Rewriting Chapter 4 ? PhL 10.31.13

In my original lines doc I had a horribly wrong expansion for things in Chapter 4. I think I realized this and here are early notes where I try to find new ways to handle the φ integrals. I see now that this led to something that is now my K integral! This whole business I think is fully cleaned up in the new Chap 4.

I write down the nice Helmholtz integral for φ, and then I power series expand q(z). If I keep just the first term in this expansion, I get

$$\varphi_1(x,y,z) = \frac{1}{4\pi\varepsilon} q(z) \int_{C_1} dx'\, dy'\, a_1(x',y') \int_{-\infty}^{\infty} dz' \frac{e^{-j\beta R}}{R} \quad . \qquad (4.1.5)$$

Now why don't I just assume small β right at this point, meaning low ω. Then

$$\beta^2 = \mu\varepsilon\omega^2 - j\omega\mu\sigma = \omega^2\mu(\varepsilon - j\sigma/\omega) = \omega^2\mu\,\xi \qquad \xi \equiv \varepsilon - j\sigma/\omega \quad . \qquad (1.5.1)$$

$$e^{-j\beta R} \approx 1 - j\beta R$$

and then

$$\varphi_1(x,y,z) = \frac{1}{4\pi\varepsilon} q(z) \int_{C_1} dx'\, dy'\, a_1(x',y') \int_{-\infty}^{\infty} dz' \,[\, 1/R_1 - j\,\beta]$$

$R_1 = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$ // where (x',y',z') lie on conductor C_1

Now consider the two dz' integrals

$$I_1 = \int_{-\infty}^{\infty} dz' \,[\, 1/R_1 \,]$$

Now write

$$R_1 = \sqrt{s^2 + (z-z')^2} \quad \text{where} \quad s \equiv \sqrt{(x-x')^2 + (y-y')^2}$$

Then obviously you can shift the z' integration range to get

$$I_1 = \int_{-\infty}^{\infty} dz'' \,[\, 1/\sqrt{s^2 + z''^2} \,]$$

Now go look up

$$\int dx/\sqrt{x^2+a^2} = \ln(x + \sqrt{x^2+a^2}) = \mathrm{sh}^{-1}(x/a) \qquad \text{// I like the last, see Sp p 30}$$

Where do you find this famous integral? Spiegel 14.182 has it as noted above. Now if we assume the integral runs -Z to Z where Z is a large cutoff, we get

$$I_1 = 2 \int_0^Z dz'' \left[1/\sqrt{s^2 + z''^2} \right] = 2\, \mathrm{sh}^{-1}(x/s)|_0^z = 2\, \mathrm{sh}^{-1}(Z/s)$$

Maybe use the other form to get

$$= 2 \ln(x + \sqrt{x^2+s^2})|_0^z = 2 \ln(Z + \sqrt{Z^2+s^2}) - 2 \ln(s) \approx 2 \ln(2Z) - 2 \ln(s)$$

$$= 2 \ln(2Z/s) = \text{dimensionless}$$

The other integral then is

$$I_2 = \int_{-\infty}^{\infty} dz' = 2Z$$

So with this cutoff we find that

$$\varphi_1(x,y,z) = \frac{1}{4\pi\varepsilon} q(z) \int_{C_1} dx'\, dy'\, a_1(x',y') \int_{-\infty}^{\infty} dz' \left[1/R_1 - j\beta \right]$$

$$= \frac{1}{4\pi\varepsilon} q(z) \int_{C_1} dx'\, dy'\, a_1(x',y') \{ I_1 - j\beta I_2 \}$$

$$= \frac{1}{4\pi\varepsilon} q(z) \int_{C_1} dx'\, dy'\, a_1(x',y') \{ 2 \ln(2Z/s) - j\beta 2Z \}$$

$$= \frac{1}{4\pi\varepsilon} q(z) \left[-2j\beta Z + \int_{C_1} dx'\, dy'\, a_1(x',y')\, 2 \ln(2Z/s) \right]$$

$$= \frac{1}{2\pi\varepsilon} q(z) \left[-j\beta Z - \int_{C_1} dx'\, dy'\, a_1(x',y') \ln(s_1/2Z) \right]$$

where $s_1 \equiv \sqrt{(x-x')^2 + (y-y')^2}$ where (x',y') at some point on C_1

and dimensions are all OK. As long as we take Z to be some large fixed value, and avoid s = 0 by picking an origin NOT on a conductor, everything here seems well-defined and finite!

What happens when $\mathbf{x} \rightarrow \mathbf{x}_1$, some point right on conductor C_1? Well, in that case there is one point in the integration where $s_1 = 0$, but that will not cause trouble as Stak showed. In fact, suppose we assert that at some point $\mathbf{x}_1$ on C_1 we have some potential V_1 which is some number. Then we can write

$$V_1 = \frac{1}{2\pi\varepsilon} q(z) \left[-j\beta Z - \int_{C_1} dx'\, dy'\, a_1(x',y') \ln(s_1/2Z) \right]$$

$$s_1 \equiv \sqrt{(x_1-x')^2 + (y_1-y')^2} \quad \text{where } (x',y') \text{ at some point on } C_1$$

Stakgold tells us that you can regard this as in integral equation to be solved for a_1 ! Let's put a hold on that idea and continue now.

Now, suppose I compute the potential only due to conductor C_2. It ought to be

$$\varphi_2(x,y,z) = \frac{1}{2\pi\varepsilon} [-q(z)] \left[-j\beta Z - \int_{C_2} dx'\, dy'\, a_2(x',y') \ln(s_2/2Z) \right]$$

Now the sum of these two guys will be

$$\varphi_{1+2}(x,y,z) = \frac{1}{2\pi\varepsilon} q(z) \left\{ - \int_{C_1} dx'\, dy'\, a_1(x',y') \ln(s_1/2Z) + \int_{C_2} dx'\, dy'\, a_2(x',y') \ln(s_2/2Z) \right\}$$

$$= -\frac{1}{2\pi\varepsilon} q(z) \left[\int_{C_1} dx'\, dy'\, a_1(x',y') \ln(s_1/2Z) - \int_{C_2} dx'\, dy'\, a_2(x',y') \ln(s_2/2Z) \right]$$

Note that the result does not depend on Z due to normalization of the a_i, but leave it there for scale.

[this looks a lot like the current K integral!]

Now let's try to defog these integrals by having named variables on each surface, to remind us of what surface we are on

$$\varphi_{1+2}(\mathbf{x}) = -\frac{1}{2\pi\varepsilon} q(z) \left[\int_{C_1} dx_1'\, dy_1'\, a_1(x_1',y_1') \ln(s_{1P}/2Z) - \int_{C_2} dx_2'\, dy_2'\, a_2(x_2',y_2') \ln(s_{2P}/2Z) \right]$$

$$s_{1P} \equiv \sqrt{(x-x_1')^2 + (y-y_1')^2} = |\, \mathbf{x} - \mathbf{x}_1' \,|$$

$$s_{2P} \equiv \sqrt{(x-x_2')^2 + (y-y_2')^2} = |\, \mathbf{x} - \mathbf{x}_2' \,|$$

where P refers to location $\mathbf{x}$ as being at some point P in the dielectric.

NOW pick point $\mathbf{x}_1$ on C_1 and $\mathbf{x}_2$ on C_2. Assume potential on C_1 is +V and on C_2 is -V. Then we have

$$V = -\frac{1}{2\pi\varepsilon} q(z) \left[\int_{C_1} dx_1'\, dy_1'\, a_1(x_1',y_1') \ln(s_{11}/2Z) - \int_{C_2} dx_2'\, dy_2'\, a_2(x_2',y_2') \ln(s_{21}/2Z) \right]$$

$$s_{11} \equiv \sqrt{(x_1-x_1')^2 + (y_1-y_1')^2} = |\, \mathbf{x}_1 - \mathbf{x}_1' \,|$$

$$s_{21} \equiv \sqrt{(x_1-x_2')^2 + (y_1-y_2')^2} = |\, \mathbf{x}_1 - \mathbf{x}_2' \,|$$

$$-V = -\frac{1}{2\pi\varepsilon} q(z) \left[\int_{C_1} dx_1' \, dy_1' \, a_1(x_1',y_1') \ln(s_{12}/2Z) - \int_{C_2} dx_2' \, dy_2' \, a_2(x_2',y_2') \ln(s_{22}/2Z) \right]$$

$$s_{12} \equiv \sqrt{(x_2-x_1')^2 + (y_2-y_1')^2} = |\, \mathbf{x}_2 - \mathbf{x}_1' \,|$$

$$s_{22} \equiv \sqrt{(x_2-x_2')^2 + (y_2-y_2')^2} = |\, \mathbf{x}_2 - \mathbf{x}_2' \,|$$

Stak would say, I think, that we have here two integral equations which are capable in theory of determining the functions a_1 and a_2 ! But I know that solving such an integral equation is nearly impossible but I did one or two examples in Stak. I did it for a metal disk with potential V_0!

So where are we supposed to go from here? For some general case we don't know a_1 and a_2 and we certainly don't want to solve the above integral equations.

Comment #1: Suppose you could somehow determine a_1 and a_2. It seems that everything is real in the above equations, so you would then obtain some kind of real capacitance. Since we assumed $\beta R << 1$, we are certainly in some low frequency limit, but that is OK.

So in the above analysis, we don't get anything complex because we never used β. But King gets something complex because he has $\frac{1}{2\pi\xi}$ sitting out front of the integrals! So now that leading factor has become suddenly more important.

What I have done here seems pretty much what King does on page 16-17. Now I am changing my tune because now I think that external ξ factor is essential to getting the right **y** admittance on page 17. For him the complex β from the exponent never made an entry, it is the external ξ that does his trick.

So how I have to somehow compare to my prototype parallel cap example, and see why it is that I am getting a real phase between V(z) and q(z) in my fancy development, but the phase is obviously not real when you just do the cap problem.