Archive off old Section 4.11 (retired on 9/23/14) PhL 9.23.14

This is just saving the original section 4.11, nothing to review here.

4.11 The Classical Transmission Line Equations

The results of the previous sections of this chapter may be succinctly summarized as:

(4.11.1)

$$\frac{1}{C'} = \frac{V(z)}{q(z)} = \frac{1}{4\pi\xi_d} K \qquad (4.4.7)$$

$$L_e = \frac{W(z)}{i(z)} = \frac{\mu_d}{4\pi} K_L \qquad (4.10.8)$$

$$K \equiv \int_{C_1} dx_1' \, dy_1' \, \alpha_1(x_1',y_1') \ln(s_{21}^2/s_{11}^2) - \int_{C_2} dx_2' \, dy_2' \, \alpha_2(x_2',y_2') \ln(s_{22}^2/s_{12}^2) \qquad (4.4.8)$$

$$K_L \equiv \int_{C_1} dx_1' \, dy_1' \, b_1(x_1',y_1') \ln(s_{21}^2/s_{11}^2) - \int_{C_2} dx_2' \, dy_2' \, b_2(x_2',y_2') \ln(s_{22}^2/s_{12}^2) \qquad (4.10.9)$$

Notice that we have made no assumptions whatsoever about the cross-sectional shape of the transmission line. We have only assumed that the transverse dimensions are small compared to the wavelength λ that corresponds to β -- this was the transmission line limit.

(a) Initial Processing

There are several equations from Chapter 1 we shall now press into service:

$$\mathbf{E} = -\text{ grad } \varphi - \partial_t \mathbf{A} \qquad (1.3.1)$$

$$\text{div } \mathbf{A} = -\mu_d \varepsilon_d \partial_t \varphi - \mu\sigma\varphi \ . \qquad \text{// the King gauge} \qquad (1.3.18)$$

In the frequency domain these become,

$$\mathbf{E} = -\text{ grad } \varphi - j\omega\mathbf{A}$$

$$\text{div } \mathbf{A} = -\, j\,(\beta_d^2/\omega)\,\varphi \ . \qquad \text{// the King gauge, see (1.5.1a) and (1.5.5) re } \beta_d^2 \qquad (4.11.2)$$

According to the Fact stated in (4.7.1), potential **A** has only component A_z, so these equations become

$$E_z(\mathbf{x}) = -\,\partial_z \varphi(\mathbf{x}) - j\omega A_z(\mathbf{x})$$

$$\partial_z A_z(\mathbf{x}) = -\, j\,(\beta_d^2/\omega)\,\varphi(\mathbf{x}) \ . \qquad (4.11.3)$$

However, as was shown at the end of Step 1 below (3.7.8), the second line of (4.11.3) can only be justified in the strong or extreme skin effect regimes, and we continue then to assume our transmission line is operating at sufficiently high ω to be in the small δ regime.

The potentials in the above equations are those due to both conductors and were denoted as φ_{12} and A_{z12} in the previous sections. We then rewrite the above as

$$E_z(\mathbf{x}) = -\partial_z\varphi_{12}(\mathbf{x}) - j\omega A_{z12}(\mathbf{x}) \qquad (4.11.4a)$$
$$\partial_z A_{z12}(\mathbf{x}) = -j\,(\beta_d^2/\omega)\varphi_{12}(\mathbf{x})\ . \qquad (4.11.4b)$$

Recall now the conductor-surface-located points $\mathbf{x}_1$ and $\mathbf{x}_2$ as shown for example in Fig 4.10. If we evaluate each of the above equations at $\mathbf{x} = \mathbf{x}_1$ and then $\mathbf{x} = \mathbf{x}_2$ and then subtract, we get

$$E_z(\mathbf{x}_1) - E_z(\mathbf{x}_2) = -\partial_z[\varphi_{12}(\mathbf{x}_1) - \varphi_{12}(\mathbf{x}_2)] - j\omega[A_{z12}(\mathbf{x}_1) - A_{z12}(\mathbf{x}_2)] \qquad (4.11.5a)$$
$$\partial_z[A_{z12}(\mathbf{x}_1) - A_{z12}(\mathbf{x}_2)] = -j\,(\beta^2/\omega)[\varphi_{12}(\mathbf{x}_1) - \varphi_{12}(\mathbf{x}_2)]\ . \qquad (4.11.5b)$$

Then using these definitions (again, we are assuming the strong or extreme skin depth regime)

$$V(z) \equiv \varphi_{12}(\mathbf{x}_1) - \varphi_{12}(\mathbf{x}_2) \qquad \textit{(4.4.1)}$$
$$W(z) \equiv A_{z12}(\mathbf{x}_1) - A_{z12}(\mathbf{x}_2) \qquad \textit{(4.10.1)}$$

we may rewrite (4.11.5) in this simple manner,

$$E_z(\mathbf{x}_1) - E_z(\mathbf{x}_2) = -\partial_z V - j\omega W \qquad (4.11.6a)$$
$$\partial_z W = -j\,(\beta_d^2/\omega)\,V\ . \qquad (4.11.6b)$$

The quantity $E_z(\mathbf{x}_1)$ is the longitudinal electric field at point $\mathbf{x}_1$ on the surface of conductor C_1. It is related to the conductor's at-the-surface current density by $J_z(\mathbf{x}_1) = \sigma E_z(\mathbf{x}_1)$. If the conductor were "perfect", we would have $\sigma = \infty$ and $E_z(\mathbf{x}_1) = 0$, but real conductors are not perfect. However, since we are assuming the strong or extreme skin effect all along here in our analysis, we do know that $E_z(\mathbf{x}_1)$ and $E_z(\mathbf{x}_2)$ are very small.

(b) Averaging Repair and the Transmission Line Equations

Our theory now has an inconsistency which needs to be fixed.

We know that for a general transmission line operating at $\omega > 0$, the current density J_z inside the conductors will not be uniformly distributed. It will be larger in the conductor region closest to the other conductor. This "proximity effect" is discussed in Appendix P from an eddy current point of view, see Fig P.13 for an example. The J_z current non-uniformity can be very dramatic as for example in a transmission line having this cross section, where J_z will be large near the gap and small far from the gap:



Fig 4.12

Since J_z is non-uniform in each conductor, so is E_z, and so we expect $E_z(\mathbf{x}_1)$ to be a strong function of the point $\mathbf{x}_1$ on the perimeter of C_1, certainly for the above cross section example. This means that the left side of (4.11.6a) is a function of $\mathbf{x}_1 = (x_1,y_1,z)$ and $\mathbf{x}_2 = (x_2,y_2,z)$ whereas the right side in our theory is a

function only of z. To remedy this inconsistency, we now have to think of V and W as having very slight dependence on $\mathbf{x}_1$ and $\mathbf{x}_2$ which we generally ignore, but which we must face up to in (4.11.6a). In reality we have $V(\mathbf{x}_1,\mathbf{x}_2)$ and $W(\mathbf{x}_1,\mathbf{x}_2)$. This is a manifestation of the fact that in reality $\varphi \approx$ constant and $A_z \approx$ constant on the boundaries (with $\approx$ and not =). In the extreme skin effect regime (think a very good conductor), the left side of (4.11.6a) can be a violent function of $\mathbf{x}_1$ and $\mathbf{x}_2$ as in the case of the above figure, but the left side is always very small, even where it is largest, and its variation can be accommodated by the right side of (4.11.6a) which is the difference of large-valued functions which vary only slightly with $\mathbf{x}_1$ and $\mathbf{x}_2$. So first rewrite (4.11.6a) as

$$E_z(\mathbf{x}_1) - E_z(\mathbf{x}_2) = -\partial_z V(\mathbf{x}_1,\mathbf{x}_2) - j\omega W(\mathbf{x}_1,\mathbf{x}_2) \ . \qquad (4.11.6a)'$$

Backing up another step, we write out of (4.11.4a) for the two perimeter points $\mathbf{x}_1$ and $\mathbf{x}_2$,

$$\begin{aligned} (1/\sigma)J_z(\mathbf{x}_1) = E_z(\mathbf{x}_1) &= -\partial_z\varphi_{12}(\mathbf{x}_1) - j\omega A_{z12}(\mathbf{x}_1) && \mathbf{x}_1 \text{ on perimeter of } C_1 \\ (1/\sigma)J_z(\mathbf{x}_2) = E_z(\mathbf{x}_2) &= -\partial_z\varphi_{12}(\mathbf{x}_2) - j\omega A_{z12}(\mathbf{x}_2) && \mathbf{x}_2 \text{ on perimeter of } C_2 \end{aligned} \qquad (4.11.4a)'$$

Calling the perimeter distances of the conductors P_1 and P_2, we then average each of these equations around its appropriate perimeter. Apply $(1/P_1)\int_{C1} ds_1$ to the first equation and $(1/P_2)\int_{C1} ds_2$ to the second to get [ds_1 is a distance element along the perimeter of C_1] ,

$$\begin{aligned} (1/\sigma)\langle J_z(\mathbf{x}_1)\rangle_{C1} &= \langle E_z(\mathbf{x}_1)\rangle_{C1} = -\partial_z\langle\varphi_{12}(\mathbf{x}_1)\rangle_{C1} - j\omega\langle A_{z12}(\mathbf{x}_1)\rangle_{C1} \\ (1/\sigma)\langle J_z(\mathbf{x}_2)\rangle_{C2} &= \langle E_z(\mathbf{x}_2)\rangle_{C2} = -\partial_z\langle\varphi_{12}(\mathbf{x}_2)\rangle_{C2} - j\omega\langle A_{z12}(\mathbf{x}_2)\rangle_{C2} \ . \end{aligned}$$

Subtract the second line from the first to get,

$$\begin{aligned} &[\langle E_z(\mathbf{x}_1)\rangle_{C1} - \langle E_z(\mathbf{x}_2)\rangle_{C2}] \\ &\quad = -\partial_z[\langle\varphi_{12}(\mathbf{x}_1)\rangle_{C1} - \langle\varphi_{12}(\mathbf{x}_2)\rangle_{C2}] - j\omega\,[\langle A_{z12}(\mathbf{x}_1)\rangle_{C1} - \langle A_{z12}(\mathbf{x}_2)\rangle_{C2}] \ . \end{aligned}$$

We now *redefine* V and W to be the averages appearing in these equations, along with E_{z1} and E_{z2} :

$$E_{z1}(z) \equiv \langle E_z(\mathbf{x}_1)\rangle_{C1} = (1/P_1)\int_{C1} ds_1\, E_z(\mathbf{x}_1)$$

$$E_{z2}(z) \equiv \langle E_z(\mathbf{x}_2)\rangle_{C2} = (1/P_2)\int_{C2} ds_2\, E_z(\mathbf{x}_2)$$

$$\begin{aligned} V(z) &\equiv \langle\varphi_{12}(\mathbf{x}_1)\rangle_{C1} - \langle\varphi_{12}(\mathbf{x}_2)\rangle_{C2} = \langle V(\mathbf{x}_1,\mathbf{x}_2)\rangle_{C1,C2} \\ W(z) &\equiv \langle A_{z12}(\mathbf{x}_1)\rangle_{C1} - \langle A_{z12}(\mathbf{x}_2)\rangle_{C2} = \langle W(\mathbf{x}_1,\mathbf{x}_2)\rangle_{C1,C2} \end{aligned} \qquad (4.11.7)$$

with this result

$$[E_{z1}(z) - E_{z2}(z)] = -\partial_z V(z) - j\omega W(z) \ . \qquad (4.11.8)$$

Meanwhile, the surface impedances on C_1 and C_2 are defined by (see C.2.1) ,

$$E_{z1}(\mathbf{x}_1) = Z_{s1}(\mathbf{x}_1)\ i_1(z)$$
$$E_{z2}(\mathbf{x}_2) = Z_{s2}(\mathbf{x}_2)\ i_2(z) \qquad \textit{(C.2.1)}$$

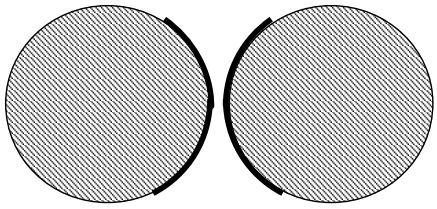
which we average in the same way to obtain

$$E_{z1}(z) = Z_{s1}\ i_1(z) \qquad Z_{s1} \equiv (1/P_1) \int_{C1} ds_1\ Z_{s1}(\mathbf{x}_1)$$
$$E_{z2}(z) = Z_{s2}\ i_2(z) \qquad Z_{s2} \equiv (1/P_2) \int_{C2} ds_2\ Z_{s2}(\mathbf{x}_2) \quad . \qquad (4.11.9)$$

There will be some location on C_1 where $E_{z1}(\mathbf{x}_1)$ and thus $Z_{s1}(\mathbf{x}_1)$ will be maximal (for example on the walls of the gap in Fig 4.12). Referring to this value as $Z_{s1,max}$ we can define

$$p_1 \equiv (Z_{s1}/Z_{s1,max})P_1$$
$$p_2 \equiv (Z_{s2}/Z_{s2,max})P_2 \qquad (4.11.10)$$

where p_1 is the effective length of the "active perimeter" of C_1. This then provides a crude model for the symbol p which appears in (2.5.1) and Fig 2.16 which we replicate here,



Fat twinlead *Fig 2.16*

Then using $i(z) = i_1(z) = -i_2(z)$ and (4.11.9), rewrite (4.11.8) and (4.11.6b) as

$$[Z_{s1} + Z_{s2}]\ i(z) = -\partial_z V(z) - j\omega W(z)$$
$$\partial_z W(z) = -j\ (\beta_d^2/\omega)\ V(z) \qquad (4.11.11)$$

where the second equation above is the $< >_{C1,C2}$ average of (4.11.6b).

Continuing this repair effort, we back up to box (4.11.1) and write

$$V(\mathbf{x}_1,\mathbf{x}_2) = q(z) / C'(\mathbf{x}_1,\mathbf{x}_2) = q(z) \left[\frac{1}{4\pi\xi_d}\ K(\mathbf{x}_1,\mathbf{x}_2) \right]$$
$$W(\mathbf{x}_1,\mathbf{x}_2) = i(z)\ L_e(\mathbf{x}_1,\mathbf{x}_2) = i(z) \left[\frac{\mu}{4\pi}\ K_L(\mathbf{x}_1,\mathbf{x}_2) \right] \qquad (4.11.12)$$

which we average in the same way to get

$$V(z) = q(z)\ \frac{1}{C'} \qquad W(z) = i(z)\ L_e$$

$$\frac{1}{C'} \equiv (1/P_1) \int_{c1} ds_1 \, (1/P_2) \int_{c2} ds_2 \, \frac{1}{C'(\mathbf{x_1},\mathbf{x_2})} = \left\langle \frac{1}{C'(\mathbf{x_1},\mathbf{x_2})} \right\rangle_{c1,c2}$$

$$L_e \equiv (1/P_1) \int_{c1} ds_1 \, (1/P_2) \int_{c2} ds_2 \, L_e(\mathbf{x_1},\mathbf{x_2}) = \langle L_e(\mathbf{x_1},\mathbf{x_2}) \rangle_{c1,c2} \; . \qquad (4.11.13)$$

The "constants" K and K_L in (4.11.1) are similarly replaced with their $\langle\rangle_{c1,c2}$ averages.

Inserting the equations on the first line of (4.11.13) into (4.11.11) we get

$$(Z_{s1} + Z_{s2})\, i(z) = - \partial_z V(z) - j\omega L_e \, i(z)$$
$$L_e \, \partial_z \, i(z) = - j \, (\beta_d^2/\omega) \, V(z)$$

which we then rearrange as

$$\partial_z V(z) = - [\, Z_{s1} + Z_{s1} + j\omega L_e] \, i(z)$$
$$\partial_z \, i(z) = - [\, j\beta_d^2/(\omega L_e)] \, V(z) \; . \qquad (4.11.14a)$$

These are the classical **transmission line equations**. They are usually written in this form: [$\partial/\partial z = d/dz$]

$$\frac{dV(z)}{dz} = - \mathbf{z} \, i(z) \qquad \frac{di(z)}{dz} = - \mathbf{y} \, V(z)$$

with

$$\mathbf{z} = R + j\omega L \qquad \mathbf{y} = G + j\omega C \; . \qquad (4.11.14b)$$

Note: We have been using bold notation only for vectors, and we now break that guideline by bolding these complex quantities **z** and **y**. Our purpose for this bolding is to distinguish them from Cartesian coordinates z and y which typically appear in the same problem. In King's books, *all* complex parameters are put in bold font, but we do this only for **z** and **y**.

If one applies ∂_z to either of the above equations and then uses the other, one obtains the corresponding wave equations (but in the ω domain, so Helmholtz equations),

$$\frac{d^2V(z)}{dz^2} - \mathbf{zy} \, V(z) = 0 \qquad \frac{d^2i(z)}{dz^2} - \mathbf{zy} \, i(z) = 0 \; . \qquad (4.11.15)$$

In the discussion below, we shall no longer mention the averaging process, but it should be understood that for closely spaced conductors the symbols Z_{s1}, Z_{s1}, L_e, C', K, K_L, V, W are the perimeter-averaged values discussed above. For widely spaced conductors, J_z is roughly uniform over the conductor cross sections and perimeters and the averaging process is not needed.

Jumping the gun a bit, if we assume now a traveling-wave z dependence $e^{j(\omega t - kz)}$ for both V(z) and i(z), where k is the wave's (possibly complex) wavenumber, then $\partial_z \rightarrow -jk$ and the transmission line equations become

$$-jk \, V(z) = - \mathbf{z} \, i(z) \qquad -jk \, i(z) = - \mathbf{y} \, V(z)$$

or

$-jk = -\,\mathbf{z}\, i(z)/V(z)$ $\qquad$ $-jk = -\,\mathbf{y}\, V(z)/i(z)$.

Equating these last two expressions gives

$$-\,\mathbf{z}\, i(z)/V(z) = -\,\mathbf{y}\, V(z)/i(z) \quad \Rightarrow \quad \mathbf{z}/\mathbf{y} = [V(z)/i(z)]^2$$

and we then have,

$$Z_0 \equiv V(z)/i(z) = \sqrt{\frac{\mathbf{z}}{\mathbf{y}}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \qquad (4.11.16)$$

where by definition Z_0 is the **characteristic impedance** of the transmission line.

The quantities **z** and **y** are called the transmission line **impedance** and **admittance**, and the four numbers R,L,G,C are defined to be the appropriate real and imaginary parts. Comparing (4.11.14a) and (4.11.14b), we may therefore conclude that:

$$\mathbf{z} = R + j\omega L = Z_{s1} + Z_{s2} + j\omega L_e \qquad (4.11.17)$$

$$\mathbf{y} = G + j\omega C = j\beta_d^2/(\omega L_e) \qquad (4.11.18)$$

where Z_{s1}, Z_{s2} and L_e are the averages shown in (4.11.9) and (4.11.13). The expression for **z** seems quite reasonable since $\omega L_e = X_L$ = inductive reactance, but the expression for **y** seems a bit unusual. This is because we still have more work to do.

There is one more equation we have not yet utilized. Recall from Chapter 1 the integral form of the equation of continuity, which in the frequency domain takes this form,

$$\text{div}\,\mathbf{J} = -\,j\omega\rho \quad \Leftrightarrow \quad -j\omega\left[\int_V \rho\, dV\right] = \int_S \mathbf{J} \bullet d\mathbf{S}\,. \qquad (1.1.35)$$

We now apply this to a Gaussian box (blue) whose faces have the same shape as the conductor cross section but are slightly larger than that cross section so as to include the conductor surface charge :

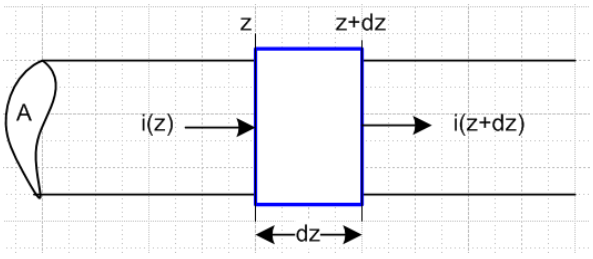


Fig 4.13

Ignoring transverse dielectric current out the radial sides of the box (since dz is tiny), we get

$-j\omega[q(z)dz]$ = i(z+dz) - i(z) = total current flowing out of the box

which then says

$$\partial_z\, i(z) = -j\omega q(z)\ . \qquad (4.11.19)$$

Jumping the gun again, if we again use $\partial_z \rightarrow -jk$ with $k = (\omega/v)$, we arrive at the intuitive relation

$$i(z) = q(z)\, v \qquad (4.11.19a)$$

which just says the charge per unit length is [in effect, see D.9 (c)] traveling down the line at phase velocity v. In the lossless case $v = v_d$ (dielectric speed of light), whereas more generally v is complex.

From summary box (4.11.1) recall that $q(z) = C'\, V(z)$ so we get from (4.11.19),

$$\partial_z\, i(z) = -\,[\, j\omega C'\,]\, V(z)\ . \qquad (4.11.20)$$

Comparing with the second equation of (4.11.14),

$$\partial_z\, i(z) = -\,[\, j\beta_d^2/(\omega L_e)\,]\ V(z)\ , \qquad \textit{(4.11.14)}$$

we get the following identity,

$$-\,[\, j\beta_d^2/(\omega L_e)\,] = -\,[\, j\omega C'\,]$$

or

$$L_e C' = \beta_d^2/\omega^2 = \mu_d \xi_d \qquad .\qquad \text{// see (1.5.1a) regarding } \beta_d^2 \qquad (4.11.21)$$

Then we can write (4.11.18) as

$$\mathbf{y} = G + j\omega C = j\beta_d^2/(\omega L_e) = j\,(\beta_d^2/\omega^2)\,(\omega/L_e) = j\,(L_e C')\,(\omega/L_e) = j\omega C'\ . \qquad (4.11.22)$$

Thus, the line capacitance C is the real part of complex capacitance, C = Re(C'), and G = - ω Im(C').

(c) Digression on the meaning of C'

Back in Section 1.5 (c) we discussed the fact that $n_c = (\xi_d/\varepsilon_d)\, n_s$ which relates actual surface charge n_s to the adjusted surface charge density n_c which allows for dielectric leakage. This relationship (1.5.17) was derived in two different ways. As noted in Comment 3 at the start of Section 4.1, and looking at (4.1.1) and (4.1.2), one sees that the linear charge density q(z) which appears in all our equations is in fact related to n_c and not n_s, so we temporarily shall refer to q(z) as $q_c(z)$. Then $q_c(z) = \int n_c\, dxdy$ = an integral over the conductor surface for length dz. The *actual* charge on the surface of this piece of conductor is $q_s(z) \equiv \int n_s\, dxdy$ and therefore $q_c/q_s = n_c/n_s = (\xi_d/\varepsilon_d)$. The capacitance C per unit length of our transmission line is defined by $q_s = C\, V(z)$. The complex capacitance C', which includes the effect of dielectric leakage current, is defined by $q_c = C'\, V(z)$. Therefore

$$C'/C = q_c/q_s = (\xi_d/\varepsilon_d) \quad . \tag{4.11.23}$$

and so then from (4.11.22), (4.11.23) and (1.5.1a),

$$\mathbf{y} = G + j\omega C = j\omega C' = j\omega(\xi_d/\varepsilon_d)C = j\omega\,(1 - j\sigma_d/\varepsilon_d\omega)C = j\omega C + (\sigma_d/\varepsilon_d)C \tag{4.11.24}$$

so that

$$G = (\sigma_d/\varepsilon_d)C \quad . \tag{4.11.25}$$

We saw an example of (4.11.23) in (1.5.19) for a parallel plate capacitor, and more generally in (4.4.10).

We may now rewrite the first equation in summary box (4.11.1) as

$$\frac{1}{C'} = \frac{1}{4\pi\xi_d}\,K \qquad \Rightarrow \qquad \frac{1}{C} = \frac{1}{4\pi\varepsilon_d}\,K \quad . \tag{4.11.26}$$

Next, combining (4.11.21) and (4.11.23) we find that

$$L_e C' = \mu_d \xi_d \tag{4.11.27}$$

$$L_e C = \mu_d \varepsilon_d = 1/v_d^2 \tag{4.11.28}$$

where v_d is the speed of light in the dielectric. Now the second equation in (4.11.1) says that

$$L_e = \frac{\mu_d}{4\pi}\,K_L \, . \tag{4.11.29}$$

Inserting (4.11.29) for L_e and (4.11.26) for C' into (4.11.27) gives

$$\left(\frac{\mu_d}{4\pi}\,K_L\right)(4\pi\xi_d/K) = \mu_d\xi_d$$

or

$$K_L = K \, . \tag{4.11.30}$$

This is a remarkable connection between our two seemingly unrelated constants K and K_L,

$$K \equiv \int_{C_1} dx_1'\, dy_1'\, \alpha_1(x_1',y_1')\, \ln(s_{21}^2/s_{11}^2) - \int_{C_2} dx_2'\, dy_2'\, \alpha_2(x_2',y_2')\, \ln(s_{22}^2/s_{12}^2) \qquad \textit{(4.4.8)}$$

$$K_L \equiv \int_{C_1} dx_1'\, dy_1'\, b_1(x_1',y_1')\, \ln(s_{21}^2/s_{11}^2) - \int_{C_2} dx_2'\, dy_2'\, b_2(x_2',y_2')\, \ln(s_{22}^2/s_{12}^2) \quad . \qquad \textit{(4.10.9)}$$

Since K involves a peripheral line integral of surface charge densities α_i whereas K_L involves a full cross sectional area integral of the current densities b_i, it seems unlikely these integrals would be equal, but they are equal.

(d) An example of $K = K_L$

The equality even seems unlikely in a case with symmetric densities on round wires, so let's do a check using our Section 4.5 example with widely-spaced round wires of unequal diameters. The first thing we need is a new picture to display the "kinematics" of the K_L integral (since densities are symmetric, one should regard this picture as having b much larger than shown relative to a_1 and a_2),

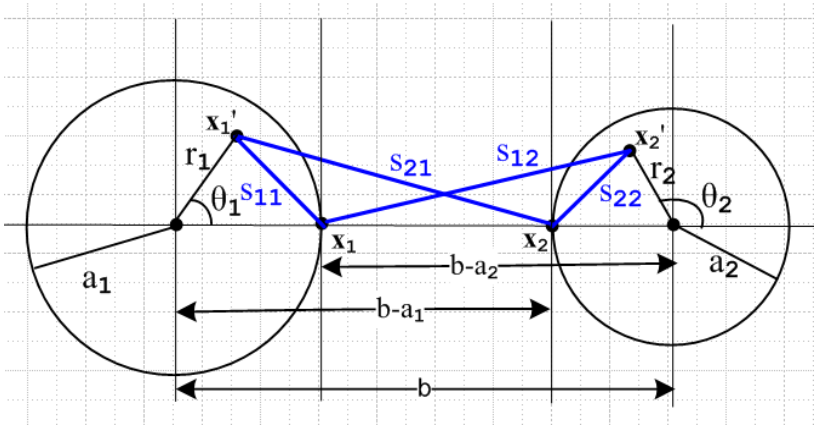


Fig 4.14

As before, we read off the four distances of interest using the law of cosines. The new distances are all different than they were before since $\mathbf{x}_1'$ and $\mathbf{x}_2'$ are now each integrated over their respective disks instead of the bounding circles.

$$s_{21}^2 = r_1^2 + (b-a_1)^2 - 2\, r_1(b-a_1)\cos(\theta_1)$$
$$s_{11}^2 = r_1^2 + a_1^2 - 2\, r_1\, a_1 \cos(\theta_1)$$
$$s_{22}^2 = r_2^2 + a_2^2 + 2\, r_2\, a_2 \cos(\theta_2)$$
$$s_{12}^2 = r_2^2 + (b-a_2)^2 + 2\, r_2(b-a_2)\cos(\theta_2)\ .$$

The integration rule is still

$$\int_0^{2\pi} d\theta \ln(A \pm B\cos\theta) = 2\pi \ln[(1/2)(A + \sqrt{A^2-B^2})] \ . \qquad (4.5.5)$$

The first integral is:

$$\int_0^{2\pi} d\theta_1 \ln(s_{21}^2) = \int_0^{2\pi} d\theta_1 \ln([r_1^2 + (b-a_1)^2 - 2\, r_1(b-a_1)\cos(\theta_1)]$$

$$A = r_1^2 + (b-a_1)^2 \qquad B = 2\, r_1(b-a_1)$$

$$A^2-B^2 = [r_1^2 + (b-a_1)^2]^2 - 4\, r_1^2(b-a_1)^2 = [r_1^2 - (b-a_1)^2]^2 \Rightarrow \sqrt{A^2-B^2} = (b-a_1)^2 - r_1^2 > 0 \qquad b >> a_1$$

$$\Rightarrow \quad \int_0^{2\pi} d\theta_1 \ln(s_{21}^2) = 2\pi \ln[(1/2)(r_1^2 + (b-a_1)^2 + (b-a_1)^2 - r_1^2) = 2\pi \ln[(b-a_1)^2]$$

But this integral is the same as before! The s_{11}^2 integral is obtained from the above with $b-a_1 \rightarrow a_1$

$$\int_0^{2\pi} d\theta_1 \ln(s_{11}^2) = 2\pi \ln(a_1^2)$$

which is also the same as before. The other two integrals are found from $1 \rightarrow 2$. Our integral summary is then exactly the same as (4.5.6),

$$\int_0^{2\pi} d\theta_1 \ln(s_{21}^2) = 2\pi \ln[(b-a_1)^2]$$

$$\int_0^{2\pi} d\theta_1 \ln(s_{11}^2) = 2\pi \ln(a_1^2)$$

$$\int_0^{2\pi} d\theta_2 \ln(s_{22}^2) = 2\pi \ln(a_2^2)$$

$$\int_0^{2\pi} d\theta_2 \ln(s_{12}^2) = 2\pi \ln[(b-a_2)^2] \quad . \qquad (4.5.6)$$

We now assume that the current densities b_i each have radial symmetry ("widely spaced wires") ,

$$b_1(r_1,\theta_1) = b_1(r_1) \qquad (4.11.31)$$

where $b_1(r_1)$ is a completely arbitrary function, with the following normalization of (4.7.4),

$$\int_0^{2\pi} d\theta_1 \int_0^{a_1} r_1 dr_1 \, b_1(r_1) = 1 \qquad \Rightarrow \qquad \int_0^{a_1} r_1 dr_1 \, b_1(r_1) = 1/2\pi \quad . \qquad (4.11.32)$$

We now proceed to calculate the constant K_L

$$K_L = \int_0^{2\pi} d\theta_1 \int_0^{a_1} r_1 dr_1 \, b_1(r_1) \ln(s_{21}^2/s_{11}^2) - \int_0^{2\pi} d\theta_2 \quad r_2 dr_2 \, b_2(r_2) \ln(s_{22}^2/s_{12}^2)$$

$$= \int_0^{a_1} r_1 dr_1 \, b_1(r_1) \int_0^{2\pi} d\theta_1 \ln(s_{21}^2/s_{11}^2) - \int_0^{a_2} r_2 dr_2 b_2(r_2) \int_0^{2\pi} d\theta_2 \ln(s_{22}^2/s_{12}^2)$$

$$= 2\pi \int_0^{a_1} r_1 dr_1 \, b_1(r_1) [\ln[(b-a_1)^2] - \ln(a_1^2)] - \int_0^{a_2} r_2 dr_2 b_2(r_2) [\ln[(b-a_2)^2] - \ln(a_2^2)]$$

$$= 2\pi [\ln[(b-a_1)^2/a_1^2] \int_0^{a_1} r_1 dr_1 \, b_1(r_1) - 2\pi [\ln[(b-a_2)^2/a_2^2] \int_0^{a_2} r_2 dr_2 \, b_2(r_2)$$

$$= [\ln[(b-a_1)^2/a_1^2] - [\ln[(b-a_2)^2/a_2^2]$$

$$= \ln \left[\frac{(b-a_1)^2 (b-a_2)^2}{a_1^2 a_2^2} \right]$$

$$= K \qquad \text{as obtained in (4.5.7)} \tag{4.11.33}$$

and we have then shown $K_L = K$ for this particular example. The key fact is that the $d\theta$ integrals *appear* to be functions of r_i , but the r_i^2 terms cancel and so the $d\theta$ integrals are independent of r_i.

(e) Summary of Results

Classical Transmission Line Equations and Parameters (ω domain) (4.11.34)

$K \equiv \int_{C_1} dx_1' \, dy_1' \, \alpha_1(x_1',y_1') \ln(s_{21}^2/s_{11}^2) - \int_{C_2} dx_2' \, dy_2' \, \alpha_2(x_2',y_2') \ln(s_{22}^2/s_{12}^2)$ *(4.4.8)*

$K_L \equiv \int_{C_1} dx_1' \, dy_1' \, b_1(x_1',y_1') \ln(s_{21}^2/s_{11}^2) - \int_{C_2} dx_2' \, dy_2' \, b_2(x_2',y_2') \ln(s_{22}^2/s_{12}^2)$, *(4.10.9)*

$K = K_L$ real and dimensionless *(4.11.30)*

$\frac{dV(z)}{dz} = -\mathbf{z}\, i(z)$ $\quad (\frac{d}{dz^2} - \mathbf{zy})\, V(z) = 0$ $\quad \mathbf{z} = R + j\omega L$ transmission line equations

$\frac{di(z)}{dz} = -\mathbf{y} V(z)$ $\quad (\frac{d}{dz^2} - \mathbf{zy})\, i(z) = 0$ $\quad \mathbf{y} = G + j\omega C$ *(4.11.14), (4.11.15)*

$\mathbf{z} = Z_{s1} + Z_{s2} + j\omega L_e$ *(4.11.17)* $\quad X_L \equiv \omega L_e$, $X_C \equiv 1/(\omega C)$

$\mathbf{y} = j\omega C' = j\omega C + (\sigma_d/\varepsilon_d)C$ *(4.11.24)* $\quad G = (\sigma_d/\varepsilon_d)C$ *(4.11.25)*

$R = \mathrm{Re}(Z_{s1} + Z_{s2})$ $\quad L = L_e + (1/\omega)\,\mathrm{Im}(Z_{s1} + Z_{s2}) = L_e + L_i$

$L_e = \frac{\mu_d}{4\pi} K$ *(4.11.29) and (4.11.30)*

$C' = 4\pi\xi_d/K$ *(4.11.26)* $\quad C' = (\xi_d/\varepsilon_d)C$ *(4.11.23)*

$C = 4\pi\varepsilon_d/K$ *(4.11.26)*

$G = 4\pi\sigma_d/K$ *(4.11.26) + (4.11.25)* $\Rightarrow$ $G/C = \sigma_d/\varepsilon_d$

$L_e C' = \mu_d \xi_d$ *(4.11.27)*

$L_e C = \mu_d \varepsilon_d = 1/v_d^2$ *(4.11.28)*

$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{\mathbf{z}}{\mathbf{y}}}$ *(4.11.16)*

$Z_0 \text{ (large } \omega) \approx \sqrt{\frac{L}{C}} \approx \sqrt{\frac{L_e}{C}} = (1/4\pi)\, K \sqrt{\mu_d/\varepsilon_d} = (K/4\pi)\, Z_m$ // See comments below

$\lambda >> D$ *(4.3.6)* assumed transmission line limit where $\beta_d = 2\pi/\lambda$

$\beta_d^2 = \mu_d\varepsilon_d\omega^2 - j\omega\mu_d\sigma_d = \omega^2\mu_d(\varepsilon_d - j\sigma_d/\omega) = \omega^2\mu_d\,\xi_d$ $\quad \xi_d \equiv \varepsilon_d - j\sigma_d/\omega$. *(1.5.1a)*

$F(z) = F(0)\, e^{-az}\, e^{-jbz}$ *see (5.3.6)* $\quad a \equiv \mathrm{Re}(\sqrt{\mathbf{zy}}) = \mathrm{Re}[\sqrt{(R+j\omega L)(G+j\omega C)}]$ attenuation $\quad b = \mathrm{Im}(\sqrt{\mathbf{zy}})$ phase

Comments:

1. In Chapter 2 we computed the surface impedance Z_s for a round wire in the case of axially symmetric current and we found that, for large ω,

$$Z_s(\omega) \approx \frac{1}{\sigma(2\pi a)\delta}\ (1+j) \qquad \textit{(2.4.16)}$$

$$\delta \equiv \sqrt{2/\omega\mu\sigma} \ = \text{skin depth} \qquad \textit{(2.2.20)}$$

so that

$$Z_s(\omega) \approx \frac{1}{2\pi a}\sqrt{\frac{\mu}{2\sigma}}\ (1+j)\sqrt{\omega} \qquad . \qquad (4.11.35)$$

Presumably the result will be $Z_s(\omega) \sim \sqrt{\omega}$ for any conductor cross section shape. Then

$$L = L_e + (1/\omega)\,\mathrm{Im}(Z_{s1} + Z_{s2}) \ = L_e + (\text{stuff})\ 1/\sqrt{\omega} \ \rightarrow L_e \qquad \text{for large } \omega \qquad (4.11.36)$$

For this reason, the high frequency characteristic impedance Z_0 can be written as shown in (4.11.34).

2. Conductors have internal inductance L_i as well as external inductance L_e. In Appendix C.3 (a) we compute the low frequency internal inductance of a round wire to be $L_i = \mu/8\pi = (\mu/\mu_0) * 50$ nH/m . Our Chapter 4 transmission line development makes no mention of L_i. This can be traced to Figure 4.11 where only the external magnetic flux is involved. In fact, L_i is accounted for in the imaginary part of the surface impedance Z_s . For example, we found that for our round wire situation,

$$Z_s(\omega) = \frac{1}{\sigma\pi a^2} + j\omega\frac{\mu}{8\pi} = R_s + j\omega L_s \qquad \text{// low frequency limit} \qquad \textit{(2.4.12)}$$

and here one sees that $L_s = \frac{\mu}{8\pi} = L_i$.

3. We have assumed that ε_d and μ_d are real. If not, the usual adjustments can be made in (4.11.34) for the interpretations of R,L,G and C. See for example (3.3.4) concerning σ being replaced by σ_{eff} if ε has an imaginary part.

4. Apart from the symmetric cases like the examples of Section 4.5 and 4.6, we do not *yet* have a way to compute K and the transmission line parameters since the charge and current distributions α_i and b_i are not known. This matter will be remedied in Chapter 5.

5. A **strip** transmission line of width w and separation s with s << w is the simplest example of the above summary:

$$E = V/s \qquad n = \varepsilon_d E = \varepsilon_d V/s \qquad q = nw \qquad C = q/V = \varepsilon_d w/s \quad \Rightarrow K = 4\pi\varepsilon_d/C = 4\pi(s/w)$$

so

$$C = 4\pi\varepsilon_d/K = \varepsilon_d\,(w/s) \qquad K = 4\pi\,(s/w)$$
$$G = 4\pi\sigma_d/K = \sigma_d\,(w/s)$$
$$L_e = (\mu_d/4\pi)\,K = \mu_d\,(s/w)$$
$$Z_0 \approx (K/\sqrt{\varepsilon_{rel}})\,30\Omega = 4\pi\,(s/w)\,(1/\sqrt{\varepsilon_{rel}})\,30\Omega = (s/w)\,(1/\sqrt{\varepsilon_{rel}})\,377\Omega \qquad (4.11.37)$$

(f) Time domain equations (telegraph equations)

The results above are all stated in the frequency domain, but it is a simple matter to convert them to the time domain using $j\omega \leftrightarrow \partial_t$. One then makes these replacements

$$\mathbf{z} = R+j\omega L \quad \rightarrow \quad L\partial_t + R$$
$$\mathbf{y} = G+j\omega C \quad \rightarrow \quad C\partial_t + G$$
$$\mathbf{zy} = (R+j\omega L)(G+j\omega C) \quad \rightarrow \quad (R + L\partial_t)(G + C\partial_t) = LC\partial_t^2 + (LG+RC)\partial_t + RG\,. \qquad (4.11.38)$$

Here then are selected equations and their translations to the time domain:

Transmission Line Equations (4.11.14b) : [coupled first order PDE's]

$$\partial_z V = -\mathbf{z}\,i \quad \Rightarrow \quad \partial_z V(z,t) = -L\partial_t i(z,t) - LR\,i(z,t)$$
$$\partial_z\, i = -\mathbf{y}\,V \quad \Rightarrow \quad \partial_z\, i(z,t) = -C\partial_t V(z,t) - CGV(z,t) \qquad (4.11.39)$$

Transmission Line Wave Equations (4.11.15) [damped wave equations]

$$(\partial_z^2 - \mathbf{zy})\,V(z) = 0 \quad \Rightarrow \quad [\partial_z^2 - LC\,\partial_t^2 - (LG+RC)\partial_t - RG]\,V(z,t) = 0$$
$$(\partial_z^2 - \mathbf{zy})\,i(z) = 0 \quad \Rightarrow \quad [\partial_z^2 - LC\,\partial_t^2 - (LG+RC)\partial_t - RG]\,i(z,t) = 0 \qquad (4.11.40)$$

If we set the loss parameters R and G both to 0 these equations become

$$\partial_z V(z,t) = -L\partial_t i(z,t) \qquad [\partial_z^2 - LC\,\partial_t^2]\,V(z,t) = 0 \qquad \text{[undamped wave equations]}$$
$$\partial_z\, i(z,t) = -C\partial_t V(z,t) \qquad [\partial_z^2 - LC\,\partial_t^2]\,i(z,t) = 0 \qquad \text{// lossless} \qquad (4.11.41)$$

At large ω one has $L \approx L_e$ (note 1 above) and since (4.11.28) says $L_eC = \mu_d\varepsilon_d = 1/v_d^2$ we conclude that the factor LC appearing in the above wave equations is $1/v_d^2$ where v_d is the dielectric wave velocity.

The various transmission line equations shown above in the time domain are often referred to as **telegraph** (telegrapher, telegrapher's) **equations**.