Save Section 3.5 and 3.6 today PhL 10.11.13

Since I am about to edit 3.5. Will toss this eventually.

3.5 The TEM mode fields and currents for an IDEAL transmission line

By "ideal" we mean that the conductors have near infinite conductivity and the dielectric has zero conductivity. Consider a cross sectional view of one conductor of a transmission line having arbitrarily shaped conductors (the shape is uniform in the z direction). At some point on the surface, define a local coordinate system where

r = radial direction = the normal outward from the surface (local x)
φ = azimuthal direction = tangential to the surface in the cross section plane (local y)
z = tangential to the surface along the transmission line (local and global z)

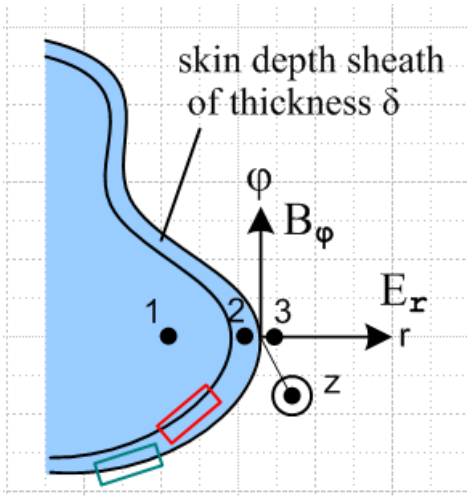


Fig 3.3

The following table shows the sizes of various components of E,B and J (conduction current) near the surface of a transmission line conductor. Several regions of space are of interest:

1. In the conductor, under the surface charge layer and under any current layer.
2. In the conductor, just under the surface charge layer, and in the current layer.
3. In the dielectric, just outside the super-thin surface charge layer.

The purpose of defining region 2 is to make the connection to a real transmission line in the next section.

The discussion below is qualitative and not really complete. One can always examine the solution for the round wire given in Appendix D as an example. Comments follow the table.

Table 1: E,B,J for an ideal transmission line

1. In the conductor, under the surface charge layer and under any current layer.

$E_r = 0$	$B_r = 0$	$J_r = 0$
$E_\varphi = 0$	$B_\varphi = 0$	$J_\varphi = 0$
$E_z = 0$	$B_z = 0$	$J_z = 0$

2. In the conductor, just under the surface charge layer, and in the current layer.

E_r = tiny	$B_r = 0$	J_r = small
$E_\varphi = 0$	B_φ = large	$J_\varphi = 0$
E_z = small	$B_z = 0$	J_z = very large

3. In the dielectric, just outside the super-thin surface charge layer (explanations below):

E_r = large	$B_r = 0$	$J_r = 0$
$E_\varphi = 0$	B_φ = large	$J_\varphi = 0$
E_z = small	$B_z = 0$	$J_z = 0$

Region 1: (the interior) In the absence of interior "applied" charges and currents, we know that **E** and **B** satisfy Helmholtz equations of the form (1.6.2). Due to the powerful exponential effect of this equation at the boundary [see skin effect discussion of Section 2.2 and (2.2.7)], we know that **E** and **B** fields cannot exist deep inside the conductor, and can exist only in the skin depth region. A "perfect conductor" has σ = extremely large, and δ = extremely small since $\delta = \sqrt{2/\mu\sigma\omega}$. Thus, conductor **E** and **B** fields can only exist close to the surface. In region 1 of the above table, we show all fields as being 0 underneath the very thin current sheath. Since **E** = 0 in the perfect conductor interior, it follows from $\mathbf{J} = \sigma\mathbf{E}$ that **J** = 0 there as well (region 1). Thus, all current is confined to the thin current sheath of regions 2. Maxwell (1.1.2) says curl **E** = $-j\omega\mathbf{B}$ in the ω domain, so if **E** = 0 in the interior, so also is **B**.

Region 2: (the current sheath) As just noted, all currents flow in a very thin sheath at the surface of thickness δ. Since the thickness is tiny, the current density J_z there is "very large" as indicated in the table. Imagine a total current I flowing down the conductor, but it is restricted to flow only in the sheath.

In this thin layer, there is some radial pumping of charge to the surface to "feed" the surface charge which is always changing in time, so we indicate a small J_r term. As noted in the previous section, this same J_r is "feeding" the total current flow through the surface, and the surface converts this total current from conduction current on the inside to displacement current on the outside. An argument is given below for why J_r is small compared with J_z.

Application of Ampere's Law (3.4.1) to the small red loop in Fig 3.3 ($B_\varphi = 0$ on the left long edge) shows that the very large J_z sheath current creates a very large B_φ field in the sheath which grows from 0 on the sheath's inner boundary to some large value at the conductor surface. Ignoring dramatic μ differences, this large B_φ then exists just outside the surface as well according to (1.1.21).

Since $\mathbf{E} = \mathbf{J}/\sigma$, even though J_z is very large, σ is extremely large, so we shall mark E_z as being "small". And since J_r is already marked "small", we mark E_r as "tiny".

The remaining four entries in the region 2 table above (B_r, E_φ, J_φ, B_z) we leave at 0, though they might have some very small values.

Region 3: (the dielectric) Since we are now outside the surface charge layer, (1.1.22) says there is a large radial electric field E_r which is supported by this charge density (Gauss's Law). Application of the integral form of (2.1.2) to the greeen loop in Fig 3.3 shows that $E_{||}$ is continuous through the boundary (since the B_φ field is not infinite there). Therefore, we give E_φ and E_z their region 2 values.

We already noted that B_φ continues being large just above the surface.

Since the dielectric has zero conductivity, the conduction current components are all zero.

Notice that just outside the surface, the **E** and **B** fields are perpendicular and both transverse to the z direction. Hence this is a TEM (Transverse Electric and Magnetic) mode of the transmission line. Their cross product is the Poynting vector **E** x **B** which is in the +z direction coming at the viewer in Fig 3.3. This is the direction of power flow along the transmission line.

3.6 The TEM mode fields and currents for a REAL transmission line

We now "turn on" the imperfections of the transmission line. As soon as σ in the conductor becomes large but finite, the infinitely thin current sheath spreads out over some reasonable skin depth δ. At very low frequencies, the current J_z is spread across the entire conductor and there is no Region 1. At higher ω there still is a Region 1, but we shall ignore it from now on. We are still interested in region 2 which is just inside the surface charge layer. Recall from Section 3.2 above that the surface charge layer remains nearly infinitely thin even for a non-perfect conductor. So here is the new table. The superscripts refer to descriptive sections below. In order to make ballpark magnitude estimates, we assume that the transmission line is 75 ohms, is properly terminated, and is driven by a voltage of amplitude 7.5 volts, so the current is 100 mA.

Table 2: E,B,J for a real transmission line

2. In the conductor, just under the surface charge layer, and in the current layer.

E_r = small [c]	B_r = 0	J_r = small [c]
E_φ = 0	B_φ = large	J_φ = 0
E_z = small [a]	B_z = small [b]	J_z = large [a]

3. In the dielectric, just outside the super-thin surface charge layer.

E_r = large [a]	B_r = 0	J_r = small [b]
E_φ = 0	B_φ = large	J_φ = 0
E_z = small [b]	B_z = small [b]	J_z = leakage [b]

(a) E_z and J_z in the conductor ; E_z and E_r outside the conductor

Inside the conductor, a non-zero E_z exists due to the current flow in the z direction and the finite conductivity of the conductor. As an estimate for a round wire not too close to the other conductor, assume that the wire has diameter 1 mm, and is operating at 1 GHz with a skin depth δ = 2 microns. The cross sectional area for current flow is then about $2\pi r\delta = 2\pi \times 10^{-9}$ m^2. If 100 mA flows through this wire, then $J_z = 0.1/(2\pi r\delta) = 1.6 \times 10^7$ amps/m^2, and this J_z is marked "large" for region 2 in the above

table. Then $E_z = J_z/\sigma = 1.6 \times 10^7 / 5.81 \times 10^7 = 0.3$ volts/meter. This E_z is marked "small" in the region 2 part of the above table. At lower frequencies where skin depth is larger, E_z is less.

Since E_z is a parallel E field, according to (1.1.20) it has the same value in region 3, so that is also marked "small" above.

In contrast, if the conductor separation is 0.5 cm, and if we crudely assume the E field is constant between the conductors, then E_r between the conductors is 7.5 volts/ 5×10^{-3} m = 1500 volts/m. This is marked "large" in region 3 above. So in region 3 just outside the conductor,

$$E_r \sim 1500 \text{ V/m} \qquad E_z \sim 0.3 \text{ V/m} \qquad \text{ratio } (E_z/E_r) \leq 2 \times 10^{-4} \tag{3.6.1}$$

(b) Leakage: J_r, B_z E_z and J_z in the dielectric

By "leakage" is meant conduction through the dielectric. As shown in (3.3.4), the effective conductivity in the dielectric is given by

$$\sigma_{eff} = (\sigma + \omega\varepsilon' \tan_L) \ . \tag{3.3.4}$$

For polyethylene, $\sigma \sim 10^{-15}$ and can be ignored, while $\varepsilon' \approx 2.3\ \varepsilon_0$ and $\tan_L \approx 2\text{x}10^{-4}$ as in (3.3.5). For a frequency of 1 GHZ, we then find

$$\sigma_{eff} \approx \omega\varepsilon' \tan_L \approx 2\pi\ 10^9 * [2.3 * 8.85 \times 10^{-12}] * 2 \times 10^{-4} \approx 2.5 \times 10^{-5} \tag{3.6.2}$$

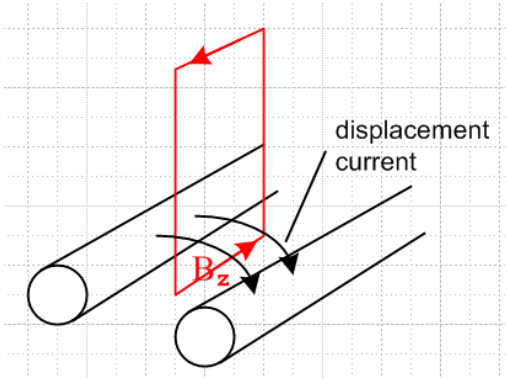
This is 12 orders of magnitude smaller than the σ of copper $\sim 10^7$, but it is 10 orders of magnitude larger than the DC conductivity of the dielectric $\sim 10^{-15}$.

To estimate the significance of this leakage at high frequencies, we can compare the ratio of the leakage current to the displacement current in the dielectric (the currents flow through the same area so ratio is J_{leak}/J_{disp})

$$\left| \frac{\text{I(leakage)}}{\text{I(displacement)}} \right| \approx \left| \frac{(\sigma + \omega\varepsilon \tan_L)E}{j\omega\varepsilon\, E} \right| \approx \tan_L \approx 2 \times 10^{-4} \ . \tag{3.6.3}$$

Thus, even at high frequencies, the effect of leakage on the current flowing through the dielectric is quite small compared to the displacement current. The "radial" current J_r has to support both the leakage current and the more significant displacement current, and we have just seen that the leakage part can be ignored. The displacement contribution will be studied below and itself turns out to be "small" and is so marked in region 3 of the above table.

Since there is a small current (mostly displacement) flowing across the transmission line, there will be a small B_z field resulting. To see why, consider a very tall red loop whose one edge lies parallel to the z direction between the conductors and whose top edge is very distant.



Consider Ampere's law (1.1.18) relative to this loop and with respect to the current flowing between the conductors,

$$\oint \mathbf{H} \bullet d\mathbf{s} \;= \int_S [\partial_t \mathbf{D} + \mathbf{J}_{\mathbf{leakage}}] \bullet d\mathbf{A} \approx \int_S \partial_t \mathbf{D} \bullet d\mathbf{A} \qquad (1.1.18)$$

Integration of the "small" displacement current $\partial_t \mathbf{D}$ passing through the loop gives some small non-zero value for the area integral on the right. The line integral on the left has cancelling contributions from the vertical loop sides, while the loop top is far away so contributes nothing. The result is some small H_z and hence small B_z in the region between the conductors. Since B_z is a parallel field, it will exist also just inside the conductor surface, as indicated by (1.1.21). Both these B_z fields are marked "small" in the above table.

Finally, we already noted a small E_z just outside the conductor, and since the dielectric has some very small leakage (σ_{eff}), there will be some small J_z in region 3 which we have marked "leakage".

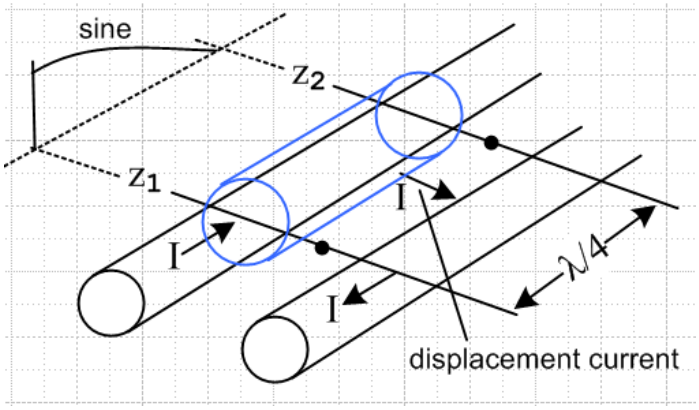
(c) E_r and J_r inside the conductor

We have already estimated that E_r inside the conductor surface is less than 10^{-6} what it is outside the surface, see Section 3.4. Thus, if E_r outside is 1500 volts/m as in our section (a) example, E_r inside is less than 1.5 mV/m at 500 GHz, and is proportionally less than this at lower frequencies, so E_r in region 2 is marked "small". In the example above we found $E_z \approx .3$ V/m inside the conductor. Thus we have $E_r << E_z$ inside the conductor which in turn means $J_r << J_z$. Since J_z is already marked "large" in region 2, we now mark J_r as "small" in region 2. Below we shall provide more support for the idea that $J_r << J_z$.

(d) J_z and J_r in the conductor.

In the table above for region 2 we have set J_z "large" and J_r "small". In this section, we show that the ratio J_r/J_z is quite small, that that is why J_r is small in the table.

Consider the following drawing of a piece of a transmission line:



In this picture the wavelength λ is highly distorted; it is intended to be much larger than the transverse dimensions of the transmission line. The picture is drawn at an instant in time when the total longitudinal current in the left conductor has its maximum value I at $z = z_1$ and vanishes at $z = z_2$. Thus, a total current of I is entering the interior of the left conductor between z_1 and z_2. This current has to go somewhere, and one can regard it as charging the capacitance of the quarter wave transmission line section between z_1 and z_2. In other words, this total current I is equal to the integral of the dielectric displacement current density over some area which divides the two conductors, such as the blue cylinder shown. As discussed above, the displacement current is fed by the radial current density J_r just inside the conductor. As a rough estimate, if the active perimeter of the left conductor is p, then

$$p * \lambda/4 * J_r * (2/\pi) \approx I \quad \Rightarrow \quad p\lambda J_r \approx 2\pi I \quad \Rightarrow \quad J_r = 2\pi I/(p\lambda) \qquad (3.6.4)$$

Here $(2/\pi)$ represents the average value of the sine-shaped displacement current curve over the z region of interest, and p would be $2\pi a$ for a wire of radius a if the two conductors were widely separated. For an arbitrary conductor shape and position, p is some effective distance associated with the transverse geometry; it is the "active" perimeter discussed earlier.

On the other hand, for a round conductor operating in the skin effect regime where $\delta < a$,

$$J_z \approx I/(p\delta) \qquad (3.6.5)$$

where p is the same active perimeter just mentioned. So

$$J_z \approx I/(p\delta)$$
$$J_r \approx 2\pi I/(p\lambda)$$

$$J_r/J_z \approx 2\pi (\delta/\lambda) . \qquad (3.6.6)$$

For δ we had

$$\delta \equiv \sqrt{2/\omega\mu\sigma} \qquad (2.1.20)$$

For λ one may write,

$$\lambda = v/f = 2\pi v/\omega \tag{3.6.7}$$

where v is the wave velocity. Then

$$(\delta/\lambda) = \sqrt{\frac{2}{\omega\mu\sigma}} \; * \; = \sqrt{\frac{2\omega}{\mu\sigma}} \; \frac{1}{2\pi v} = \sqrt{\frac{4\pi f}{\mu\sigma}} \; \frac{1}{2\pi v} \; . \tag{3.6.8}$$

Setting $v \approx c$ and $\mu = \mu_0 = 4\pi \times 10^{-7}$ and $\sigma = 5.81 \times 10^{7}$ (copper) and $f = 10^{9} f(\text{Ghz})$ we get

$$(\delta/\lambda) \approx \sqrt{\frac{4\pi f}{\mu\sigma}} \; \frac{1}{2\pi v} = \sqrt{\frac{4\pi \, 10^{9} \, f(\text{GHz})}{4\pi \times 10^{-7} \times 5.81 \times 10^{7}}} \; \frac{1}{2\pi} \; \frac{1}{3 \times 10^{8}}$$

$$= \sqrt{\frac{10^{10} \, f(\text{GHz})}{58.1}} \; \frac{1}{2\pi} \; \frac{1}{3 \times 10^{8}} = \sqrt{\frac{f(\text{GHz})}{58.1}} \; \frac{1}{6\pi} \; 10^{-3} = 7 \times 10^{-6} \sqrt{f(\text{GHz})}$$

and so

$$J_r/J_z \approx (2\pi)(\delta/\lambda) \approx 4.4 \times 10^{-5} \sqrt{f(\text{GHz})} \tag{3.6.9}$$

For $f \leq 10$ GHz we then find

$$J_r/J_z \leq 1.4 \times 10^{-4} \; . \qquad f \leq 10 \text{ GHz} \tag{3.6.10}$$

A round conductor is in the low-frequency non-skin-effect regime when $\delta > a$, where a is the wire radius. This means

$$\sqrt{2/\omega\mu\sigma} > a \qquad \Rightarrow \qquad \omega < 2/(\mu\sigma a^{2}) \qquad \text{or} \qquad \omega a/2 < 1/(\mu\sigma a) \tag{3.6.11}$$

In this low frequency regime we must replace (3.6.5) by

$$J_z \approx I/(\pi a^{2}) \tag{3.6.12}$$

Since (3.6.4) is still valid, we find now that

$$J_z \approx I/(\pi a^{2})$$
$$J_r \approx 2\pi I/(p\lambda) \approx 2\pi I/(2\pi a\lambda) \approx I/(a\lambda)$$

$$J_r/J_z \approx \pi(a/\lambda) \approx (\pi a)(\omega/2\pi v) \approx \omega a/2v = (\omega a/2)(1/v) \tag{3.6.13}$$

Using ** this says

$$J_r/J_z < 1/(\mu\sigma a v) \tag{3.6.14}$$

With $\mu = \mu_0 = 4\pi \times 10^{-7}$, $\sigma = 5.81 \times 10^{7}$ (copper) and $v = c = 3 \times 10^{8}$ we find for a wire of radius 1 mm,

$$J_r/J_z < \frac{1}{4\pi * 5.81 * 10^{-3} * 3 * 10^{8}} = \frac{10^{-5}}{4\pi * 5.81 * 3} = 4.6 \times 10^{-8} \qquad (3.8.15)$$

Therefore, our high frequency result of 1.4×10^{-4} for $f \leq 10$ GHz represents a much worse case, so we will stick with that number in our work below.

I have finally reached this point at 4 PM Thurs Oct 10.

We now display once again the table for a real transmission line, showing relative estimates of the sizes of things . A few primes have been added to remove ambiguities. [huh?]

Table 3: E, B and J for a real transmission line

2. In the conductor, just under the surface charge layer, and in the current layer.

$E_r' \leq 10^{-6}\ E_r$ (c)	$B_r = 0$	$J_r \leq 2 \times 10^{-5}\ J_z$ (d)
$E_\varphi = 0$	B_φ = large	$J_\varphi = 0$
$E_z \leq 2\times10^{-4}\ E_r$ (a)	$B_z \leq 2 \times 10^{-5}\ B_\varphi$ (b)	J_z = large [$\sim 10^{7}$ A/m^2]

3. In the dielectric, just outside the super-thin surface charge layer.

E_r = large [1500 v/m]	$B_r = 0$	$J'_r \leq 2\times10^{-4}\ J_r$ (d)
$E_\varphi = 0$	B_φ = large	$J_\varphi = 0$
$E_z \leq 2\times10^{-4}\ E_r$ (a)	$B_z \leq 2 \times 10^{-5}\ B_\varphi$ (b)	$J_z' \approx 10^{-12}\ J_z$ (b)