This is the Title PhL 3.26.05

Review the following carefully, then I think it can replace the existing Section 3.1.

This has been installed!

3.1 Why is there no free charge inside a conductor or a dielectric?

Imagine that at time t = 0 there was some free charge ρ inside a medium having conductivity σ. What would this free charge do?

As discussed in Appendix E, for a non-neutral medium, Ohm's Law takes the form

$$\mathbf{J} = \sigma\mathbf{E} - D \text{ grad } \rho \qquad (3.1.1)$$

where **J** is conduction current and the second term, associated with Fick's Law, is non-zero when the free charge density ρ is non-zero. This second term is a diffusion term, D is the (electron) diffusion constant for the medium at hand, and the diffusion current flows from a region of high charge density to one of lower density, hence the minus sign. Taking the divergence of the above equation, we find

$$\text{div } \mathbf{J} = \sigma \text{ div } \mathbf{E} - D\,\nabla^2 \rho$$

or

$$-\partial_t\rho = \sigma\,\rho/\varepsilon - D\,\nabla^2\rho \qquad \text{// using (1.1.25) for div } \mathbf{J}, \text{ and (1.1.3) with (1.1.6) for div } \mathbf{E}$$

or

$$\partial_t\rho - D\,\nabla^2\rho + (\sigma/\varepsilon)\rho = 0 \qquad (3.1.2)$$

or

$$\partial_t\rho - a\nabla^2\rho - b\rho = 0 \qquad a = D, \quad b = -(\sigma/\varepsilon)\ . \qquad (3.1.3)$$

Now let $\rho' = \rho\, e^{-bt}$ be an "adjusted" charge density. Then, since $\rho = \rho' e^{bt}$, (3.1.3) becomes

$$[(\partial_t\rho')e^{bt} + \rho' b e^{bt}] - a\, e^{bt}\nabla^2\rho' - b e^{bt}\rho' = 0$$

or

$$\partial_t\rho' - a\nabla^2\rho' = 0 \qquad (3.1.4)$$

which is the normal heat/diffusion equation. If one starts at t = 0 with a point charge $\rho' = q\,\delta(\mathbf{r})$ at the origin, and if one assumes an infinite isotropic medium, one finds that at time t the charge density is given by

$$\rho'(r,t) = q\,\exp(-r^2/4at) / (4\pi at)^{3/2}\ . \qquad (3.1.5)$$

This is just the 3D causal free-space propagator (Green function) for the heat equation. It is the solution of

$$(\partial_t - a\,\nabla^2)\,\rho'(r,t) = \delta(\mathbf{r})\delta(t) \qquad \rho'(r,t) = 0 \text{ for } t<0\ . \qquad (3.1.6)$$

See Stakgold (5.133) and (5.136). In n dimensions, the propagator is as in (3.1.5) with 3/2 → n/2 and is derived in the text leading up to (5.140).

Therefore, if we consider (3.1.3) for charge density ρ

$$(\partial_t - a\nabla^2 - b)\rho(r,t) = \delta(\mathbf{r})\delta(t) \qquad (3.1.7)$$

replacing $\rho = \rho' e^{bt}$ gives

$$\partial_t\rho' - a\nabla^2\rho' = \delta(\mathbf{r})\delta(t)e^{-bt} = \delta(\mathbf{r})\delta(t)$$

which is the same as (3.1.6). Therefore, the solution of (3.1.7) is

$$\rho(r,t) = \rho' e^{bt} = q\, e^{bt}\exp(-r^2/4at) / (4\pi at)^{3/2} \qquad \rho(r,0) = q\,\delta(\mathbf{r})$$

or

$$\rho(r,t) = q\, e^{-(\sigma/\varepsilon)t}\exp(-r^2/4Dt) / (4\pi Dt)^{3/2} \qquad \rho(r,0) = q\,\delta(\mathbf{r}) \quad . \qquad (3.1.8)$$

The first factor $e^{-(\sigma/\varepsilon)t}$ says that ρ(r,t) decays exponentially in time in a uniform manner over space, while the second term says that the rough radius of the diffusing charge cloud is given by $r = \sqrt{4at}$.

The main point of all this math is the following: if there is any free charge in a medium, it goes away in a timely manner. In our idealized analysis above, it runs off to r = ∞, but in a finite medium it runs off to the boundary surface of the medium and becomes surface charge.

Let's now look at two extreme cases.

For a good conductor with a low diffusion rate, equation (3.1.2) becomes

$$\partial_t\rho + (\sigma/\varepsilon)\rho = 0 \qquad (3.1.9)$$

which has the obvious solution $\rho(r,t) = \rho(r,0)\, e^{-(\sigma/\varepsilon)t}$ which replicates the first factor of (3.1.8). The charge just "flows away" due to the large σ.

For a dielectric with a very small conductivity, equation (3.1.2) instead becomes

$$D\nabla^2\rho - \partial_t\rho = 0 \qquad (3.1.10)$$

which is just the heat equation whose impulse response solution is (3.1.8) with σ = 0, as was shown in (3.1.5). In this case, the charge at least has time to diffuse out before it goes away!

There are then two time constants involved. The first is for the $e^{-(\sigma/\varepsilon)t}$ factor where $\tau = \varepsilon/\sigma$. We can estimate this time constant for a conductor and dielectric using $\varepsilon \approx \varepsilon_0$, and

copper $\sigma = 5.81 \times 10^7$ mho/m

$\varepsilon_0 = 8.8541877 \times 10^{-12}$ farad/m (from 1.1.28)

$$\tau = \varepsilon/\sigma \approx 10^{-11} / 10^8 \approx 10^{-18} \text{ sec} \qquad (3.1.11)$$

so in copper, free charge runs off to the surface in one thousandth of a femtosecond, so we don't worry about the diffusion time constant.

For a dielectric with $\sigma = 10^{-15}$ mho/m we get τ larger by 10^{23} which is then 10^5 seconds or about a day. But in this case, the diffusion mechanism wins out. As an example, for pure silicon, $D \approx 40$ cm^2/sec = 4×10^{-3} m^2/sec. The time to diffuse from a delta function out to say r = 1 mm is given by

$$r = \sqrt{4Dt} \quad = \quad t = r^2/(4D) = (10^{-3}\text{m})^2 / (4*10^{-3}\ \text{m}^2/\text{sec}) = (1/4) \times 10^{-3}\ \text{sec} \tag{3.1.12}$$

so in this case the charge is pretty much gone in a quarter of a millisecond.

We arrive then at this fact:

Fact 1: In a transmission line, charge exists only on the surface of conductors.

Comment: If one wants an initial charge distribution $\rho'(r,0)$ to be something *other* than a delta function, one may use this solution to the heat equation (3.1.4),

$$\rho'(r,t) = [2r\sqrt{\pi a t}\,]^{-1} \int_0^\infty r'dr'\, \rho'(r',0)\, \{\ \exp[-(r-r')^2/4at]\ -\ \exp[-(r+r')^2/4at]\ \} \tag{3.1.13}$$

which appears in Polyanin 1.2.3-10. Setting $\rho'(r',0) = q\,\delta(\mathbf{r'}) = q\,\delta(r')/(4\pi r'^2)$ then replicates the earlier result (3.1.5), after using L'Hôpital's Rule on the integrand. The reason $\delta(\mathbf{r}) = \delta(r)/4\pi r^2$ is that it makes $\int dV\, \delta(\mathbf{r}) = 1$ when integrated over a sphere of any radius.