$$

where we have now labeled the conduction current and displacement current terms. Using the numbers above one finds,

$$\underset{\text{cond}}{1} + \underset{\text{disp}}{5000\, j}\,]\, J_{n1} = [\underset{\text{cond}}{1} + \underset{\text{disp}}{10^{-18} f}\,]\, J_{n2} \qquad (3.4.12)$$

For f in the range (10 Hz, 10^{16} Hz) it is clear that in the dielectric, essentially all the current is displacement current, while in the conductor it is essentially all conduction current. We conclude:

Fact 3: For any practical frequency and good dielectric, the total current in the dielectric is almost all displacement current, while that in the conductor it is almost all conduction current. (3.4.13)

Imagine now a tiny patch of area (bordered in red) on the surface between a conductor and a dielectric,

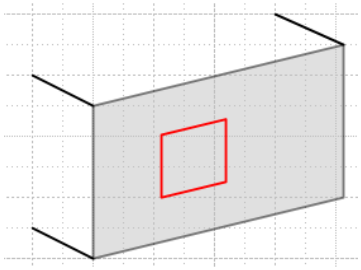


Fig 3.2

Defining a total current $J_{tot,n} \equiv j\omega\varepsilon E_n + \sigma E_n$, as in (2.2.1), we have shown that this total current flows right through the patch but changes its nature from nearly all conduction current in the conductor to nearly all displacement current in the dielectric. In the next section, we identify the normal direction with the local radial direction. Then the total current passing through a tiny square patch like that in Fig 3.2 can be regarded as being "fed" by the radial conduction current J_r just inside the conductor where $J_r = \sigma E_r$. This current feeds the surface charge on the boundary which creates a large E field and thus a large displacement current in the dielectric. We sometimes refer to this mechanism as "charge pumping".

Recall now some results from Section 1.5,

$$J_{n1} = n_s(\sigma_1/\varepsilon_1) \qquad \textit{(1.5.15)}$$

$$J_{2n} = (\xi_1/\varepsilon_1)\,(j\omega)\,n_s = [\,j\omega + (\sigma_1/\varepsilon_1)\,]\,n_s \quad . \qquad \textit{(1.5.16)}$$

It follows that (remember that these J's are conduction currents)

$$J_{2n} = j\omega n_s + J_{n1} \quad . \qquad (3.4.14)$$

We interpret this to say that during the charge pumping process, some of J_{2n} is used to feed the change in the surface charge n_s (think $\partial_t n_s$), and the rest flows through into the dielectric as J_{n1}. This last equation is just an application of continuity (1.1.35) at the boundary.