Bug with small beta limit in Ch 4 PhL 1.3.14

My old Chapter 4 did have a big flaw. I want to be talking small β, but my main expansion assumed large β as you see in (4.2.2) quoted below. This resulted in full rework of Chapter 4 and it used to have an Appendix G for support, but that is now gone and replaced with old Appendix M.

Nothing like a brand new and fatal bug to deal with after working 3+ months on this monster.

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Plan A. My expression for V(z) is this,

$$V(z) = \frac{1}{4\pi\xi} \sum_{m=0}^{\infty} \frac{2\,(2m-1)!!}{(2m)!} \frac{1}{(j\beta)^m} q^{(2m)}(z) *$$

$$\{ \int_{C_1} dx'\, dy'\, a_1(x',y') [\, s_1^m K_m(j\beta s_1) - s_2^m K_m(j\beta s_2)]$$

$$- \int_{C_2} dx'\, dy'\, a_2(x',y') [\, s_1^m K_m(j\beta s_1) - s_2^m K_m(j\beta s_2)] \quad \} \qquad (4.2.2)$$

where $s_i = |\, \mathbf{x}_i - \mathbf{x}' \,|$ and $\mathbf{x}_1$ is a point on C_1 and x_2 a point on C_2. I am trying to discuss the small β limit of this expression and trying to show that beyond the leading term, things are order β^2. But as I examine things, this seems completely wrong. If I look at any of the "four terms" in the above, say for m > 0, I see this

$$\frac{1}{(j\beta)^m} K_m(j\beta s_1) \approx \frac{1}{(j\beta)^m} 2^{m-1}(m-1)!\, z^{-m}$$

$$= \frac{1}{(j\beta)^m} 2^{m-1}(m-1)!\, (j\beta s_1)^{-m} = \frac{1}{(j\beta)^m} 2^{m-1}(m-1)! \frac{1}{(j\beta s_1)^m}$$

$$= \frac{1}{(j\beta)^{2m}} 2^{m-1}(m-1)!\, s_1^{-m}$$

So the larger m is, the more **this thing blows up in my face! It's as if the expansion I used is completely wrong. [yup]**

Let's back up a bit to this point, where I have one of my "four terms" :

$$\varphi_1(x,y,z) = \frac{1}{4\pi\xi} \sum_{n=0}^{\infty} (1/n!)\, q^{(n)}(z) \int_{C_1} dx'\, dy'\, a_1(x',y') \int_{-\infty}^{\infty} dz'\, (z'-z)^n \frac{e^{-j\beta R}}{R} . \qquad (4.1.7)$$

$$R = \sqrt{(z'-z)^2 + s^2}$$

It seems quite clear that as $\beta \to 0$, the dz' integral blows up something like this

$$\int_{-\infty}^{\infty} dz'\, (z'-z)^{n-1}$$

and that of course is what my exact integration shows.

$$\int_{-\infty}^{\infty} dz'\, (z'-z)^n \frac{e^{-j\beta R}}{R}$$

$$= s^n f_n(\beta s) = \begin{cases} 0 & \text{for odd n} \\ 2s^n\, (n-1)!!\, K_{n/2}(j\beta s) / (j\beta s)^{n/2} & \text{for n even} \end{cases} \qquad (G.10)$$

My expansion of q(z) just makes things worse, bringing in those powers $(z'-z)^n$.

Is it possible that my $K_m(j\beta s)$ limit is wrong? I have to allow that β can be completely real in a good dielectric, so we really are going right ON the imaginary axis with the argument.

According to green Jackson page 75, we can write this in terms of $H_m^{(1)}(-\beta s)$. Do I know anything about that function for small β ? It seems that $Y_m(-\beta s)$ will diverge.

....

$$H_\nu^{(1)}(z) \sim -H_\nu^{(2)}(z) \sim -(i/\pi)\,\Gamma(\nu)(\tfrac{1}{2}z)^{-\nu}, \quad \Re\nu > 0.$$

So we still have our power divergence for small z no matter how you write it.

It must be that you should not do this dz' integral at this point in the calculation! King deals with this question on his page 16, and I thought I was cleverly generalizing his work.

First of all, King only deals with 2 terms instead of my four terms, because he assumes "a balanced line". That is to say, he assumes $\varphi_2 = -\varphi_1$ so that then $V = \varphi_1 - \varphi_2 = 2\varphi_1$ as in his top page 15. Of course he is already "into" his simple example.

Suppose I hold off on the expansion of q(z) so that, for some observation point **x** in the dielectric,

$$\varphi_1(x,y,z) = \frac{1}{4\pi\xi} \int_{C_1} dx'\, dy'\, a_1(x',y') \int_{-\infty}^{\infty} dz'\, q_1(z') \frac{e^{-j\beta R}}{R} . \qquad (4.1.5)$$

$$\varphi_2(x,y,z) = \frac{1}{4\pi\xi} \int_{C_2} dx'\, dy'\, a_2(x',y') \int_{-\infty}^{\infty} dz'\, q_2(z') \frac{e^{-j\beta R}}{R} . \qquad (4.1.5)$$

Now assume only that $q_1 = q$ and $q_2 = -q_1 = -q$ so we then have

$$\varphi_1(x,y,z) = \frac{1}{4\pi\xi} \int_{C_1} dx'\, dy'\, a_1(x',y') \int_{-\infty}^{\infty} dz'\, q(z') \frac{e^{-j\beta R}}{R} \quad . \tag{4.1.5}$$

$$\varphi_2(x,y,z) = \frac{1}{4\pi\xi} \int_{C_2} dx'\, dy'\, a_2(x',y') \int_{-\infty}^{\infty} dz'\, [-q(z')] \frac{e^{-j\beta R}}{R} \quad .$$

where $R = |\mathbf{x}-\mathbf{x}'|$ but of course $\mathbf{x}'$ takes different values for the different integrals. I am really stuck at this point, I don't see how to do anything with the subtraction. Maybe slide dz' integration to the left,

$$\varphi_1(x,y,z) = \frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz'\, q(z') \int_{C_1} dx'\, dy'\, a_1(x',y') \frac{e^{-j\beta R}}{R} \quad . \tag{4.1.5}$$

$$\varphi_2(x,y,z) = \frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz'\, [-q(z')] \int_{C_2} dx'\, dy'\, a_2(x',y') \frac{e^{-j\beta R}}{R} \quad .$$

Maybe I can say something about the transverse integral in the transmission line limit where $\lambda >> d$.

$$\varphi_1(x,y,z) = \frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz'\, q(z') \int_{C_1} dx_1'\, dy_1'\, a_1(x_1',y_1') \frac{e^{-j\beta R1}}{R_1} \quad .$$

$$R_1^2 = (x-x_1')^2 + (y-y_1')^2 + (z-z_1')^2$$

$$\varphi_2(x,y,z) = \frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz'\, [-q(z')] \int_{C_2} dx_2'\, dy_2'\, a_2(x_2',y_2') \frac{e^{-j\beta R2}}{R_2} \quad .$$

$$R_1^2 = (x-x_2')^2 + (y-y_2')^2 + (z-z_2')^2$$

Both integrations have a common z' coordinate, so no need for z_1' and z_2'.

Now it is true that, in the TL limit, we will have $\beta R_1 << 1$ and $\beta R_2 << 1$ so we could do expansions:
[we can do these expansions even if βR is NOT $<< 1$ because ∞ disk of convergence]

$$\varphi_1(x,y,z) = \frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz'\, q(z') \int_{C_1} dx_1'\, dy_1'\, a_1(x_1',y_1')\, R_1^{-1} \Sigma_{n=0}^{\infty} (-j\beta R_1)^n/n!$$

$$\varphi_2(x,y,z) = \frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz'\, [-q(z')] \int_{C_2} dx_2'\, dy_2'\, a_2(x_2',y_2')\, R_2^{-1} \Sigma_{n=0}^{\infty} (-j\beta R_2)^n/n! \quad .$$

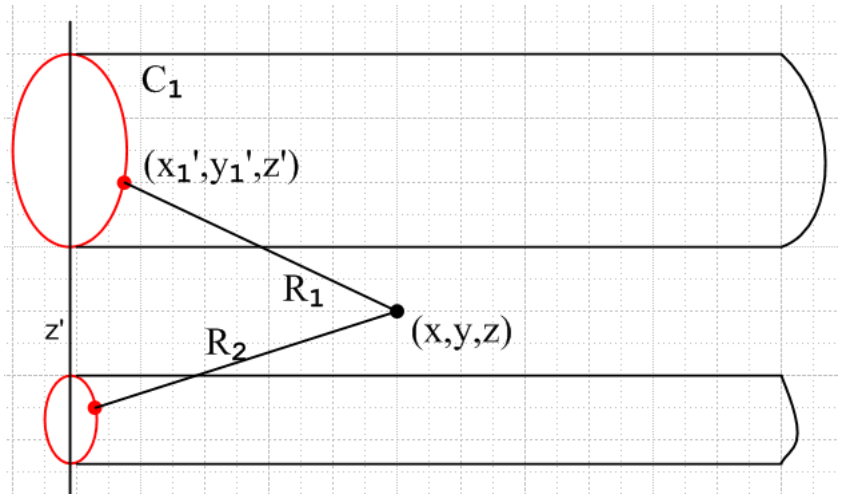
$$\varphi_1(x,y,z) = \frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz'\, q(z') \Sigma_{n=0}^{\infty} \frac{1}{n!} (-j\beta)^n \int_{C_1} dx_1'\, dy_1'\, a_1(x_1',y_1')\, R_1^{n-1}$$

$$\varphi_2(x,y,z) = -\frac{1}{4\pi\xi}\int_{-\infty}^{\infty} dz'\, q(z')\, \Sigma_{n=0}^{\infty}\frac{1}{n!}\,(-j\beta)^n \int_{C_2} dx_2'\, dy_2'\, a_2(x_2',y_2')\, R_2^{n-1}$$

Now I claim that,

$$\int_{C_1} dx_1'\, dy_1'\, a_1(x_1',y_1')\, R_1^{n-1} \;=\; f^1{}_n(x,y,z;\, z')$$

which is a function only of geometry and certainly does not depend on β. What exactly does this integral represent? Here is a picture, where we are integrating the red dot around the red boundary for each φ_i



If we regard $a_1(x_1',y_1')$ as a normalized weighting function, it is like a probability and then what we have is

$$\int_{C_1} dx_1'\, dy_1'\, a_1(x_1',y_1')\, R_1^{n-1} \;=\; \langle R_1^{n-1}\rangle \;=\; \text{expected value of } R_1^{n-1} \;=\; \text{average value of } R_1^{n-1}$$

You can see that if the two points have a very large z separation compared to other separations (that is, if the angle R_1 makes is close to parallel with the conductor) then

$$R_1^2 = (x-x_1')^2 + (y-y_1')^2 + (z-z_1')^2 \;\approx\; (z-z_1')^2 \qquad \Rightarrow \qquad R_1 \;\approx\; |z-z_1|$$

$$\langle R_1^{n-1}\rangle \;\approx\; |z-z_1|^{n-1}$$

So then we have

$$\varphi_1(x,y,z) = \frac{1}{4\pi\xi}\int_{-\infty}^{\infty} dz'\, q(z')\, \Sigma_{n=0}^{\infty}\frac{1}{n!}\,(-j\beta)^n\, f^1{}_n(x,y,z;\, z')$$

$$\varphi_2(x,y,z) = -\frac{1}{4\pi\xi}\int_{-\infty}^{\infty} dz'\, q(z')\, \Sigma_{n=0}^{\infty}\frac{1}{n!}\,(-j\beta)^n\, f^2{}_n(x,y,z;\, z')$$

At least we seem to have a small-β expansion now. Then,

$$\varphi_{12} = \varphi_1 - \varphi_2 = \frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz'\, q(z') \sum_{n=0}^{\infty} \frac{1}{n!} (-j\beta)^n [\, f^1_n(x,y,z;\, z') - f^2_n(x,y,z;\, z') \,]$$

We can write out some of the terms:

$$\varphi_{12} = \frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz'\, q(z') [\, f^1_0(x,y,z;\, z') - f^2_0(x,y,z;\, z') \,]$$
$$+ \frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz'\, q(z') \sum_{n=2}^{\infty} \frac{1}{n!} (-j\beta)^n [\, f^1_n(x,y,z;\, z') - f^2_n(x,y,z;\, z') \,]$$

which has the form

$$\varphi_{12} = \frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz'\, q(z') [\, f^1_0(x,y,z;\, z') - f^2_0(x,y,z;\, z') \,] + \text{order}(\beta^2)$$

Write the first two non-vanishing terms only as

$$\varphi_{12} = \frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz'\, q(z') [\, f^1_0(x,y,z;\, z') - f^2_0(x,y,z;\, z') \,]$$

$$+ \frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz'\, q(z') (1/2) (-j\beta)^2 [\, f^1_2(x,y,z;\, z') - f^2_2(x,y,z;\, z') \,]$$

Now look back at

$$\int_{C_1} dx_1'\, dy_1'\, a_1(x_1',y_1')\, R_1^{n-1} = f^1_n(x,y,z;\, z')$$

For large z-z', we have roughly that $R_1^2 = (z-z_1')^2$ so this becomes

$$f^1_n(x,y,z;\, z') \text{ for large } |z-z'| \approx \int_{C_1} dx_1'\, dy_1'\, a_1(x_1',y_1')\, |z-z'|^{n-1} = |z-z'|^{n-1}$$

$$f^2_n(x,y,z;\, z') \text{ for large } |z-z'| \approx \int_{C_2} dx_2'\, dy_2'\, a_1(x_2',y_2')\, |z-z'|^{n-1} = |z-z'|^{n-1}$$

This really means we can write

$$\varphi_{12} = \varphi_1 - \varphi_2 = \frac{1}{4\pi\xi} \int_{-\Lambda}^{\Lambda} dz'\, q(z') \sum_{n=0}^{\infty} \frac{1}{n!} (-j\beta)^n [\, f^1_n(x,y,z;\, z') - f^2_n(x,y,z;\, z') \,]$$

where perhaps $\Lambda \geq$ 100 d where d is the diameter of a tube which can contain both conductors. The main point is that each term in the n sum has a convergent dz' integral AND we have things in a power series expansion in β. The first two series terms are then

$$\varphi_{12} = \frac{1}{4\pi\xi} \int_{-\Lambda}^{\Lambda} dz'\, q(z') \left[f^1{}_0(x,y,z;\, z') - f^2{}_0(x,y,z;\, z') \right]$$

$$- (1/2)\, \beta^2 \frac{1}{4\pi\xi} \int_{-\Lambda}^{\Lambda} dz'\, q(z') \left[f^1{}_2(x,y,z;\, z') - f^2{}_2(x,y,z;\, z') \right]$$

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Plan B. Let's go back to

$$\varphi_{12} = \varphi_1 - \varphi_2 = \frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz'\, q(z') \sum_{n=0}^{\infty} \frac{1}{n!} (-j\beta)^n \left[f^1{}_n(x,y,z;\, z') - f^2{}_n(x,y,z;\, z') \right]$$

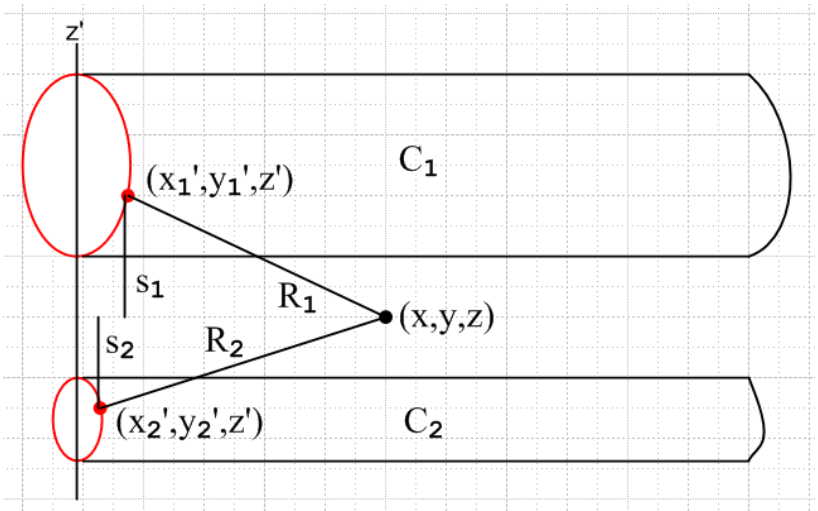
or

$$\varphi_{12} = \varphi_1 - \varphi_2 = \frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz'\, q(z') \sum_{n=0}^{\infty} \frac{1}{n!} (-j\beta)^n \left[\langle R_1^{n-1} \rangle - \langle R_2^{n-1} \rangle \right]$$

$$\int_{C_1} dx_1'\, dy_1'\, a_1(x_1',y_1')\, R_1^{n-1} = f^1{}_n(x,y,z;\, z') = \langle R_1^{n-1} \rangle$$

$$R_1^2 = (x-x_1')^2 + (y-y_1')^2 + (z-z')^2 = s_1^2 + (z-z')^2$$
$$R_2^2 = (x-x_2')^2 + (y-y_2')^2 + (z-z')^2 = s_2^2 + (z-z')^2$$

and maybe do something with this difference function. s_1 s_2



Let's try an expansion for where |z-z'| is large.

$$R_1^2 = (z-z')^2 + s_1^2 = (z-z')^2 \left[1 + s_1^2/(z-z')^2 \right]$$

$$R_1 = |z-z'| \left(1 + s_1^2/|z-z'|^2 \right)^{1/2}$$

$$R_1^{n-1} = |z-z'|^{n-1} \left(1 + s_1^2/|z-z'|^2 \right)^{(n-1)/2} \qquad // (1+x)^n = 1^n x + n\, 1^{n-1} x + \ldots.$$

or

$$R_1^{n-1} \approx |z-z'|^{n-1}\left(1 + \frac{n-1}{2}\frac{s_1^2}{|z-z'|^2}\right)$$

$$R_2^{n-1} \approx |z-z'|^{n-1}\left(1 + \frac{n-1}{2}\frac{s_2^2}{|z-z'|^2}\right)$$

Then write

$$\langle R_1^{n-1}\rangle - \langle R_2^{n-1}\rangle = \int_{C_1} dx_1'\, dy_1'\, a_1(x_1',y_1')\, R_1^{n-1} - \int_{C_2} dx_2'\, dy_2'\, a_2(x_2',y_2')\, R_2^{n-1}$$

$$= \int_{C_1} dx_1'\, dy_1'\, a_1(x_1',y_1') \left\{ |z-z'|^{n-1}\left(1 + \frac{n-1}{2}\frac{s_1^2}{|z-z'|^2}\right)\right\}$$

$$- \int_{C_2} dx_2'\, dy_2'\, a_2(x_2',y_2') \left\{ |z-z'|^{n-1}\left(1 + \frac{n-1}{2}\frac{s_2^2}{|z-z'|^2}\right)\right\}$$

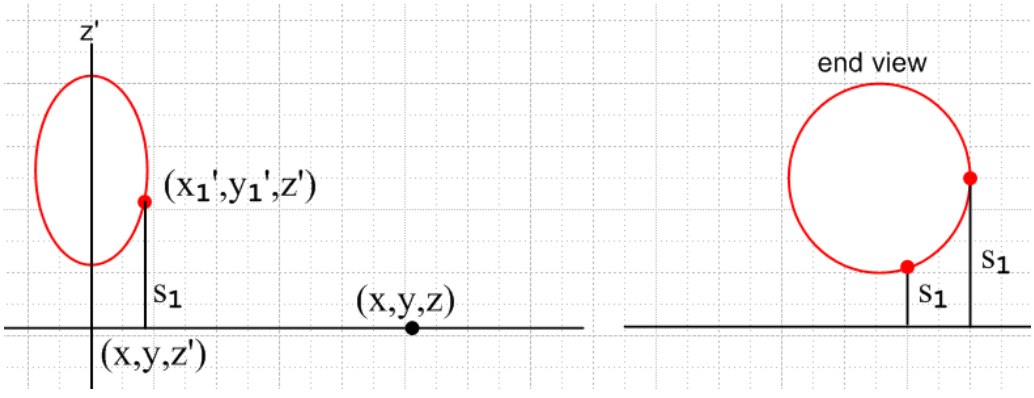
Now the "1" contributions exactly cancel due to the normalization of a_1 and a_2 , so that is the main cancellation idea realized yesterday. We then have

$$\langle R_1^{n-1}\rangle - \langle R_2^{n-1}\rangle \approx \int_{C_1} dx_1'\, dy_1'\, a_1(x_1',y_1') \left\{ |z-z'|^{n-1}\left(\frac{n-1}{2}\frac{s_1^2}{|z-z'|^2}\right)\right\}$$

$$- \int_{C_2} dx_2'\, dy_2'\, a_2(x_2',y_2') \left\{ |z-z'|^{n-1}\left(\frac{n-1}{2}\frac{s_2^2}{|z-z'|^2}\right)\right\}$$

$$= \frac{n-1}{2}\,|z-z'|^{n-3}\left\{ \int_{C_1} dx_1'\, dy_1'\, a_1(x_1',y_1')\, s_1^2 - \int_{C_2} dx_2'\, dy_2'\, a_2(x_2',y_2')\, s_2^2 \right\}$$

This appears to be a step in the right direction. I could write more terms in this expansion, but let's hold off on that for the moment. What is the meaning of the integrals appearing here? For a given point **x** in the dielectric, the first integral is a mean value of s_1^2 from the line specified by x,y as z varies.



So I have then shown that

$$\langle R_1^{n-1}\rangle - \langle R_2^{n-1}\rangle \approx \frac{n-1}{2} |z-z'|^{n-3} [\langle s_1^2\rangle - \langle s_2^2\rangle]$$

Then

$$\varphi_{12} = \varphi_1 - \varphi_2 = \frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz' \, q(z') \sum_{n=0}^{\infty} \frac{1}{n!} (-j\beta)^n [\langle R_1^{n-1}\rangle - \langle R_2^{n-1}\rangle]$$

$$\approx \frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz' \, q(z') \sum_{n=0}^{\infty} \frac{1}{n!} (-j\beta)^n \frac{n-1}{2} |z-z'|^{n-3} [\langle s_1^2\rangle - \langle s_2^2\rangle]$$

$$\approx \frac{1}{4\pi\xi} [\langle s_1^2\rangle - \langle s_2^2\rangle] \sum_{n=0}^{\infty} \frac{1}{n!} (-j\beta)^n \frac{n-1}{2} \int_{-\infty}^{\infty} dz' \, q(z')|z-z'|^{n-3}$$

OK, at least I have done *something*. But for larger n, the dz' integral still diverges, so what went wrong?

Plan C

Well, suppose we "back up" even more like so:

$$\varphi_1(x,y,z) = \frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz' \, q(z') \int_{C_1} dx_1' \, dy_1' \, a_1(x_1',y_1') \frac{e^{-j\beta R_1}}{R_1} .$$

$$\varphi_2(x,y,z) = \frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz' \, [-q(z')] \int_{C_2} dx_2' \, dy_2' \, a_2(x_2',y_2') \frac{e^{-j\beta R_2}}{R_2} .$$

And supposed to DON'T expand the exponential yet. Then

$$\varphi_{12}(\mathbf{x}) = \varphi_1(\mathbf{x}) - \varphi_2(\mathbf{x}) =$$

$$\frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz' \, q(z') \{ \int_{C_1} dx_1' \, dy_1' \, a_1(x_1',y_1') \frac{e^{-j\beta R_1}}{R_1} - \int_{C_2} dx_2' \, dy_2' \, a_2(x_2',y_2') \frac{e^{-j\beta R_2}}{R_2} \}$$

Now let's try an expansion of $\frac{e^{-j\beta R_1}}{R_1}$. As above we write,

$$R_1 = |z-z'| \left(1 + \frac{s_1^2}{|z-z'|^2} \right)^{1/2}$$

$$\frac{e^{-j\beta R_1}}{R_1} = \exp\left[-j\beta |z-z'| \left(1 + \frac{s_1^2}{|z-z'|^2} \right)^{1/2} \right] / \; |z-z'| \left(1 + \frac{s_1^2}{|z-z'|^2} \right)^{1/2}$$

$$\approx \exp\left[-j\beta |z-z'| \left(1 + \frac{1}{2} \frac{s_1^2}{|z-z'|^2} \right) \right] / \; |z-z'| \left(1 + \frac{1}{2} \frac{s_1^2}{|z-z'|^2} \right)$$

$$\approx \exp[\,-j\beta\,|z-z'| \;\; -j\beta\,\frac{1}{2}\,\frac{s_1^2}{|z-z'|}\,)\,]\;/\;\;|z-z'|\,(\;1\;+\frac{1}{2}\,\frac{s_1^2}{|z-z'|^2}\,)$$

$$\approx \exp[\,-j\beta\,|z-z'|\,]\,\exp[\;\;-j\beta\,\frac{1}{2}\,\frac{s_1^2}{|z-z'|}\,)\,]\;/\;\;|z-z'|\,(\;1\;+\frac{1}{2}\,\frac{s_1^2}{|z-z'|^2}\,)$$

$$= \frac{e^{-j\beta|z-z'|}}{|z-z'|} \;\; * \frac{\exp[\,-j\beta\,\frac{1}{2}\,\frac{s_1^2}{|z-z'|}\,]}{1\;+\frac{1}{2}\,\frac{s_1^2}{|z-z'|^2}} \qquad (*)$$

If I do this as well for $\frac{e^{-j\beta R_2}}{R_2}$, then the "leading term" exactly cancels in φ_{12}, as happened in my last attempt. Let's go ahead and write it all out

$$\varphi_{12} \approx \frac{1}{4\pi\xi}\int_{-\infty}^{\infty} dz'\, q(z') \int_{C_1} dx_1'\, dy_1'\, a_1(x_1',y_1')\,[\frac{e^{-j\beta|z-z'|}}{|z-z'|} \;\; * \frac{\exp[\,-j\beta\,\frac{1}{2}\,\frac{s_1^2}{|z-z'|}\,]}{1\;+\frac{1}{2}\,\frac{s_1^2}{|z-z'|^2}}\,]$$

$$-\frac{1}{4\pi\xi}\int_{-\infty}^{\infty} dz'\, q(z') \int_{C_2} dx_2'\, dy_2'\, a_2(x_2',y_2')\,[\frac{e^{-j\beta|z-z'|}}{|z-z'|} \;\; * \frac{\exp[\,-j\beta\,\frac{1}{2}\,\frac{s_2^2}{|z-z'|}\,]}{1\;+\frac{1}{2}\,\frac{s_2^2}{|z-z'|^2}}\,]$$

or

$$\varphi_{12} \approx \frac{1}{4\pi\xi}\int_{-\infty}^{\infty} dz'\, q(z')\,\frac{e^{-j\beta|z-z'|}}{|z-z'|} \int_{C_1} dx_1'\, dy_1'\, a_1(x_1',y_1')\,[\frac{\exp[\,-j\beta\,\frac{1}{2}\,\frac{s_1^2}{|z-z'|}\,]}{1\;+\frac{1}{2}\,\frac{s_1^2}{|z-z'|^2}}\,]$$

$$-\frac{1}{4\pi\xi}\int_{-\infty}^{\infty} dz'\, q(z')\,\frac{e^{-j\beta|z-z'|}}{|z-z'|} \int_{C_2} dx_2'\, dy_2'\, a_2(x_2',y_2')\,[\frac{\exp[\,-j\beta\,\frac{1}{2}\,\frac{s_2^2}{|z-z'|}\,]}{1\;+\frac{1}{2}\,\frac{s_2^2}{|z-z'|^2}}\,]$$

I think there is a finite $\beta = 0$ term sitting here:

$$\varphi_{12}^{\beta=0} = \frac{1}{4\pi\xi}\int_{-\infty}^{\infty} dz'\, q(z')\,\frac{1}{|z-z'|}\,\{ \int_{C_1} dx_1'\, dy_1'\, a_1(x_1',y_1')\,[\frac{1}{1\;+\frac{1}{2}\,\frac{s_1^2}{|z-z'|^2}}\,]$$

$$-\frac{1}{4\pi\xi}\int_{-\infty}^{\infty} dz'\, q(z')\,\frac{1}{|z-z'|} \int_{C_2} dx_2'\, dy_2'\, a_2(x_2',y_2')\,[\frac{1}{1\;+\frac{1}{2}\,\frac{s_2^2}{|z-z'|^2}}\,]$$

or

$$\varphi_{12}^{\beta=0} \approx \frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz'\, q(z') \frac{1}{|z-z'|} \{ \int_{C_1} dx_1'\, dy_1'\, a_1(x_1',y_1') [1 - \frac{1}{2} \frac{s_1^2}{|z-z'|^2}] - \frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz'\, q(z') \frac{1}{|z-z'|} \int_{C_2} dx_2'\, dy_2'\, a_2(x_2',y_2') [1 - \frac{1}{2} \frac{s_2^2}{|z-z'|^2}]$$

The leading terms cancel due to a_i normalization and we are left with

$$\varphi_{12}^{\beta=0} \approx (-\frac{1}{2}) \frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz'\, q(z') \frac{1}{|z-z'|^3} \{ \langle s_1^2 \rangle - \langle s_2^2 \rangle \}$$

$$= (-\frac{1}{2}) \frac{1}{4\pi\xi} \{ \langle s_1^2 \rangle - \langle s_2^2 \rangle \} \int_{-\infty}^{\infty} dz'\, q(z') \frac{1}{|z-z'|^3}$$

Now suppose we put in a power series for q(z'),

$$q(z') = q(z) + q'(z)\,(z'-z) + ..$$

Then the leading term diverges at the end near z' = 0,'

$$\int_{-\infty}^{\infty} dz'\, q(z) \frac{1}{|z-z'|^3} = q(z) \int_{-\infty}^{\infty} dz' \frac{1}{|z-z'|^3} = q(z) \int_{-\infty}^{\infty} dx \frac{1}{|x|^3} = 2\, q(z) \int_{0}^{\infty} dx\, x^{-3} = \text{diverges}$$

This I guess is due to my expansion for large |z-z'| being bad for small |z-z'|. Ignoring this for the moment:

OK, now maybe is the time to do our exponential expansion

$$\exp[-j\beta \frac{1}{2} \frac{s_1^2}{|z-z'|}] = \Sigma_{n=0}^{\infty} (1/n!) (j\beta \frac{1}{2} \frac{s_1^2}{|z-z'|})^n$$

$$= 1 + \Sigma_{n=1}^{\infty} (1/n!) (j\beta \frac{1}{2} \frac{s_1^2}{|z-z'|})^n$$

But since I only expanded two terms in (*) above, the higher terms here are not too meaningful. So maybe only keep the first two terms:

$$\exp[-j\beta \frac{1}{2} \frac{s_1^2}{|z-z'|}] \approx 1 + (1/2)\, j\beta \frac{1}{2} \frac{s_1^2}{|z-z'|}$$

Then we get

$$\frac{\exp[-j\beta \frac{1}{2} \frac{s_1^2}{|z-z'|}]}{1 + \frac{1}{2} \frac{s_1^2}{|z-z'|^2}} \approx [1 + (1/2)\, j\beta \frac{1}{2} \frac{s_1^2}{|z-z'|}] [1 - \frac{1}{2} \frac{s_1^2}{|z-z'|^2}] \approx 1 + (j\beta/4) \frac{s_1^2}{|z-z'|} + \text{order}(\frac{1}{|z-z'|^2})$$

Then put pieces together again

$$\varphi_{12} \approx \frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz'\, q(z') \frac{e^{-j\beta|z-z'|}}{|z-z'|} \{ \int_{C_1} dx_1'\, dy_1'\, a_1(x_1',y_1') [1 + (j\beta/4)\frac{s_1^2}{|z-z'|}]$$

$$- \frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz'\, q(z') \frac{e^{-j\beta|z-z'|}}{|z-z'|} \{ \int_{C_2} dx_2'\, dy_2'\, a_2(x_2',y_2') [1 + (j\beta/4)\frac{s_2^2}{|z-z'|}]$$

As already noted, the leading terms cancel and we then have

$$\varphi_{12} \approx \frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz'\, q(z') \frac{e^{-j\beta|z-z'|}}{|z-z'|} \{ \int_{C_1} dx_1'\, dy_1'\, a_1(x_1',y_1') [(j\beta/4)\frac{s_1^2}{|z-z'|}]$$

$$- \frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz'\, q(z') \frac{e^{-j\beta|z-z'|}}{|z-z'|} \{ \int_{C_2} dx_2'\, dy_2'\, a_2(x_2',y_2') [(j\beta/4)\frac{s_2^2}{|z-z'|}]$$

$$= (j\beta/4) \frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz'\, q(z') \frac{e^{-j\beta|z-z'|}}{|z-z'|^2} \{ \langle s_1^2 \rangle_t - \langle s_2^2 \rangle_t \}$$

or

$$\varphi_{12}(x,y,z) \approx (j\beta/4) \frac{1}{4\pi\xi} \{ \langle s_1^2 \rangle_t - \langle s_2^2 \rangle_t \} \int_{-\infty}^{\infty} dz'\, q(z') \frac{e^{-j\beta|z-z'|}}{|z-z'|^2}$$

Now the dz' integral seems a little more convergent. But the leading term seems order β and not β^2 ? What result am I eventually expecting?

$$V(z) = \frac{2}{4\pi\varepsilon} q(z) \{ \int_{C_1} dx'\, dy'\, a_1(x',y') \ln(s_2/s_1) - \int_{C_2} dx'\, dy'\, a_2(x',y') \ln(s_2/s_1) \} \qquad (4.2.5)$$

This is of course another step of differencing, but it has no β dependence.

Plan D Start again,

$$\varphi_{12}(\mathbf{x}) = \varphi_1(\mathbf{x}) - \varphi_2(\mathbf{x}) =$$

$$\frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz'\, q(z') \{ \int_{C_1} dx_1'\, dy_1'\, a_1(x_1',y_1') \frac{e^{-j\beta R_1}}{R_1} - \int_{C_2} dx_2'\, dy_2'\, a_2(x_2',y_2') \frac{e^{-j\beta R_2}}{R_2} \}$$

$$R_1^2 = (x-x_1')^2 + (y-y_1')^2 + (z-z')^2 = s_1^2 + (z-z')^2$$
$$R_2^2 = (x-x_2')^2 + (y-y_2')^2 + (z-z')^2 = s_2^2 + (z-z')^2$$

Again, for **x** somewhere in the middle, this thing has to be finite for any β. If $a_1 = a_2$ as in King's example, then you get the difference function $(\frac{e^{-j\beta R_1}}{R_1} - \frac{e^{-j\beta R_2}}{R_2})$ which is how he deals with things. His symmetry then allows consideration of only 2 terms instead of 4 terms for V(x).

What happens if I start off with the expansion

$$q(z') = q(z) + q'(z)\,(z'-z) + ..$$

This will only be valid when z' is close to z, so why is this useful? One argument is this: when z' is far from z, contributions to the above integrals are small because the two R_i are large. So the main contribution comes from the region where z' is near z, and then this expansion is reasonable. then

$$\varphi_{12}(\mathbf{x}) = \varphi_1(\mathbf{x}) - \varphi_2(\mathbf{x}) =$$

$$q(z)\frac{1}{4\pi\xi}\int_{-\infty}^{\infty} dz' \left\{ \int_{C_1} dx_1'\, dy_1'\, a_1(x_1',y_1')\frac{e^{-j\beta R_1}}{R_1} - \int_{C_2} dx_2'\, dy_2'\, a_2(x_2',y_2')\frac{e^{-j\beta R_2}}{R_2} \right\}$$

$$+\ q'(z)\frac{1}{4\pi\xi}\int_{-\infty}^{\infty} dz'\ (z'-z) \left\{ \int_{C_1} dx_1'\, dy_1'\, a_1(x_1',y_1')\frac{e^{-j\beta R_1}}{R_1} - \int_{C_2} dx_2'\, dy_2'\, a_2(x_2',y_2')\frac{e^{-j\beta R_2}}{R_2} \right\}$$

The problem is then to show that this second line is small compared to the first line in the transmission line limit. But if I make the ansatz that

$$q(z) = q(z_0)e^{-j\beta z} \qquad \text{for z dependence}$$

then

$$q'(z) = -j\beta q(z)$$ and then we have

$$\varphi_{12}(\mathbf{x}) = \varphi_1(\mathbf{x}) - \varphi_2(\mathbf{x}) =$$

$$q(z)\frac{1}{4\pi\xi}\int_{-\infty}^{\infty} dz' \left\{ \int_{C_1} dx_1'\, dy_1'\, a_1(x_1',y_1')\frac{e^{-j\beta R_1}}{R_1} - \int_{C_2} dx_2'\, dy_2'\, a_2(x_2',y_2')\frac{e^{-j\beta R_2}}{R_2} \right\}$$

$$-j\beta q(z)\frac{1}{4\pi\xi}\int_{-\infty}^{\infty} dz'\ (z'-z) \left\{ \int_{C_1} dx_1'\, dy_1'\, a_1(x_1',y_1')\frac{e^{-j\beta R_1}}{R_1} - \int_{C_2} dx_2'\, dy_2'\, a_2(x_2',y_2')\frac{e^{-j\beta R_2}}{R_2} \right\}$$

and then the second term is down by a factor of β (at least, maybe it will be more)

Idea: in this second term, each R_i is even in (z'-z). Thus, $\frac{e^{-j\beta R_1}}{R_1}$ is even in (z'-z) as is $\frac{e^{-j\beta R_2}}{R_2}$. Thus, the entire integrand {...} is even in (z'-z), and thus the entire second integral vanishes exactly!! If we then do the next term in the power series

$$q(z') = q(z) + q'(z)(z'-z) + (1/2)q''(z)\,(z'-z)^2 + ...$$

$$q(z) = q(z_0)e^{-j\beta z}$$

$$q'(z) = (-j\beta)q(z)$$

$$q''(z) = (-j\beta)^2 q(z)$$

Then the next term will be

$$(-j\beta)^2 q(z) \frac{1}{8\pi\xi} \int_{-\infty}^{\infty} dz' (z'-z)^2 \{ \int_{C_1} dx_1' dy_1' a_1(x_1',y_1') \frac{e^{-j\beta R_1}}{R_1} - \int_{C_2} dx_2' dy_2' a_2(x_2',y_2') \frac{e^{-j\beta R_2}}{R_2} \}$$

and this term is then order β^2 in agreement with King.

So at this point, although pieces are not all there, we have shown that if we drop β^2 and higher terms, we end up with

$$\varphi_{12}(\mathbf{x}) = q(z) \frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz' \{ \int_{C_1} dx_1' dy_1' a_1(x_1',y_1') \frac{e^{-j\beta R_1}}{R_1} - \int_{C_2} dx_2' dy_2' a_2(x_2',y_2') \frac{e^{-j\beta R_2}}{R_2} \}$$

So then I only have this thing to deal with, but of course I have to get to the four terms.

$V(z) = \varphi_{12}(\mathbf{x}_1) - \varphi_{12}(\mathbf{x}_2)$ = potential between conductors.

When things are symmetric as in King's example, then we get

$V(z) = \varphi_{12}(\mathbf{x}_1) - \varphi_{12}(\mathbf{x}_2) = 2\,\varphi_{12}(\mathbf{x}_1)$

or

$$V(z) = q(z) \frac{1}{2\pi\xi} \int_{-\infty}^{\infty} dz' \{ \int_{C_1} dx_1' dy_1' a_1(x_1',y_1') \frac{e^{-j\beta R_1}}{R_1} - \int_{C_2} dx_2' dy_2' a_2(x_2',y_2') \frac{e^{-j\beta R_2}}{R_2} \}$$

where now

$$R_1^2 = (x_1-x_1')^2 + (y_1-y_1')^2 + (z-z')^2 = s_{11}^2 + (z-z')^2$$
$$R_2^2 = (x_1-x_2')^2 + (y_1-y_2')^2 + (z-z')^2 = s_{12}^2 + (z-z')^2$$

Now I think if the wires are thin, and we take our touch points to be the nearest points, and we assume the average value for x_1' is the wire center, then we get

$$R_1^2 \approx a^2 + (z-z')^2$$

$$R_2^2 \approx (x_1-x_2)^2 + (y_1-y_2)^2 + (z-z')^2 \approx b^2 + (z-z')^2$$

where d is the separation of the thin wires. I think in this case, the transverse integration is 1 and we have

$$V(z) = q(z) \frac{1}{2\pi\xi} \int_{-\infty}^{\infty} dz' \{ \frac{e^{-j\beta R_1}}{R_1} - \frac{e^{-j\beta R_2}}{R_2} \}$$

Now the leading term with $\beta = 1$ would be

$$V(z) = q(z)\frac{1}{2\pi\xi}\int_{-\infty}^{\infty} dz' \left\{ \frac{1}{R_1} - \frac{1}{R_2} \right\}$$

$$= q(z)\frac{1}{2\pi\xi}\int_{-\infty}^{\infty} dz' \left\{ \frac{1}{(z-z')^2} - \frac{1}{d^2 + (z-z')^2} \right\}$$

$$= q(z)\frac{1}{2\pi\xi}\int_{-\infty}^{\infty} dx \left\{ \frac{1}{\sqrt{a^2+x^2}} - \frac{1}{\sqrt{b^2+x^2}} \right\}$$

According to Maple,

```
Int(1/sqrt(a^2+x^2)- 1/sqrt(b^2+x^2),x=-infinity..infinity);
```

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{a^2+x^2}} - \frac{1}{\sqrt{b^2+x^2}}\,dx$$

```
int(1/sqrt(a^2+x^2)- 1/sqrt(b^2+x^2),x=-infinity..infinity);
```

$$-\ln(a^2) + \ln(b^2)$$

so this integral is

$$\int_{-\infty}^{\infty} dz' \left\{ \frac{1}{R_1} - \frac{1}{R_2} \right\} = \int_{-\infty}^{\infty} dx \left\{ \frac{1}{\sqrt{a^2+x^2}} - \frac{1}{\sqrt{b^2+x^2}} \right\} = \ln(b^2/a^2) = 2\ln(b/a)$$

in agreement with King, and then we get

$$V(z) = q(z)\frac{1}{2\pi\xi}\, 2\ln(b/a) = q(z)\frac{1}{\pi\xi}\ln(b/a) = Q/C'$$

$$C' = \pi\xi/\ln(b/a)$$

Taking the ε part we get

$C = \pi\varepsilon/\ln(b/a)$ // agrees with King (30b), and so on for R.

So at least I see how the King thing works, vaguely. He makes a lot of approximations, I hoped to do better.

Plan E I now assume that higher terms are β^2 by the argument given in Plan D, and now I concentrate on evaluation of the main term. Here is my potential at an arbitrary central point,

$$\varphi_{12}(\mathbf{x}) = \varphi_1(\mathbf{x}) - \varphi_2(\mathbf{x}) =$$

$$q(z)\frac{1}{4\pi\xi}\int_{-\infty}^{\infty} dz' \left\{ \int_{C_1} dx_1'\, dy_1'\, a_1(x_1',y_1')\frac{e^{-j\beta R_1}}{R_1} - \int_{C_2} dx_2'\, dy_2'\, a_2(x_2',y_2')\frac{e^{-j\beta R_2}}{R_2} \right\}$$

But if I am doing $\beta = 0$, this is really

$$\varphi_{12}(\mathbf{x}) = \varphi_1(\mathbf{x}) - \varphi_2(\mathbf{x}) =$$

$$q(z)\frac{1}{4\pi\xi}\int_{-\infty}^{\infty} dz' \left\{ \int_{C_1} dx_1'\, dy_1'\, a_1(x_1',y_1')\frac{1}{R_1} - \int_{C_2} dx_2'\, dy_2'\, a_2(x_2',y_2')\frac{1}{R_2} \right\}$$

I am done with all expansion things, and I only have to compute this term. I think I should draw a picture that shows all the R's that will be involved.

$$\varphi_{12}(\mathbf{x}_1) = q(z)\frac{1}{4\pi\xi}\int_{-\infty}^{\infty} dz' \left\{ \int_{C_1} dx_1'\, dy_1'\, a_1(x_1',y_1')\frac{1}{R_{11}} - \int_{C_2} dx_2'\, dy_2'\, a_2(x_2',y_2')\frac{1}{R_{12}} \right\}$$

$$R_{11}^2 = (x_1-x_1')^2 + (y_1-y_1')^2 + (z-z')^2 = s_{11}^2 + (z-z')^2 \qquad s_{11}^2 = (x_1-x_1')^2 + (y_1-y_1')^2$$

$$R_{12}^2 = (x_1-x_2')^2 + (y_1-y_2')^2 + (z-z')^2 = s_{12}^2 + (z-z')^2 \qquad s_{12}^2 = (x_1-x_2')^2 + (y_1-y_2')^2$$

$$\varphi_{12}(\mathbf{x}_2) = q(z)\frac{1}{4\pi\xi}\int_{-\infty}^{\infty} dz' \left\{ \int_{C_1} dx_1'\, dy_1'\, a_1(x_1',y_1')\frac{1}{R_{21}} - \int_{C_2} dx_2'\, dy_2'\, a_2(x_2',y_2')\frac{1}{R_{22}} \right\}$$

$$R_{22}^2 = (x_2-x_2')^2 + (y_2-y_2')^2 + (z-z')^2 = s_{22}^2 + (z-z')^2 \qquad s_{22}^2 = (x_2-x_2')^2 + (y_2-y_2')^2$$

$$R_{21}^2 = (x_2-x_1')^2 + (y_2-y_1)^2 + (z-z')^2 = s_{21}^2 + (z-z')^2 \qquad s_{21}^2 = (x_2-x_1')^2 + (y_2-y_1)^2$$

$$R_{22}^2 = |\mathbf{r}_2 - \mathbf{r}_2'|^2$$

Then finally we have

$$V(z) = \varphi_{12}(\mathbf{x}_1) - \varphi_{12}(\mathbf{x}_2)$$

$$= q(z)\frac{1}{4\pi\xi}\int_{-\infty}^{\infty} dz' \left\{ \int_{C_1} dx_1'\, dy_1'\, a_1(x_1',y_1')\frac{1}{R_{11}} - \int_{C_2} dx_2'\, dy_2'\, a_2(x_2',y_2')\frac{1}{R_{12}} \right\}$$
$$- q(z)\frac{1}{4\pi\xi}\int_{-\infty}^{\infty} dz' \left\{ \int_{C_1} dx_1'\, dy_1'\, a_1(x_1',y_1')\frac{1}{R_{21}} - \int_{C_2} dx_2'\, dy_2'\, a_2(x_2',y_2')\frac{1}{R_{22}} \right\}$$

$$= q(z)\frac{1}{4\pi\xi}\int_{-\infty}^{\infty} dz' \left\{ \int_{C_1} dx_1'\, dy_1'\, a_1(x_1',y_1')\left(\frac{1}{R_{11}} - \frac{1}{R_{21}}\right) - \int_{C_2} dx_2'\, dy_2'\, a_2(x_2',y_2')\left(\frac{1}{R_{12}} - \frac{1}{R_{22}}\right) \right\}$$

Now what??? I could move the dz' integrations to the right. Then I need, for example,

$$\int_{-\infty}^{\infty} dz' \left(\frac{1}{R_{11}} - \frac{1}{R_{21}}\right)$$

$$= \int_{-\infty}^{\infty} dz' \left(\frac{1}{\sqrt{s_{11}^2 + (z-z')^2}} - \frac{1}{\sqrt{s_{21}^2 + (z-z')^2}}\right)$$

But I think I know this integral! Recall from above that

$$\int_{-\infty}^{\infty} dx \left\{ \frac{1}{\sqrt{a^2 + x^2}} - \frac{1}{\sqrt{b^2 + x^2}} \right\} = \ln(b^2/a^2) = 2\ln(b/a)$$

So then we should have

$$\int_{-\infty}^{\infty} dz' \left(\frac{1}{R_{11}} - \frac{1}{R_{21}} \right) = \int_{-\infty}^{\infty} dz' \left(\frac{1}{\sqrt{s_{11}^2 + (z-z')^2}} - \frac{1}{\sqrt{s_{21}^2 + (z-z')^2}} \right) = 2\ln(s_{21}/s_{11})$$

and then

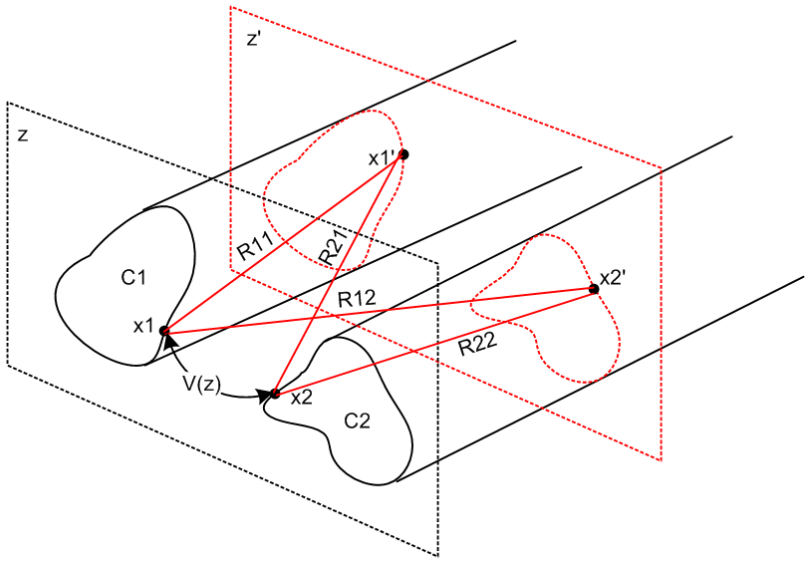
$$\int_{-\infty}^{\infty} dz' \left(\frac{1}{R_{12}} - \frac{1}{R_{22}} \right) = 2\ln(s_{22}/s_{12})$$

and then I arrive here:

$$V(z) = q(z) \frac{1}{2\pi\xi} \left\{ \int_{C_1} dx_1' \, dy_1' \, a_1(x_1', y_1') \ln(s_{21}/s_{11}) - \int_{C_2} dx_2' \, dy_2' \, a_2(x_2', y_2') \ln(s_{22}/s_{12}) \right\}$$

I think then this is a more precise version of what I have in my lines doc. Here is a picture to go with the original equations

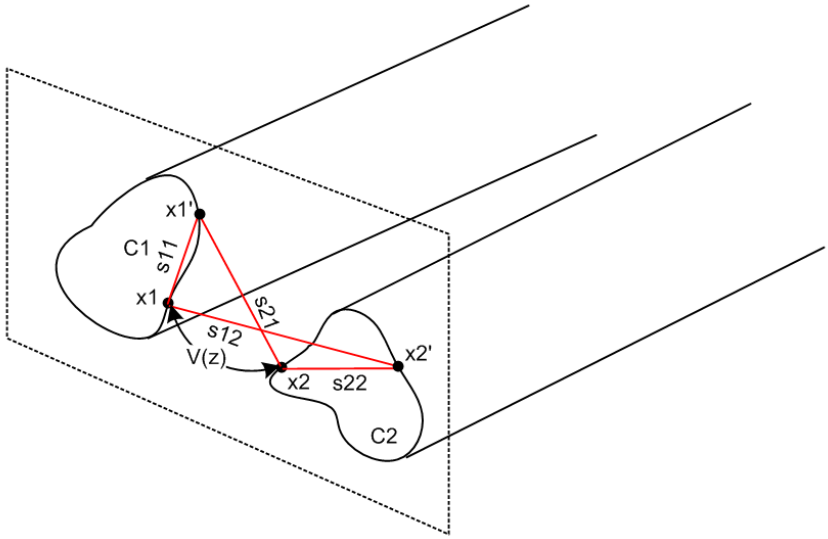
$$= q(z) \frac{1}{4\pi\xi} \int_{-\infty}^{\infty} dz' \left\{ \int_{C_1} dx_1' \, dy_1' \, a_1(x_1', y_1') \left(\frac{1}{R_{11}} - \frac{1}{R_{21}} \right) - \int_{C_2} dx_2' \, dy_2' \, a_2(x_2', y_2') \left(\frac{1}{R_{12}} - \frac{1}{R_{22}} \right) \right\}$$



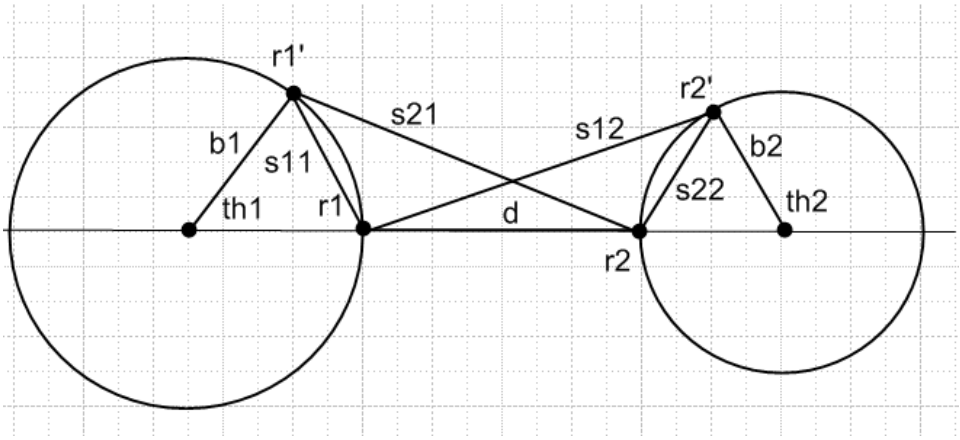
The last equation has a different picture all in a constant z plane:

$$V(z) = q(z)\frac{1}{2\pi\xi}\ \{\ \int_{C_1} dx_1'\, dy_1'\, a_1(x_1',y_1')\ln(s_{21}/s_{11}) - \int_{C_2} dx_2'\, dy_2'\, a_2(x_2',y_2')\ln(s_{22}/s_{12})\ \}$$

$$s_{11}^2 = (x_1-x_1')^2 + (y_1-y_1')^2 \qquad s_{21}^2 = (x_2-x_1')^2 + (y_2-y_1)^2$$
$$s_{12}^2 = (x_1-x_2')^2 + (y_1-y_2')^2 \qquad s_{22}^2 = (x_2-x_2')^2 + (y_2-y_2')^2$$



Let's now try the two circles:



STOP! On the next line, I assume incorrectly that the surface charge is uniform around the round wires, but that is of course only valid for d >> b_1, b_2 so THAT is my error!! That is why the result I get below disagrees with King, because it makes a false assumption right at the start

So that above equation is valid in the sense that, if you KNEW $a_1(x_1',y_1')$ and $a_2(x_1',y_1')$, you then have a result. So what good is this whole approach since we DONT know these things except in the large separation case?

$$V(z) = q(z)\frac{1}{2\pi\xi}\ \{\ \int_{C_1} dx_1'\, dy_1'\, \delta(r - b_1)/(2\pi b_1)\ln(s_{21}/s_{11}) - \int_{C_2} dx_2'\, dy_2'\, \delta(r - b_2)/(2\pi b_2)\ln(s_{22}/s_{12})\ \}$$

$$= q(z)\frac{1}{4\pi^2\xi}\ \{\ \int_0^{\infty} dr_1'\ \int_0^{2\pi} d\theta'_1\ \delta(r_1' - b_1)\ \ln(s_{21}/s_{11}) - \int_0^{\infty} dr_2'\ \int_0^{2\pi} d\theta'_2\ \delta(r_2' - b_2)\ \ln(s_{22}/s_{12})\ \}$$

$$= q(z)\frac{1}{4\pi^2\xi}\ \{\ \int_0^{2\pi} d\theta'_1\ \ln(s_{21}/s_{11})|^{r1'=b1} - \int_0^{2\pi} d\theta'_2\ \ln(s_{22}/s_{12})|^{r2'=b2}\ \}$$

Now

$$s_{11}^2 = (x_1-x_1')^2 + (y_1-y_1')^2 = b_1^2 + b_1^2 - 2b_1b_1\cos(\theta_1) = 2b_1^2\ [\ 1-\cos(\theta_1)]$$
$$s_{12}^2 = (x_1-x_2')^2 + (y_1-y_2')^2 = (d+b_2)^2 + b_2^2 - 2(d+b_2)\ b_2\cos(\pi-\theta_2)$$

$$s_{21}^2 = (x_2-x_1')^2 + (y_2-y_1)^2 = (d+b_1)^2 + b_1^2 - 2(d+b_1)\ b_1\cos(\theta_1)$$
$$s_{22}^2 = (x_2-x_2')^2 + (y_2-y_2')^2 = 2b_2^2(\ 1+\cos(\theta_1))$$ seem OK

Inserting these, we get

$$V(z) = q(z)\frac{1}{4\pi^2\xi}\ \{\ \int_0^{2\pi} d\theta'_1\ \ln(s_{21}/s_{11})|^{r1'=b1} - \int_0^{2\pi} d\theta'_2\ \ln(s_{22}/s_{12})|^{r2'=b2}\ \}$$

$$= q(z)\frac{1}{4\pi^2\xi}\ (1/2)\ *\ (T1 + T2 + T3 + T4)$$

$$T1 = + \int_0^{2\pi} d\theta'_1\ \ln(s_{21}^2) = \int_0^{2\pi} d\theta'_1\ \ln\ [(d+b_1)^2 + b_1^2 - 2(d+b_1)\ b_1\cos(\theta_1')]$$

$$T2 = - \int_0^{2\pi} d\theta'_1\ \ln(s_{11}^2) = - \int_0^{2\pi} d\theta'_1\ \ln\ [2b_1^2\ (\ 1-\cos(\theta_1'))]$$

$$T3 = - \int_0^{2\pi} d\theta'_2\ \ln(s_{22}^2) = - \int_0^{2\pi} d\theta'_2\ \ln\ [2b_2^2(\ 1+\cos(\theta_2'))\ ]$$

$$T4 = + \int_0^{2\pi} d\theta'_2\ \ln(s_{12}^2) = \int_0^{2\pi} d\theta'_2\ \ln[(d+b_2)^2 + b_2^2 + 2(d+b_2)\ b_2\cos(\theta_2')]$$

This seems more complicated than the example I did in lines doc. Well, I allow arbitrary b_1, b_2 and d in this example, whereas I did not in the text. now I need to know this integral

$$\int_0^{2\pi} dx\ \ln\ (A + B\cos x) = 2\pi\ \ln[(1/2)(A + \sqrt{A^2-B^2}\ )]$$ following from GF7 p531

$$9.^8 \qquad \int_0^{\pi} \ln\,(a + b\cos x)\ dx = \pi \ln \frac{a + \sqrt{a^2 - b^2}}{2} \qquad [a \geq |b| > 0] \qquad \text{GW (322)(15)}$$

Let's assume the condition is met for all four terms, check this later. Then

T1: $A = (d+b_1)^2 + b_1^2$ $B = -2(d+b_1)\, b_1$

$A^2-B^2 = [(d+b_1)^2 + b_1^2]^2 - 4(d+b_1)^2\, b_1^2$

$= (d+b_1)^4 + b_1^4 - 2(d+b_1)^2\, b_1^2 = [(d+b_1)^2 - b_1^2]^2$

Then

$T1 = 2\pi\ln[(1/2)((d+b_1)^2 + b_1^2) + (d+b_1)^2 - b_1^2)] = 2\pi \ln[(d+b_1)^2] = 4\pi \ln(d+b_1)$

T2: $A = 2b_1^2$ $B = -2b_1^2$

$A^2-B^2 = 0$

Then

$T2 = -2\pi\ln[b_1^2] = -4\pi \ln(b_1)$

T3: $A = 2b_2^2$ $B = 2b_2^2$

$A^2-B^2 = 0$

Then

$T3 = -2\pi \ln[b_2^2] = -4\pi \ln(b_2)$

T4: $A = (d+b_2)^2 + b_2^2$ $B = 2(d+b_2)\, b_2$

$T4 = +4\pi \ln(d+b_2)$

To summarize:

$T1 = 4\pi \ln(d+b_1)$
$T2 = -4\pi \ln(b_1)$
$T3 = -4\pi \ln(b_2)$
$T4 = 4\pi \ln(d+b_2)$

Then we get

$$V(z) = q(z)\frac{1}{4\pi^2\xi}(1/2) * (T1 + T2 + T3 + T4)$$

$$= q(z)\frac{1}{\pi\xi}(1/2)* (\ln(d+b_1) + \ln(d+b_2) - \ln(b_1) - \ln(b_2)$$

$$= q(z)\,\frac{1}{2\pi\xi}\,\ln\left[\frac{(d+b_1)\,(d+b_2)}{b_1\,b_2}\right]$$ d = inner touch distance b_i = radii

For very large d this says

$$V(z) = q(z)\,\frac{1}{2\pi\xi}\,\ln\left[\frac{d^2}{b_1\,b_2}\right]$$

If we change to diameters instead of radius this becomes

$$V(z) = q(z)\,\frac{1}{2\pi\xi}\,\ln\left[\frac{2d^2}{d_1\,d_2}\right]$$

The capacitance is then

$$1/C = \frac{1}{2\pi\varepsilon}\,\ln\left[\frac{2d^2}{d_1\,d_2}\right]$$

How can I verify this result? It says that

$$C = 2\pi\varepsilon/\ln\left[\frac{2d^2}{d_1\,d_2}\right]$$

Suppose we had equal radii. Then we would have

$$C = 2\pi\varepsilon/\ln\left[\frac{d^2}{b_1\,b_2}\right] = 2\pi\varepsilon/\ln\left[\frac{d^2}{b_1^2}\right] = \pi\varepsilon/\ln(d/b)$$

and this does agree with King p 17 (30b). So at least I have a verification for equal radii which means my overall constant is right.

Now go back to my more general result:

$$V(z) = q(z)\,\frac{1}{2\pi\xi}\,\ln\left[\frac{(d+b_1)\,(d+b_2)}{b_1\,b_2}\right]$$

Let's change symbols on my result by using

B = distance between centers $= d + b_1 + b_2$

$d + b_1 = B - b_2$ $\qquad$ $d + b_2 = B - b_1$

Then my result becomes

$$V(z) = q(z)\,\frac{1}{2\pi\xi}\,\ln\left[\frac{(B-b_2)\,(B-b_1)}{b_1\,b_2}\right]$$

The capacitance is then

$$1/c = \frac{1}{2\pi\varepsilon} \ln\left[\frac{(B-b_2)(B-b_1)}{b_1 b_2}\right]$$

$$c = 2\pi\varepsilon \; 1/\ln\left[\frac{(B-b_2)(B-b_1)}{b_1 b_2}\right]$$

This agrees with King p 28 if I can show that:

$$(ch^{-1}\psi_1 + ch^{-1}\psi_2) = \ln\left[\frac{(B-b_2)(B-b_1)}{b_1 b_2}\right]$$

He has, converted to my notation,

$$\psi_1 = (B^2+b_2^2-b_1^2)/(2Bb_1)$$
$$\psi_2 = (B^2+b_1^2-b_2^2)/(2Bb_2)$$

Let's try some numbers

$B = 9 \qquad b_1 = 2 \qquad b_2 = 3 \qquad d = 4$

```
evalf(arccosh(psi1) + arccosh(psi2));
                                    2.391506684

evalf(ln((B-b2)*(B-b1)/(b1*b2)));
                                    1.945910149
```

Result: My result disagrees with King! No need to go any further. A third source is that I do this problem myself later in lines where I find that

$$K = f(\mathbf{x}_1) - f(\mathbf{x}_2) = B_1 - B_2 = \ln\left[(d/a_1) + \sqrt{(d/a_1)^2 + 1}\right] + \ln\left[(d/a_2) + \sqrt{(d/a_2)^2 + 1}\right]$$

where a_i are the radii and 2d is the distance between the foci. My drawing is very unclear. I am doing the potential 2D method, so my result should be same as King's. My b is distance between centers. I claim that

$$d = (1/2b)\sqrt{b^4 + (a_1^2 - a_2^2)^2 - 2b^2(a_1^2 + a_2^2)} \qquad (6.3.7)$$

But things then look pretty messy for the above K. But my result has no verification, so it is just another floating calculation, not useful to me right now!

Matick has a result on page 317 for two equal radii with S as the center separation. So let's compare the three results for that case:

King: $\psi_1 = \psi_2 = B/2b$ so then

$$[\ \psi_1 + \sqrt{\psi_1^2-1}\ ]\ [\ \psi_2 + \sqrt{\psi_2^2-1}\ ]\ = [\ \psi_1 + \sqrt{\psi_1^2-1}\ ]^2$$

$$= \psi_1^2 + \psi_1^2 - 1 + 2\ \psi_1\sqrt{\psi_1^2-1}\ = 2\psi_1^2 - 1 + 2\ \psi_1\sqrt{\psi_1^2-1}$$

$$= 2(B/2b)^2 - 1 + 2(B/2b)\ \sqrt{(B/2b)^2-1}$$

So King's result is then related to

$$\ln\ [2(B/2b)^2 - 1 + 2(B/2b)\ \sqrt{(B/2b)^2-1}\ ]$$

Me: $$\ln\ [\frac{(B-b_2)\ (B - b_1)}{b_1\ b_2}\ ]\ = \ln\ [\frac{(B-b)^2}{b^2}\ ]\ \ =2\ \ln\ [\frac{(B-b)}{b}\ ]$$

Matick: $$\ln \frac{B+\sqrt{B^2-4b^2}}{2b}$$

So we have three *different* results! I will put in my numbers B = 9 and b = 2, and this shows that King and Matick I think agree, because I find numerically that

$$(\frac{B+\sqrt{B^2-4b^2}}{2b}\ )^2\ = 2(B/2b)^2 - 1 + 2(B/2b)\ \sqrt{(B/2b)^2-1}$$

which is the same analytically as

$$[(B/2b) + \sqrt{(B/2b)^2-1}\ ]^2\ =\ 2(B/2b)^2 - 1 + 2(B/2b)\ \sqrt{(B/2b)^2-1}$$

I predict LHS = RHS. Eval

$$\text{LHS} = (B/2b)^2\ \ +\ \ (B/2b)^2 - 1 + 2(B/2b)\ \sqrt{(B/2b)^2-1}$$

$$= 2\ (B/2b)^2\ \ - 1 + 2(B/2b)\ \sqrt{(B/2b)^2-1}$$ // yes

So I think King and Mat are compatible. It is MY result that is clearly wrong, it is "too simple".

Then

$$(ch^{-1}\psi_1 + ch^{-1}\psi_2)\ =\ \ln\ [\ \psi_1 + \sqrt{\psi_1^2-1}\ ]\ +\ \ln\ [\ \psi_2 + \sqrt{\psi_2^2-1}\ ]$$

$$= \ln\ \{\ [\ \psi_1 + \sqrt{\psi_1^2-1}\ ]\ [\ \psi_2 + \sqrt{\psi_2^2-1}\ ]\ \}$$

So what I want to show is this:

$$\ln \{ [\psi_1 + \sqrt{\psi_1^2 - 1}] [\psi_2 + \sqrt{\psi_2^2 - 1}] \} = \ln [\frac{(B - b_2)(B - b_1)}{b_1 b_2}]$$

which is to say, I want to show that

$$[\psi_1 + \sqrt{\psi_1^2 - 1}] [\psi_2 + \sqrt{\psi_2^2 - 1}] = \frac{(B - b_2)(B - b_1)}{b_1 b_2}$$

But numerically I have already shown above that this does not work! MY result after my huge calculation is wrong somehow. There will be many things to check, Manana!

The capacitance result is not stated in my Radio Eng book, but page 592 shows a result for Z which has some relevance. Meanwhile, King does this problem and gets a result on page 28 which indicates

$$1/c = (1/2\pi\varepsilon) [\cosh^{-1} \frac{D^2 - b_2^2 + b_1^2}{2 b_1 D} + \cosh^{-1} \frac{D^2 - b_1^2 + b_2^2}{2 b_2 D}]$$

where D = distance between centers $= d + b_1 + b_2$.

Theorem: $\cosh^{-1} x = \ln[x + \sqrt{x^2 - 1}\,] \equiv \ln(y)$

Proof: Show that

$$\cosh(\ln[x + \sqrt{x^2 - 1}\,) = x$$

Evaluate the LHS :

$$\cosh(\ln[x + \sqrt{x^2 - 1}\,) = \cosh(\ln(y)) = (1/2) [e^{\ln y} + e^{-\ln y}] = (1/2) [y + 1/y]$$

$$= (1/2) [x + \sqrt{x^2 - 1} + x - \sqrt{x^2 - 1}\,] = x \quad \text{QED}$$

This is Spiegel 8.56, duh huh

Now I just showed the following on scratch paper:

$$y = \text{ch}^{-1} \psi_1 + \text{ch}^{-1} \psi_2$$

$$\Rightarrow$$

$$y = \ln [\psi_1 \psi_2 + \psi_2 \sqrt{\psi_1^2 - 1} + \psi_1 \sqrt{\psi_2^2 - 1} + \sqrt{\psi_1^2 - 1}\, \sqrt{\psi_2^2 - 1}]$$

so at least I can make some connection to logs. Just out of curiosity,

$$\psi_1 = \frac{D^2 - b_2^2 + b_1^2}{2 b_1 D}$$

$$\psi_1^2 - 1 = [(D^2-b_2^2+b_1^2)^2 - (2b_1D)^2] / (2b_1D)^2$$

Consider

$$(x^2-y^2+z^2)^2 - 4\,z^2x^2$$

```
f := (x^2-y^2+z^2)^2 - 4*z^2*x^2;
```

$$f := (x^2-y^2+z^2)^2 - 4z^2x^2$$

```
expand(%);
```

$$x^4-2x^2y^2-2z^2x^2+y^4-2y^2z^2+z^4$$

```
factor(%);
```

$$(x+y-z)(y+x+z)(x-y+z)(x-y-z)$$

Then

$$(x^2-y^2+z^2)^2 - 4\,z^2x^2 = [\,(x+y)^2 - z^2]\,[\,(x-y)^2 - z^2]$$

Thus suggests that we get this not-very-simple expression,

$$\psi_1^2 - 1 = [\,(D+b_2)^2 - b_1^2]\,[\,(D-b_2)^2 - b_1^2] / (2b_1D)^2$$

$$\sqrt{\psi_1^2-1} = \sqrt{(D+b_2)^2 - b_1^2}\,\sqrt{(D-b_2)^2 - b_1^2} / (2b_1D)$$

Then I guess

$$\sqrt{\psi_2^2-1} = \sqrt{(D+b_1)^2 - b_2^2}\,\sqrt{(D-b_1)^2 - b_2^2} / (2b_2D)$$

$$\psi_1\psi_2 + \psi_2\sqrt{\psi_1^2-1} + \psi_1\sqrt{\psi_2^2-1} + \sqrt{\psi_1^2-1}\,\sqrt{\psi_2^2-1}$$

$$= (\psi_1+\sqrt{\psi_1^2-1}\,)(\,\psi_2+\sqrt{\psi_2^2-1}\,)$$

$$\psi_1+\sqrt{\psi_1^2-1} = \frac{D^2-b_2^2+b_1^2}{2b_1D} + \sqrt{(D+b_2)^2 - b_1^2}\,\sqrt{(D-b_2)^2 - b_1^2} / (2b_1D)$$

$$= [\,D^2-b_2^2+b_1^2 + \sqrt{(D+b_2)^2 - b_1^2}\,\sqrt{(D-b_2)^2 - b_1^2}\,] / (2b_1D)$$

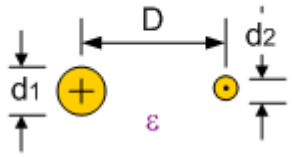
Does this simplify?

Web Scan. It is not easy to find what I want. I search on "unequal diameters" and get a hit

Two Conductors in Parallel (Balanced)
of Unequal Diameters

$$Z_0 = (60/\varepsilon^{1/2}) \cosh^{-1}(N)$$

$$N = \tfrac{1}{2}[(4D^2/d_1d_2) - (d_1/d_2) - (d_2/d_1)]$$



http://www.rfcafe.com/references/electrical/transmission-lines.htm

This seems different from King's result.

Why don't I say anything about "impedance" of a line? I just added some notes above the start of lines 4.6 and concluded there that at large ω

$$Z_0 = \sqrt{\mu\varepsilon}\ 1/C$$

For my two round wire line with large spacing I get

$$C = \pi\varepsilon / \ln(b/\sqrt{a_1a_2}) \qquad (4.6.5)$$

That would imply that

$$Z_0 = (1/\pi\varepsilon)\sqrt{\mu\varepsilon}\ \ln(b/\sqrt{a_1a_2}) = (1/\pi)\sqrt{\mu/\varepsilon}\ \ln(b/\sqrt{a_1a_2})$$

Now assume $\mu = \mu_0$ and $\varepsilon = (\varepsilon/\varepsilon_0)\varepsilon_0$ so then

$$Z_0 = (1/\pi)\sqrt{\frac{\mu}{\varepsilon}}\ \ln(b/\sqrt{a_1a_2}) = (1/\pi)\sqrt{\frac{\mu_0}{(\varepsilon/\varepsilon_0)\varepsilon_0}}\ \ln(b/\sqrt{a_1a_2})$$

$$= (1/\pi)\sqrt{\frac{\mu_0}{\varepsilon_0}}\ \frac{1}{\sqrt{\varepsilon/\varepsilon_0}}\ \ln(b/\sqrt{a_1a_2}) = (1/\pi)\ 376.73032 \text{ ohms} * \frac{1}{\sqrt{\varepsilon/\varepsilon_0}}\ \ln(b/\sqrt{a_1a_2})$$

$$= 119.9169852\ \frac{1}{\sqrt{\varepsilon/\varepsilon_0}}\ \ln(b/\sqrt{a_1a_2})$$

$$= 119.9169852\ \frac{1}{\sqrt{\varepsilon/\varepsilon_0}}\ \ln(2b/\sqrt{d_1d_2})$$

$$\approx 120\ \frac{1}{\sqrt{\varepsilon/\varepsilon_0}}\ \ln(2b/\sqrt{d_1d_2}) \qquad (**)$$

Now

$y = \ln x \qquad e^{y} = e^{\ln x} = x$

$z = \log x \qquad 10^{z} = 10^{\log x} = x \qquad 10^{z} = e^{y}$

$z = \log(e^{y}) = y\log(e)$

$y = \ln(e^{y}) = \ln(10^{z}) = z \ln(10)$

$\log x = \log(e)\ln x$

$\ln x = \ln(10)\log x$

$\log(e) = 0.434294482$

$\ln(10) = 2.302585093$

These are inverses of each other. It is hard to find this simple fact on the web!

$\ln x = 2.302585093 \log x$ // write this down somewhere for next time.

Continuing the above

$$Z_0 = (1/\pi)^* 376.73032 \text{ ohms} * \frac{1}{\sqrt{\varepsilon/\varepsilon_0}} 2.302585093 \log_{10}\left(b/\sqrt{a_1 a_2} \right)$$

```
evalf((1/Pi)*376.73032*2.302585093);
                                  276.1190626
```

$$= 276.1190626 \frac{1}{\sqrt{\varepsilon/\varepsilon_0}} \log_{10}\left(b/\sqrt{a_1 a_2} \right)$$

Here a_i are radius values. Write $a_1 = d_1/2$ to get

$$\sqrt{a_1 a_2} = \sqrt{d_1 d_2}\ /2$$

and then I get

$$Z_0 = 276.1190626 \text{ ohms} \frac{1}{\sqrt{\varepsilon/\varepsilon_0}} \log_{10}\left(2b/\sqrt{d_1 d_2} \right)$$

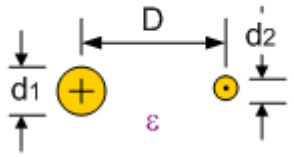
This agrees with radio book page 592! Finally something agrees with something! That book clearly states that it is assuming b is >> the radii.

Now how does this fit into the picture,

Two Conductors in Parallel (Balanced) of Unequal Diameters

$$Z_0 = (60/\varepsilon^{1/2}) \cosh^{-1}(N)$$

$$N = \tfrac{1}{2}[(4D^2/d_1d_2) - (d_1/d_2) - (d_2/d_1)]$$



For large D, the claim is that $N = 2D^2/d_1d_2$ and then they claim that

$$Z_0 = 60 \frac{1}{\sqrt{\varepsilon/\varepsilon_0}} \cosh^{-1}(2D^2/d_1d_2)$$

Now consider

$y = \cosh^{-1}(x)$ $\quad$ $x = \cosh y$ $\quad$ x = large and y = large

$x = (1/2)[\, e^y + e^{-y}] \approx (1/2)e^y$

$\ln(2x) = y$ $\quad$ $y = \ln(2x)$

So their result becomes

$$Z_0 = 60 \frac{1}{\sqrt{\varepsilon/\varepsilon_0}} \ln(4D^2/d_1d_2) = 120 \frac{1}{\sqrt{\varepsilon/\varepsilon_0}} \ln(\, 2D/\sqrt{d_1d_2}\,)$$

This agrees with my (**) above, so now we have two agreements.