Retired Chapters 4,5,6 PhL 1.14.14

Chapter 4: Transmission Line Equations

In this Chapter we use the potential integral expressions derived in Chapter 1 to derive the classic transmission line equations. We learn that all "external" transmission line parameters are determined by a single geometric integral K. The approximations assumed are clearly stated.

4.1 Computation of potential φ due to one conductor of a transmission line

Our starting point is the φ expression given in box (1.5.23) for the potential at some arbitrary point **x** in the dielectric *due to conductor* C_1 ,

$$\varphi_1(\mathbf{x},\omega) = \frac{1}{4\pi\xi}\int_{C_1} \rho_1(x',y',z',\omega)\frac{e^{-j\beta R}}{R}\,dx'dy'dz' \quad . \qquad R = |\mathbf{x} - \mathbf{x}'| \qquad (4.1.1)$$

Here the point **x'** = (x',y',z') runs over the surface of C_1 and R is the distance between the observation point **x** in the dielectric and the integration point **x'**. Parameters β and ξ are for the dielectric.

Comments on ρ_1 :

(1) ρ_1 is the volume charge density associated with "surface charge" n_1 according to $\rho_1 dV' = n_1 dS'$.

(2) ρ_1 is a distribution. For example, for a round wire of radius a we expect ρ_1 to be proportional to δ(r'-a) where $r' = \sqrt{x'^2+y'^2}$.

(3) Recall from Section 1.5 (c) and (1.5.17) the fact that there are two distinct areal charge distributions called n_c and n_s which are related by $n_c = (\xi/\varepsilon)n_s$. Here n_s is the *actual* charge distribution, whereas n_c is an *adjusted* charge density which is directly associcated with the current I in the conductor and which accounts for possible leakage in the dielectric. Our n_1 and ρ_1 are associated with this n_c adjusted charge distribution, not with n_s. That is why the external factor is $1/4\pi\xi$ instead of $1/4\pi\varepsilon$.

Consider now this charge density $\rho_1(\mathbf{x})$. Following a standard methodology, we make the assumption that its functional form may be factored in the following manner,

$$\rho_1(x,y,z) = a_1(x,y)\; q_1(z) \quad . \qquad (4.1.2)$$
$$C/m^3 \qquad 1/m^2 \quad C/m$$

The dimensions of the functions in this factorization are as indicated, so the charge goes with q_1. Moreover, without any loss of generality we select the relative scale of the two factors such that the integral of $a_1(x,y)$ over a slice of conductor C_1 at any z is unity,

$$\int_{C_1} dx\, dy\, a_1(x,y) = 1 \; . \qquad (4.1.3)$$

Therefore, we can interpret $q_1(z)$ as the *total charge per unit length* on C_1 at location z :

$$\int_{C_1} dx\, dy\, \rho_1(x,y,z) = q_1(z) \int_{C_1} dx\, dy\, a_1(x,y) = q_1(z) \bullet 1 = q_1(z) .$$

Assume that $q_2(z)$ is the charge on the other conductor C_2. If $q_1(z) + q_2(z) \neq 0$, then we have a net charge per unit length and the transmission line is acting as a radiating antenna as well as a transmission line. From now on, we ignore this superposed problem and assume that at each value of z, the net charge on both conductors is 0. This means that

$$q_2(z) = - q_1(z) \equiv -q(z) \quad . \tag{4.1.4}$$

To simplify notation, we now dispense with the subscript and denote $q_1(z) = q(z)$. However, we maintain the subscript on $a_1(x,y)$ to emphasize that the two conductors can have completely different cross sectional shapes. The shape of the transverse distribution of charge on C_1 is determined by $a_1(x,y)$, but the total charge is q(z) per unit length.

How can we justify assumption (4.1.2)? This is "separation of variables". The idea is that we assume it without any justification, and then we try to find a solution to our problem which is consistent with the assumption. All we really want is to find a solution to our basic differential equations with their boundary conditions, and any assumptions we make can be justified in the end once we have found a solution. On the other hand, if an assumption like (4.1.2) does not lead to a solution, then it must have been a bad assumption.

Now insert (4.1.2) into (4.1.1) to get,

$$\varphi_1(x,y,z) = \frac{1}{4\pi\xi} \int_{C_1} dx'\, dy'\, a_1(x',y') \int_{-\infty}^{\infty} dz'\, q(z') \frac{e^{-j\beta R}}{R} \quad . \tag{4.1.5}$$

The next move is to write a power series expansion for q(z') about the point z:

$$q(z') = q(z) + q'(z)\,(z'-z) + \ldots = \sum_{n=0}^{\infty} q^{(n)}(z)\,(z' - z)^n / n! \quad . \tag{4.1.6}$$

where we mean by $q^{(n)}(z)$ the n^{th} derivative with respect to z. Sticking this expansion into φ_1 gives

$$\varphi_1(x,y,z) = \frac{1}{4\pi\xi} \sum_{n=0}^{\infty} (1/n!)\, q^{(n)}(z) \int_{C_1} dx'\, dy'\, a_1(x',y') \int_{-\infty}^{\infty} dz'\, (z'-z)^n \frac{e^{-j\beta R}}{R} . \tag{4.1.7}$$

In Appendix G we investigate the dz' integral on the right. If we define the following symbol (the transverse part of the distance between **x** and **x'**),

$$s \equiv \sqrt{(x-x')^2 + (y-y')^2} \tag{4.1.8}$$

then Appendix G shows the integral has the following functional form,

$$\int_{-\infty}^{\infty} dz' \, (z'-z)^n \frac{e^{-j\beta R}}{R} = s^n \, f_n(\beta s) \, . \tag{4.1.9}$$

One could arrive at this general form based on the fact that the integral must have dimensions of (distance)^n. The Appendix shows that the integral is zero for odd n, so it is then convenient to change to summation variable m = n/2. Then (4.1.7) becomes,

$$\varphi_1(x,y,z) = \frac{1}{4\pi\xi} \sum_{m=0}^{\infty} \frac{1}{(2m)!} \, q^{(2m)}(z) \int_{C_1} dx' \, dy' \, a_1(x',y') \, s^{2m} \, f_{2m}(\beta s) \, . \tag{4.1.10}$$

The integral of (4.1.9) is evaluated in Appendix G with the result (G.7) that

$$f_{2m}(\beta s) = 2 \, (2m-1)!! \, K_m(j \, \beta s) / (j\beta s)^m \tag{4.1.11}$$

where $K_m(z)$ is a modified Bessel function. Inserting (4.1.11) into φ_1 gives,

$$\varphi_1(x,y,z) = \frac{1}{4\pi\xi} \sum_{m=0}^{\infty} \frac{2 \, (2m-1)!!}{(2m)!} \, \frac{1}{(j\beta)^m} \, q^{(2m)}(z) \int_{C_1} dx' \, dy' \, a_1(x',y') \, s^m \, K_m(j\beta s) \, . \tag{4.1.12}$$

We are now done. Note that we have made no assumptions whatsoever other than the separation of variables in (4.1.2). If we knew the way the charge was transversely distributed on conductor C_1, as indicated by $a_1(x',y')$, we would have an exact closed form solution for φ_1 at any point in space. Notice that the general form is the following:

$$\varphi_1(x,y,z) = k_0(s)q(z) + k_1(s)q''(z) + k_2(s)q^{(4)}(z) + ... \tag{4.1.13}$$

We can now consider applying our first approximation:

The Transmission Line Limit: In this limit, we assume that wavelength λ is large compared to the transverse dimensions of the transmission line as characterized by variable s. Since $\beta = 2\pi/\lambda$, this means that (βs) is small, so we can then use the small-argument limit of the Bessel function $K_m(j\beta s)$. If we do this, what we find is that the series (4.1.13) is not rapidly convergent and may even diverge. If we make the reasonable assumption that the charge density q(z) behaves as a wave of wavevector β along the line, $\exp(-j\beta z)$, we find that the terms in (4.1.12) drop off on the order of 1/m. This suggests a possible formal logarithmic divergence of (4.1.12). In any event, it is clear that the q"(z) term certainly cannot be "neglected" in (4.1.13).

In our transmission line analysis, what we really want is the potential due to *both* conductors C_1 and C_2, call this φ_{12}. Moreover, we are interested in the difference $V(z) = \varphi_{12}(\mathbf{x}_1) - \varphi_{12}(\mathbf{x}_2)$, where $\mathbf{x}_1$ is a point

on C_1, and $\mathbf{x_2}$ a point on C_2, such that these two points have the same value of z. This difference V(z) is the normal "voltmeter voltage" between the two conductors of the transmission line at z. In the next section we will show that, when this four-term combination V(z) is constructed, the series corresponding to (4.1.12) or (4.1.13) is highly convergent when we assume the transmission line limit. In fact, if we drop terms that are of order β^2 and smaller, we will find that

$$V(z) = \kappa_0 \, q(z)$$

and we will then interpret κ_0 as the inverse (complex) capacitance of the line per unit length based on the usual notion that Q = CV.

If we do *not* assume the transmission line limit, then the result for V(z) is more like (4.1.13), and we then have some entity that is more complex than a "standard transmission line", since it has some sort of higher "capacitive moments" k_i that have to be kept track of. We do not in this case obtain the classic transmission line equations. This is a whole painful world we plan to steer clear of. We are happy to live within the transmission line limit. Moreover, this limit serves to rule out possible waveguide modes of the transmission line. As discussed in Appendix F, these TE and TM modes all have cutoff frequencies ω_m below which the mode cannot operate, and our long λ transmission line limit is also a low ω frequency limit.

An implication of the above discussion is that there is a region of frequency ω where the waveguide modes have not yet been activated, but in which the TEM mode does not really obey the classic transmission line equations. This happens when $\lambda/2$ is slightly larger than the transverse dimensions of the transmission line.
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4.2 Computation of V(z)

First, as outlined above, we form $\varphi_{12}(x)$ as the potential of both conductors. We get basically two terms each of the form of (4.1.12). The relative minus sign is due to $q_2(z) = -q(z)$:

$$\varphi_{12}(x,y,z) = \frac{1}{4\pi\xi} \sum_{m=0}^{\infty} \frac{2\,(2m-1)!!}{(2m)!} \frac{1}{(j\beta)^m} q^{(2m)}(z) *$$

$$\{ \int_{C_1} dx'\, dy'\, a_1(x',y')\, s^m K_m(j\beta s) - \int_{C_2} dx'\, dy'\, a_2(x',y')\, s^m K_m(j\beta s) \} \qquad (4.2.1)$$

with distance s still given by (4.1.8). Next, we form

$$V(z) = \varphi_{12}(\mathbf{x}_1) - \varphi_{12}(\mathbf{x}_2)$$

to get,

$$V(z) = \frac{1}{4\pi\xi} \sum_{m=0}^{\infty} \frac{2\,(2m-1)!!}{(2m)!} \frac{1}{(j\beta)^m} q^{(2m)}(z) *$$

$$\{ \int_{C_1} dx'\, dy'\, a_1(x',y') [\, s_1^m K_m(j\beta s_1) - s_2^m K_m(j\beta s_2)]$$

$$- \int_{C_2} dx'\, dy'\, a_2(x',y') [\, s_1^m K_m(j\beta s_1) - s_2^m K_m(j\beta s_2)] \ \} \qquad (4.2.2)$$

where now $s_i = |\, \mathbf{x}_i - \mathbf{x}' \,|$.

One might wonder why it is that V(z) is independent of the location of the contact points $\mathbf{x}_1$ and $\mathbf{x}_2$, since this dependence seems to be present on the right side of (4.2.2). If the charge distributions a_1 and a_2 were prescribed by fiat, V(z) given by (2) *would* be a function of $\mathbf{x}_1$ and $\mathbf{x}_2$. However, the charge distributions in fact arrange themselves in such a way as to cause each conductor to be an equipotential surface at any given z. This was Fact 6 of Section 3.8. Since the sliced surfaces are equipotentials, the potential difference cannot possibly depend on where contact is made on each surface, assuming both contacts are in the same z plane.

We now make the following claim, and relegate its proof to Appendix 4.1:

Fact: In (4.2.2), each term m=1 and higher makes a contribution to the sum which is of order (β^2) or smaller. In the small-β limit which is the transmission line limit, we can neglect all these terms, so that the only term left is the term with m=0.

Here then is the m=0 term in (4.2.2): (as discussed in Appendix F, (-1)!! = (0)!! = 1)

$$V(z) = \frac{1}{4\pi\xi}\ 2\ q(z)\ *$$

$$\{\ \int_{C_1} dx'\ dy'\ a_1(x',y')\ [\ K_0(j\beta s_1)\ -\ K_0(j\beta s_2)]$$

$$-\ \int_{C_2} dx'\ dy'\ a_2(x',y')\ [\ K_0(j\beta s_1)\ -\ K_0(j\beta s_2)]\ \ \}\qquad (4.2.3)$$

where $s_i = |\ \mathbf{x}_i - \mathbf{x}'\ |$. Since we have already assumed βs_i is small, we can take the small-z limit of the $K_0(z)$ Bessel function which is,

$$K_0(z) \approx -\ln(z/2)\ [\ 1 + (z/2)^2 + \text{order}(z^4)\ ] + \psi(1) + (z/2)^2\ \psi(2) + \text{order}(z^4\ )\qquad (4.2.4)$$

The main item here is -ln(z/2), but we have shown the non-leading terms as well. The $\psi(i)$ are certain constants relating to the gamma function $\Gamma(z)$. We can see what happens with these non-leading terms in (4.2.3). A power z^2 causes a factor of β^2 or $\beta^2\ln(\beta)$ so we can throw these terms out, and the z^4 terms are even smaller. The $\psi(1)$ term cancels in each square bracket difference. The bottom line is then this relatively simple result:

$$V(z) = \frac{2}{4\pi\varepsilon}\ q(z)\{\ \int_{C_1} dx'\ dy'\ a_1(x',y')\ \ln(s_2/s_1) - \int_{C_2} dx'\ dy'\ a_2(x',y')\ \ln(s_2/s_1)\ \ \}\qquad (4.2.5)$$

where $s_i = |\ \mathbf{x}_i - \mathbf{x}'\ |$.

At this point, we invoke the discussion of Appendix B about the complex dielectric constant to make this replacement shown in (B.6) where q(z) is the integral of the surface charge n(x,y,z) around a band of conductor C_1 of width dz,

$$q(z)/\varepsilon = q_{\mathbf{eff}}(z)/\xi.\qquad (4.2.6)$$

As shown in (B.8), $q_{\mathbf{eff}}(z)/\ V(z) = C'$, the complex capacitance per unit length of the transmission line. This is nothing fancy, we are just saying that if the dielectric has some conductivity $\sigma > 0$, then this effective C' is complex, so it includes the parts normally called G and C (G = "conductance")

$$C' = C + G/j\omega\ .\qquad (4.2.7)$$

Since

$$q(z) = (\varepsilon/\xi)\ q_{\mathbf{eff}}(z) = (\varepsilon/\xi)\ C'\ V(z) \qquad \Rightarrow \qquad 1/C' = (\varepsilon/\xi)\ V(z)/q(z)$$

we find from (4.2.5) that

$$1/C' = (1/2\pi\xi)\ \{\ \int_{C_1} dx'\ dy'\ a_1(x',y')\ \ln(s2/s1) - \int_{C_2} dx'\ dy'\ a_2(x',y')\ \ln(s2/s1)\ \}\qquad (4.2.8)$$

This is our first major result. We have used Maxwell's equations and have ended up with an expression for the C and G of a transmission line as a purely geometric transverse integral of the charge densities on the two conductors. Later we will show how these charge distributions can be computed for very general situations. For simple situations, such as twin lead and coaxial cables, we can (and will) use (4.2.8) "as is" to get all the familiar results.

4.3 Computation of A_z due to one conductor of a transmission line

In this section, we quickly develop a set of expressions for A_z which are entirely analogous to those for φ. The similarity is not coincidental but is in fact necessary due to the fact that A_z and φ are components of the same Lorentz 4-vector, and we have selected a reasonably covariant gauge. This was discussed in Section 1.3. [wrong! The King gauge is not covariant. I am now assuming the Lorenz gauge]

In Section 3.5 Fact 5 it was noted that A_z is the only significant component of **A**. Our starting point then is the z-component of (1.5.8), which expresses the potential A_z at some arbitrary point **x** in space *due to conductor* C_1 as an integral over the current density J_z on the surface of conductor C_1:

$$A_{z1}(\mathbf{x}) = \frac{\mu}{4\pi} \int_{C_1} J_{z1}(\mathbf{x}') \frac{e^{-j\beta R}}{R} dx'dy'dz' \qquad R = |\mathbf{x} - \mathbf{x}'| \tag{4.3.1}$$

We then make the same assumption of separation of variables to write

$$\begin{array}{lll} J_{z1}(x,y,z) = b_1(x,y)\, i_1(z) & & \\ \quad A/m^2 \qquad\quad 1/m^2 \quad A & & \end{array} \tag{4.3.2}$$

where i_1 is scaled such that

$$\int_{C_1} dx\, dy\, b_1(x,y) = 1 \; . \tag{4.3.3}$$

As before, we can now interpret $i_1(z)$ as the *total current* in C_1 at z. Again assuming that there is no net superposed radiating antenna current, we have equal and opposite currents in the two conductors,

$$i_2(z) = -\, i_1(z) = -i(z) \; . \tag{4.3.4}$$

We are of course led at once to an analogous version of (4.1.5),

$$A_{z1}(x,y,z) = \frac{\mu}{4\pi} \int_{C_1} dx\, dy\, b_1(x',y') \int_{-\infty}^{\infty} dz'\, i(z') \frac{e^{-j\beta R}}{R} \tag{4.3.5}$$

This is *identical* to (4.1.5) with these replacements:

$$\begin{array}{ll} \varphi_1 \rightarrow A_{z1} & q(z) \rightarrow i(z) \\ a_1 \rightarrow b_1 & (1/\varepsilon) \rightarrow (\mu) \end{array} \tag{4.3.6}$$

We can now dispense with duplicating the next several steps and jump right to the bottom line,

$$A_{z1}(x,y,z) = \frac{\mu}{4\pi} \sum_{m=0}^{\infty} \frac{2^{1-m}}{(m)!} (j\beta)^{-m} i^{(2m)}(z) \int_{C_1} dx'\, dy'\, b_1(x',y')\, s^m K_m(j\beta s) \text{ repair} \qquad (4.3.7)$$

4.4 Computation of W(z)

Our analogous treatment continues. We first construct A_{z12} (x) as the potential of both conductors C_1 and C_2 which gives a result identical to (4.2.1) with changes (4.3.6). We then take the difference of this potential evaluated at the two conductor surfaces,

$$W(z) \equiv A_{12,z}(\mathbf{x}_1) - A_{12,z}(\mathbf{x}_2) \qquad (4.4.1)$$

to get,

$$W(z) = \frac{\mu}{4\pi} \sum_{m=0}^{\infty} \frac{2^{1-m}}{(m)!} (j\beta)^{-m} i^{(2m)}(z) * \qquad (4.4.2)$$

$$\{ \int_{C_1} dx'\, dy'\, b_1(x',y') [s_1^m K_m(j\beta s_1) - s_2^m K_m(j\beta s_2)]$$

$$- \int_{C_2} dx'\, dy'\, b_2(x',y') [s_1^m K_m(j\beta s_1) - s_2^m K_m(j\beta s_2)] \} \text{ repair}$$

where $s_i = | \mathbf{x}_i - \mathbf{x}' |$. We go on to assume the transmission line limit so β is small. This leads to a result similar to (4.2.3). We then install the limit (4.2.4) to end up with this final result:

$$W(z) = \frac{2\mu}{4\pi} i(z) \{ \int_{C_1} dx'\, dy'\, b_1(x',y') \ln(s2/s1) - \int_{C_2} dx'\, dy'\, b_2(x',y') \ln(s2/s1) \} \text{ repair?} \qquad (4.4.3)$$

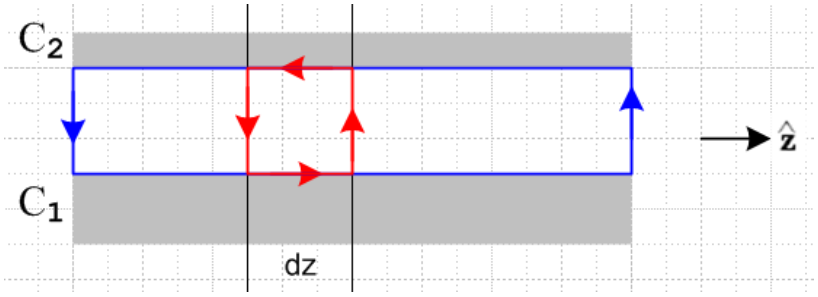
where $s_i = | \mathbf{x}_i - \mathbf{x}' |$.

Recall that functions $b_i(x,y)$ describe how the current J_z is distributed in the conductors, and the leading factor of 2 is left over from the 2^{1-m} factor in (4.4.2).

The Stokes theorem applied to **B** = curl **A** says

$$\text{curl } \mathbf{A} = \mathbf{B} \qquad \Leftrightarrow \qquad \oint \mathbf{A} \bullet d\mathbf{s} = \int_S \mathbf{B} \bullet d\mathbf{A} \ . \qquad (4.4.4)$$

Consider the red loop shown in this top view of the two transmission line conductors. The loop is intended to have a tiny width dz, and the top view obscures the fact that each conductor has an arbitrary cross section.



Since we neglect any transverse components of **A**, the Stokes theorem says

$$[A_{z1}(\text{bottom}) - A_{z2}(\text{top})]\, dz = [\text{ magnetic flux through red loop}] \qquad (4.4.5)$$

We can imagine the entire transmission line as forming a long thin horizontal loop of wire as shown in blue. The long loop has some total external inductance which a has this definition: (flux through blue loop) = $L_{tot,ext}$ I where I is the loop current. This is an external inductance only because it does not account for the stored field energy inside the conductors, as we saw in the case of a round wire in Appendix C.3. The blue loop is a superposition of many red loops, and for the red loop shown we write

$$[\text{magnetic flux through red loop}] = (L_e dz)\, i(z) \qquad (4.4.6)$$

where recall i(z) is the loop current. Now L_e is the external inductance per unit length of the transmission line. Combining the above and cancelling the dz's we find from (4.4.1) that

$$W(z) = L_e\, i(z) \qquad (4.4.7)$$

In the context of the red loop bordered by two current carrying "wires", one would refer to L_e as a mutual inductance, although it is part of the self-inductance of the entire transmission line.

Therefore, from (4.4.3) we have

$$L_e = \frac{\mu}{2\pi} \left\{ \int_{C_1} dx'\, dy'\, b_1(x',y') \ln(s2/s1) - \int_{C_2} dx'\, dy'\, b_2(x',y') \ln(s2/s1) \right\} \qquad (4.4.8)$$

As noted in the V(z) discussion, if we do not assume the transmission line limit, we end up with a transmission line described by several "inductive moments" in addition to (4.4.8). These are the coefficients of the derivatives of i(z), that is, we have a situation similar to (4.1.13). As before, we steer clear of this situation and cling happily to the transmission line limit.

Observation : If the dielectric between the conductors has any kind of hysteresis, which is not usual for a dielectric, this would be reflected by μ(ω) in (4.4.8) being complex. In this case, L_e has a slight imaginary part which we interpret as a resistive loss effect due to the hysteresis. In general, we shall consider L_e to be completely real.

4.5 The Classic Transmission Line Equations

The results of the previous sections of this chapter may be succinctly summarized as:

$$q_{eff}(z) = C' V(z) \quad \text{or} \qquad q(z) = C\, V(z)$$

$$W(z) = L_e\, i(z) \tag{4.5.1}$$

where C' is given by (4.2.8) and L_e by (4.4.8). Notice that we have made no assumptions whatsoever about the cross-sectional shape of the transmission line. We have only assumed that the transverse dimensions are small compared to the wavelength λ that corresponds to β -- this was the transmission line limit.

There are two equations from Chapter 1 which we now wish to press into service:

$$\mathbf{E} = -\,\text{grad}\,\varphi - \partial\mathbf{A}/\partial t \qquad \text{div}\,\mathbf{A} = -\,\sigma\,\mu\,\varphi - \frac{\mu\varepsilon}{c^2}\frac{\partial\varphi}{\partial t} \qquad \text{delete red term} \tag{4.5.2}$$

If we regard **B** = curl **A** as the definition of **A**, then the first equation above can be regarded as the definition of φ. The second equation above is our "modified Lorenz gauge" condition. In the frequency domain these equations become

$$\mathbf{E} = -\,\text{grad}\,\varphi - j\omega\mathbf{A} \qquad \text{div}\,\mathbf{A} = -\,j\,(\beta^2/\omega)\,\varphi \tag{4.5.3}$$

Since **A** has only component A_z, these equations become,

$$E_z = -\,\partial\varphi/\partial z - j\omega A_z \qquad \partial A_z/\partial z = -\,j\,(\beta^2/\omega)\,\varphi \tag{4.5.4}$$

The potentials in the above equations are those due to both conductors and were denoted as φ_{12} and A_{z12} in the previous sections. Looking back out our definitions of V(z) and W(z) as differences, we can rewrite the above as:

$$E_{z1} - E_{z2} = -\,\partial V/\partial z - j\omega W \qquad \partial W/\partial z = -\,j\,(\beta^2/\omega)\,V \tag{4.5.5}$$

The quantity E_{z1} is the longitudinal electric field at the surface of conductor C_1. It is related to the conductor's surface current density by $J_{z\cdot} = \sigma E_{z\cdot}$. If the conductor were "perfect", we would have σ = ∞ and $E_{z1} = 0$. Real conductors are of course not perfect. As shown in (2.4.5), E_{z1} can be related to the total current in the conductor i(z) by a quantity known as the *surface impedance*, so

$$E_{z1} = Z_{i1}\; i_1(z) \qquad E_{z2} = Z_{i2}\; i_2(z) \tag{4.5.6}$$

The surface impedance of a perfect conductor is zero. Since $i_1(z) = -i_2(z) = i(z)$, we rewrite(4.5.5) as,

$$(Z_{i1} + Z_{i2})\, i(z) = -\,\partial V/\partial z - j\omega W \qquad \partial W/\partial z = -\,j\,(\beta^2/\omega)\,V \tag{4.5.7}$$

tentatively ok to here and β is the real β !!

The circle now closes when we insert into (4.5.7) the second expression in (4.5.1):

$$\frac{\partial V(z)}{\partial z} = -\left[Z_{i1} + Z_{i1} + j\omega L_e \right] i(z) \qquad \frac{\partial i(z)}{\partial z} = -\left[j\beta^2/(\omega L_e) \right] V(z) \tag{4.5.8}$$

These are the classic transmission line equations. They are usually written in this form:

$$\frac{\partial V(z)}{\partial z} = -\mathbf{z}\, i(z) \qquad \frac{\partial i(z)}{\partial z} = -\mathbf{y}\, V(z) \tag{4.5.9}$$

where

$$\mathbf{z} = R + j\omega L \qquad \mathbf{y} = G + j\omega C \tag{4.5.10}$$

Here, **z** and **y** are called the transmission line impedance and admittance, and the four numbers R,L,G,C are defined to be the appropriate real and imaginary parts. Looking at (4.5.8), we may therefore conclude that:

$$\mathbf{z} = R + j\omega L = Z_{i1} + Z_{i2} + j\omega L_e \tag{4.5.11}$$

$$\mathbf{y} = G + j\omega C = j\beta^2/(\omega L_e) \tag{4.5.12}$$

The expression for **z** seems quite reasonable, but the one for **y** seems a bit unusual. This is because we still have more work to do.

There is one more equation we have not yet made use of. It is the equation of continuity applied to either conductor. In differential form this is div **J** = $-\partial\rho/\partial t = -j\omega\rho$. When this is applied to a slice of conductor of thickness dz, we conclude that

$$\frac{\partial i(z)}{\partial z} = -\partial q_{eff}/\partial t = -j\omega\, q_{eff}(z) \tag{4.5.13}$$

Here we assume that positive current flows in the z direction in conductor C_1. If i(z+dz) is larger than i(z), then the net effective charge $q_{eff}(z)$ within dz must be decreasing. Inserting the first of equations (4.5.1) into (4.5.13) we get,

$$\frac{\partial i(z)}{\partial z} = -\left[j\omega C' \right] V(z) \tag{4.5.14}$$

Comparison with the second equation (4.5.8) results in the following identity,

$$L_e C' = \beta^2/\omega^2 = \mu\xi \tag{4.5.15}$$

As shown in Appendix 1.2, Eq. (4), we know that $C' = (\xi/\varepsilon)\, C$. And from (1.5.3) we have $\beta^2 = \omega^2\, \mu\, \xi$. Thus we find,

$$L_e C = \mu\varepsilon\, \mu_0\varepsilon_0 = \mu\varepsilon/\, c^2 = 1/v^2 \qquad (4.5.16)$$

This tells us that that $1/\sqrt{L_e C} = v$ = the speed of light in the dielectric medium. Look back now at our expressions for C' in (4.2.8) and L_e in (4.4.4),

$$2\pi\xi\, /C' = \{ \int_{C_1} dx'\, dy'\, a_1(x',y')\, \ln(s2/s1) - \int_{C_2} dx'\, dy'\, a_2(x',y')\, \ln(s2/s1) \}$$

$$2\pi L_e\, /\, \mu = \{ \int_{C_1} dx'\, dy'\, b_1(x',y')\, \ln(s2/s1) - \int_{C_2} dx'\, dy'\, b_2(x',y')\, \ln(s2/s1) \}$$

According to the identity (4.5.15), we conclude that the right hand sides of the two equations above are exactly the same! This seems rather amazing, see further comments in Chapter 5. For now, we simply conclude that the basic parameters of a transmission line, apart from the internal impedances, are entirely determined by the following dimensionless transverse integral,

$$K = \{ \int_{C_1} dx'\, dy'\, a_1(x',y')\, \ln(s2/s1) - \int_{C_2} dx'\, dy'\, a_2(x',y')\, \ln(s2/s1) \} \qquad (4.5.17)$$

In terms of this integral we may write:

$$L_e = (\mu/2\pi)K \qquad (4.5.18)$$

$$C' = 2\pi\xi/K = C + G/j\omega$$

Since $\xi = \varepsilon + \sigma/j\omega$, we can decompose the last into two equations,

$$C = 2\pi\varepsilon/K \qquad (4.5.19)$$

$$G = 2\pi\sigma/K \qquad (4.5.20)$$

Thus, the three traditional parameters of a transmission line L_e, C and G are all determined by the same geometric constant K.

Here then is a summary of the results of this section:

Transmission Line Equations

$$\frac{\partial V(z)}{\partial z} = -\, \mathbf{z}\, i(z) \qquad \frac{\partial i(z)}{\partial z} = -\, \mathbf{y}\, V(z)$$

$$\mathbf{z} = R + j\omega L = Z_{i1} + Z_{i2} + j\omega L_e \qquad \mathbf{y} = G + j\omega C$$

$$L_e = (\mu/2\pi)K \qquad C = 2\pi\varepsilon/K \qquad G = 2\pi\sigma/K$$

$$L_e C = \mu\varepsilon\mu_0\varepsilon_0 = \mu\varepsilon/c^2 = 1/v^2 \qquad (4.5.21)$$

$$K = \{ \int_{C_1} dx'\, dy'\, a_1(x',y') \ln(s2/s1) - \int_{C_2} dx'\, dy'\, a_2(x',y') \ln(s2/s1) \}, \quad s_i = | \mathbf{x}_i - \mathbf{x}' |$$

Comment: According to Matick p 34, $Z_0 = \sqrt{\mathbf{z}/\mathbf{y}}$ is the characteristic impedance of a transmission line, something I never seem to address anywhere, it is more a circuit theory thing. Then it would seem that

$$Z_0^2 = \frac{\mathbf{z}}{\mathbf{y}} = \frac{Z_{i1} + Z_{i2} + j\omega L_e}{G + j\omega C} = \frac{Z_{i1} + Z_{i2} + j\omega\,(\mu/2\pi)K}{2\pi\sigma/K + j\omega 2\pi\varepsilon/K}$$

$$= \frac{(Z_{i1} + Z_{i2})K + j\omega\,(\mu/2\pi)K^2}{2\pi\sigma + j\omega 2\pi\varepsilon} = (1/2\pi)\frac{(Z_{i1} + Z_{i2})K + j\omega\,(\mu/2\pi)K^2}{\sigma + j\omega\varepsilon}$$

As Matick says, this thing is in general complex, If you go to high frequency, then you get just

$$Z_0^2 = \frac{\mathbf{z}}{\mathbf{y}} = \frac{j\omega L_e}{j\omega C} = \frac{L_e}{C} = \frac{(\mu/2\pi)K}{2\pi\varepsilon/K} = (\mu/\varepsilon)\,(1/2\pi)^2\,K^2$$

$$Z_0 = \sqrt{(\mu/\varepsilon)}\;K/2\pi = \sqrt{(\mu/\varepsilon)}\;[2\pi\varepsilon/C]/2\pi = \sqrt{(\mu/\varepsilon)}\;\varepsilon/C = \sqrt{\mu\varepsilon}\;1/C$$

4.6 Example: the wide-spaced, two-wire transmission line.

Here we assume two round wires of radius a_1 and a_2 whose centers are separated a distance b. In order to evaluate the integral K in the above box, we must assume that b is much larger than both wire radii. In this case, we can assume that the charge distribution on the wire surfaces is azimuthally symmetric. The more general case of arbitrary a_1, a_2, b will have to wait.

The calculation is incredibly easy. For each of the two integrals in K, we assume cylindrical coordinates (r,θ) about the respective conductor. The normalized charge density functions are:

$$a_1(r,\theta) = \delta(r - a_1)/(2\pi a_1) \qquad a_2(r,\theta) = -\delta(r - a_2)/(2\pi a_2) \qquad (4.6.1)$$

The normalization factors are such that the transverse integral for each charge density is 1, as required in (4.1.3). We need only do the C_1 integral, then the C_2 one will be obvious. For the C_1 integral we use the following geometry: [new picture, show θ, show values for s_1^2 and s_2^2]

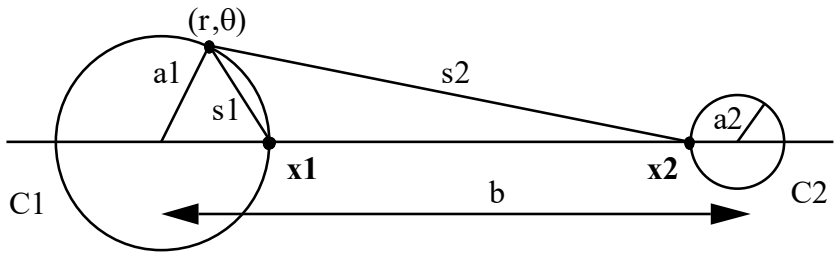


Fig 1: Geometry for computing K contribution from C_1.

Notice that points $\mathbf{x_1}$ and $\mathbf{x_2}$ have been chosen to lie on the points of closest approach. The C_1 integral is:

$$K_1 = \int_0^\infty r\,dr \int_0^{2\pi} d\theta\, [\,\delta(r-a_1)/2\pi a_1]\,(1/2)\ln[\,b^2/2a_1^2(1-\cos\theta)\,]$$

$$= (1/4\pi) \int_0^{2\pi} d\theta \ln[\,b^2/2a_1^2(1-\cos\theta)\,]$$

$$= (1/2)\ln[\,b^2/2a_1^2\,] - (1/4\pi)\{\int_0^{2\pi} d\theta \ln[\,1-\cos\theta\,]\}\ . \qquad (4.6.2)$$

The integral in { } is a standard definite integral equaling $2\pi \ln(1/2)$, so the second term gives $-(1/2)\ln(1/2)$ which exactly cancels the 2 of $2a_1^2$ in the first term. The result is then

$$K_1 = \ln(b/a_1)\ . \qquad (4.6.3)$$

We evaluate the C_2 integral in the same way. The minus sign in the expression for K cancels with the minus sign in the charge density for C_2 (equal and opposite charges recall), so the result is then

$$K = K_1 + K_2 = \ln(b/a_1) + \ln(b/a_2) = \ln(b^2/a_1a_2) = 2\ln(\,b/\sqrt{a_1a_2}\,) \qquad (4.6.4)$$

Therefore, as shown in the above box, we obtain these results for the widely-spaced two-wire transmission line:

$$K = 2\ln(\,b/\sqrt{a_1a_2}\,) \qquad C = \pi\varepsilon / \ln(\,b/\sqrt{a_1a_2}\,)$$

$$L_e = (\mu/\pi)\ln(\,b/\sqrt{a_1a_2}\,) \qquad G = \pi\sigma / \ln(\,b/\sqrt{a_1a_2}\,) \qquad (4.6.5)$$

In these formulas, μ, ε and σ of course refer to the dielectric, not the conductors. Normally μ=1 and σ is extremely small. ε is always in the range 1 to 10.

What about the internal inductance and resistance?

We have solved this problem exactly in Chapter 2, Section 2.3. The exact answer for either wire (n=1,2) is given by (2.4.7):

$$R_{in} + j\omega L_{in} = Z_s(\omega) = (-j\omega\mu'\mu_0/ 2\pi a_n) J_0(\beta a_n) / J_1(\beta a_n) \qquad (4.6.6)$$

where

$$\beta = (\sqrt{2} / \delta) \sqrt{-j} \qquad \delta = \sqrt{2/\omega\mu'\mu_0\sigma'} \ . \qquad (4.6.7)$$

Here we are using primed symbols to stand for quantities for the conductor. Normally $\mu' = 1$ since one seldom uses ferromagnetic (iron) conductors.

In the low frequency limit we get from (2.4.9), combining both wires,

$$R_i(\omega=0) = \frac{1}{\pi\sigma'} \left[\frac{1}{a_1^2} + \frac{1}{a_2^2} \right] \qquad (4.6.8)$$

$$L_i(\omega=0) = \mu'\mu_0 / 4\pi \qquad (4.6.9)$$

In the high frequency limit, where skin depth $\delta << a_1$ and a_2, the result is (2.4.15),

$$R_i(\omega) = \frac{1}{2\pi\sigma'\delta} \left[\frac{1}{a_1} + \frac{1}{a_2} \right] = \frac{1}{2\pi\sigma'} \left[\frac{1}{a_1} + \frac{1}{a_2} \right] \sqrt{\frac{\omega\mu'\mu_0\sigma'}{2}}$$

$$L_i(\omega) = R_i(\omega)/\omega = \frac{1}{2\pi\sigma'} \left[\frac{1}{a_1} + \frac{1}{a_2} \right] \sqrt{\frac{\mu'\mu_0\sigma'}{2\omega}}$$

Since the constant in both expressions is the same in this limit, we can write,

$$R_i(\omega) = X_i(\omega) = k\sqrt{\omega} \qquad (4.6.10)$$

$$L_i(\omega) = k\,(1/\sqrt{\omega}\,)$$

$$k = \frac{1}{2\pi\sigma'} \left[\frac{1}{a_1} + \frac{1}{a_2} \right] \sqrt{\frac{\mu'\mu_0\sigma'}{2\omega}} \quad \text{ohm-sec}^{1/2}$$

As the frequency increases, the internal resistance R_i and the internal inductive reactance X_i of the two wires both scale as $\sqrt{\omega}$.

Recall that the transmission line parameters are

$$\mathbf{z} = R + j\omega L \qquad \mathbf{y} = G + j\omega C$$

In the case that the dielectric constant has a slight imaginary part, we can, according to Appendix 1.2 (9), write

$$\varepsilon(\omega) = \mathrm{Re}[\varepsilon(\omega)]\,[\,1 - j\,\tan_{\mathbf{L}}(\omega)] \qquad (4.6.11)$$

where the imaginary part is represented by the loss tangent of the dielectric. In this case, we modify the above C and G as follows:

$$C = \pi\,\mathrm{Re}[\varepsilon]\,\varepsilon_0 / \ln\left(b/\sqrt{a_1 a_2}\right) \qquad (4.6.12)$$

$$G = \pi\,\sigma_{\mathbf{eff}} / \ln\left(b/\sqrt{a_1 a_2}\right)$$

where

$$\sigma_{\mathbf{eff}} = \sigma + \varepsilon_0\,\omega\,\mathrm{Re}[\varepsilon]\,\tan_{\mathbf{L}}$$

We can now combine all these results:

Two-wire transmission line, $b >> a_{\mathbf{i}}$, in the skin effect limit

$$L = L_{\mathbf{e}} + L_{\mathbf{i}} = (\mu/\pi)\ln\left(b/\sqrt{a_1 a_2}\right) + k\,(1/\sqrt{\omega})$$

$$R = k\sqrt{\omega} \quad k = \frac{1}{2\pi\sigma'}\left[\frac{1}{a_1} + \frac{1}{a_2}\right]\sqrt{\frac{\mu'\mu_0\sigma'}{2\omega}} \qquad \sigma',\ \mu' = \text{conductor}$$

$$C = \pi\,\mathrm{Re}[\varepsilon]\,\varepsilon_0 / \ln\left(b/\sqrt{a_1 a_2}\right) \qquad \mu,\ \sigma,\ \varepsilon,\ \tan_{\mathbf{L}} = \text{dielectric}$$

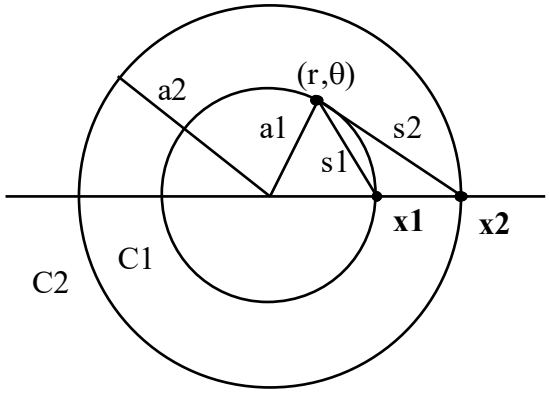
$$G = \pi\,\sigma_{\mathbf{eff}} / \ln\left(b/\sqrt{a_1 a_2}\right) \qquad \sigma_{\mathbf{eff}} = \sigma + \varepsilon_0\,\omega\,\mathrm{Re}[\varepsilon]\,\tan_{\mathbf{L}}$$

(4.6.13)

4.7 Example: the coaxial cable.

This calculation is very similar to the previous example. First, we need a picture:

Fig 1: The coax cable for K_1



The distances are given by:

$$s_1^{\,2} = 2\, a_1^{\,2}\,(1 - \cos\theta) \tag{4.7.1}$$

$$s_2^{\,2} = a_1^{\,2} + a_2^{\,2} - 2a_1a_2\cos\theta \tag{4.7.2}$$

We use the exact same charge distributions as in the previous example, namely (4.6.1). Thus, the K_1 integral becomes,

$$K_1 = \int_0^{\infty} r\,dr \int_0^{2\pi} d\theta\,[\,\delta(r-a_1)/2\pi a_1]\,(1/2)\,\ln\left[\frac{a_1^{\,2} + a_2^{\,2} - 2a_1a_2\cos\theta}{2a_1^{\,2}\,(1-\cos\theta)}\right]$$

$$= (1/4\pi)\int_0^{2\pi} d\theta\,\ln\left[\frac{a_1^{\,2} + a_2^{\,2} - 2a_1a_2\cos\theta}{2a_1^{\,2}\,(1-\cos\theta)}\right] \tag{4.7.3}$$

This is really two integrals of the same form, [Ref]

$$\int_0^{2\pi} d\theta\,\ln[\,A + B\cos\theta\,] = 2\pi\,\ln\left[\frac{A^2 + \sqrt{A^2 - B^2}}{2}\right] \tag{4.7.4}$$

Thus, keeping track of the two separate integrals we get

$$K_1 = (1/4\pi)\,\{\;2\pi\,\ln\left[\frac{a_1^{\,2} + a_2^{\,2} + |\,a_1^{\,2} - a_2^{\,2}\,|}{2}\right] - 2\pi\,\ln\left[\frac{2a_1^{\,2}}{2}\right]\} \tag{4.7.5}$$

Since $a_2 > a_1$, the first term becomes $2\pi\,\ln\left[\frac{2a_2^{\,2}}{2}\right]$, and the overall result is then

$$K_1 = (1/2) \ln [\, a_2^2 \,/\, a_1^2] = \ln (a_2/a_1) \,. \qquad (4.7.6)$$

This integral has a fairly striking similarity to (4.6.3). In fact, (4.6.3) is the correct result for the example of Section 4.6 even when b is small, if we could force the charge distributions to be symmetric, and provided we reinterpret b as being the distance from the center of C_1 to point $\mathbf{x}_2$. We know this is true because, if we distort the C_2 circle in 4.6 Fig 1, we get 4.7 Fig 1, and we have just done the exact computation for 4.7 Fig 2 and we got $K_1 = \ln (a_2/a_1)$.

What about K_2? Just looking at Fig 1 above, we can see that $K_2 = 0$. K_2 represents the contribution to K from the outer sheath C_2. We can get the geometry by just distorting Fig 1 and taking $a_1 \leftrightarrow a_2$. This contribution to K represents the potential at point $\mathbf{x}_1$ due to C_2. Since C_2 is a cylindrical shell of uniform charge density, we know that the potential at any interior point is 0. To see this, put a Gaussian shell inside. Since no charge is in the shell, the electric field on this shell is zero, and therefore the potential is constant.

If all this makes no sense, we will now prove it in one sentence. To get K_2, make the change $a_1 \leftrightarrow a_2$ in (4.7.5) and now the second term cancels the first term, so $K_2 = 0$.

We are done! The result for the coaxial cable is $K = \ln(a_2/a_1)$. Since the result in Section 4.6 was $K = 2 \ln (b/\sqrt{a_1 a_2})$, and since the conductors are still round wires, we can obtain results for the coax cable by replacing K in the box in Section 4.6 with our new K.

The only question one might ask is whether the surface impedance of the outer conductor as a cylindrical shell is the same as for a solid wire of the same radius. In light of the discussion surrounding (2.5.1) , we may conclude that this is indeed precisely the case, provided that the outer sheath has a thickness which is many times the skin depth. In this case, we can regard the sheath as being infinitely thick, and then there are no other dimensions to enter the answer. In a nutshell, we are saying that $D = 2\pi a_2$ in (2.5.1), which seems quite reasonable.

One could solve this problem exactly by selecting the solution $Y_0(x)$ in (2.2.16) in place of $J_0(x)$. We shall not bother doing this.

In the low frequency limit, for a non-infinite sheath C_2, the DC resistance and inductance will be different from (4.6.8) and (4.6.9). If the thickness of the sheath is $t \ll a_2$, then we can replace (4.7.8) with:

$$R_i(\omega=0) = \frac{1}{\pi a_1^2 \sigma'} + \frac{1}{2\pi a_2 \sigma' t} \,. \qquad (4.7.7)$$

Here we have just installed the appropriate cross sectional area, as per Appendix 2.1 2(2).

As for the DC inductance of a thin sheath, we can make the following computation based on Appendix 2.1, which is accurate if $t \ll a$. Outside the sheath we know that $H = I/2\pi a$, and inside $H = 0$, so we can take the average value in the sheath to be $I/4\pi a$. The energy stored in the sheath is $U = (1/2)\mu H^2 V$ where

V is the volume of the sheath V = 2πat dz, and H = I/4πa. Setting this equal to (1/2)L_i I^2 we find this result,

L_i (thin shell of radius a, thickness t) = (μ/8π) (t/a) (4.7.8)

This is the DC inductance of a wire of any radius times (t/a). As t → 0, L_i goes to zero because the field is finite, but the volume goes to 0.

So here is our modified DC inductance for the coaxial cable with a thin sheath:

$$L_i(\omega=0) = (\mu'\mu_0 / 4\pi) [1 + t/a] \qquad (4.7.9)$$

We conclude with the skin-effect limit results, as transcribed from the box at the end of Section 4.6 replacing K with our new K:

Coaxial transmission line, in the skin effect limit

$$L = L_e + L_i = (\mu /2\pi) \ln (a_2/a_1) + k (1/\sqrt{\omega})$$

$$R = k\sqrt{\omega} \quad k = \frac{1}{2\pi\sigma'} \left[\frac{1}{a_1} + \frac{1}{a_2} \right] \sqrt{\frac{\mu'\mu_0\sigma'}{2}} \qquad \sigma', \mu' = \text{conductor}$$

$$C = 2\pi \mathrm{Re}[\varepsilon]\, \varepsilon_0 / \ln (a_2/a_1) \qquad \mu, \sigma, \varepsilon, \tan_L = \text{dielectric}$$

$$G = 2\pi\, \sigma_{eff} / \ln (a_2/a_1) \qquad \sigma_{eff} = \sigma + \varepsilon_0\, \omega\, \mathrm{Re}[\varepsilon]\, \tan_L$$

(4.7.10)

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Chapter 5: The Transverse Problem

5.1 Philosophy

In Chapter 4 we derived the general structure of the behavior of a TEM mode wave on a transmission line. The situation is summarized in the box at the end of Section 4.5. Under the assumption that the wavelength λ = 2π/β is large compared with the transverse dimensions of the transmission line, we ended up with the classic transmission line equations. It was shown that the usual parameters L_e, C, and G are all related to each other by the factor K.

It is now time to look back at what we really did. It all goes back to Maxwell's equation (1.1.1). There is a current term $\mathbf{J_a}$ sitting in this equation which makes its way into (1.5.8) which was really the basis of Chapter 4. Similarly, the charge density ρ_a in (1.1.3) ended up in (1.5.9).
[these "applied" sources are long gone]

We referred in Section 1.1 to these two sources $\mathbf{J_a}$ and ρ_a as being "externally applied". The approach philosophy was to remove the conductors from Maxwell's world and pretend that currents and charges could be prescribed in and on the conductors. These sources then created potentials φ and **A**, and these then determined the **E** and **B** fields and the problem was seemingly solved.
[but I didn't do that, for example charge density α_i is undetermined]

In retrospect, we took this plan to its logical conclusion outlined in the box at the end of Section 4.5. Our big problem is that we have no way to compute the K integral shown in the box, except in certain highly symmetrical situations such as those treated in Sections 4.6 and 4.7. The reason is that we do not know how the sources (charges and currents) are distributed on the conductors in the transverse direction, and this information is what is needed to compute K. [exactly correct]

We must now change our philosophy. We must think of all space as being Maxwell's province. This means the inside of the conductors as well as the dielectric between the conductors. From this point of view, there is no "externally applied" current $\mathbf{J_a}$, and no "externally applied" charge density ρ_a. We should set $\mathbf{J_a}$ and ρ_a equal to zero in Maxwell's equations (1.1.1) and (1.1.3). There is of course a current **J**, and it has already been accounted for in (1.1.1) by the term $\sigma\mathbf{E}$. There will also be a charge density ρ which will arise from the divergence of **E**, as in (1.1.3).
[I think this is the messy situation that I cleaned up now in the current section 1 3 (c) with drawing]

The idea is that we are now looking for a self-consistent TEM wave mode travelling down the conductors. The fields are generated by the current and charge distributions, and the distributions are in turn generated by the fields. For example, the principle current in the conductors J_z is due to the fact that there is some electric field E_z which is creating this current according to Ohm's law $\mathbf{J} = \sigma\ \mathbf{E}$. At the same time, this created current produces a magnetic field which in turn has an associated electric field according to Maxwell (1.1.2). The entire situation is a self-consistent closed loop. If we can find such a solution, then the solution must exist. [I agree that things must be self consistent]

Whenever there is a normal component of electric field at the surface of a conductor, there is an associated charge density. This is the ρ appearing in (1.1.3). This charge distribution is associated with the current distribution according to the continuity equation $\mathrm{div}(\sigma\mathbf{E}) = -\partial\rho/\partial t$. The charge density arises as part of the self-consistent solution to the problem. Like **J**, ρ is not "externally applied". [more of same]

This situation is not uncommon in electromagnetic problems. For example, an accelerating particle radiates, but then the radiation acts back on the particle in a phenomenon known as radiation damping. An antenna radiates, but the radiation has an effect on the current distribution in the antenna. These all serve as reminders that Maxwell's equations, although simply stated, are incredibly complex, and with very few exceptions, no problems have ever been solved exactly. [more on finding self consistent solutions]

In Chapter 4, the integral expressions (1.5.8) and (1.5.9) formed the basis for everything! We had prescribed $\mathbf{J_a}$ and ρ_a , and we used these expressions to compute the potentials φ and $\mathbf{A}$. In our new point of view, these integral expressions need to be re-interpreted. Equation (1.5.8) would seem to indicate that $\mathbf{A} = 0$, since $\mathbf{J_a} = 0$. In fact, (1.5.8) is only a "particular" solution of the differential equation (1.5.1). Now we have to find the "homogeneous solutions" of (1.5.1), which is of the classic Helmholtz form $(\nabla^2 + \beta^2)\mathbf{A} = 0$. Equation (1.5.9) which now contains the unknown charge density ρ can be interpreted as an integral equation which is equivalent to the differential equation (1.5.2). So in our new mindset, the integral formulas are not incorrect, they are just not too useful. [I am struggling to justify ρ_a and J_a, I think I really finally conquered this painful aspect of my original doc]

The time has now come to face the differential equations directly and find solutions. One might at this point think of Chapter 4 as a great waste, but this is not true. Chapter 4 did provide the overall structure of things, and this will be heeded in solving the differential equations.

[So basically all I am saying is: Let's look at the Helmholtz wave equations.]

5.2 The Helmholtz Equations and Separation of Variables

Since $\mathbf{J_a}$ and ρ_a no longer exist, as described above, we set $\mathbf{J_a} = 0$ in (1.5.1) and $\rho_a = 0$ in (1.5.2), so we have this pair of differential equations,

$$(\nabla^2 + \beta^2) \mathbf{A} = \mathbf{0} \tag{5.2.1}$$

$$(\nabla^2 + \beta^2) \varphi = - (1/\varepsilon) \rho \tag{5.2.2}$$

where

$$\beta \equiv \omega \sqrt{(\mu)\xi} \tag{5.2.3}$$

with

$$\xi \equiv [\varepsilon + \sigma/(j\omega)] \tag{5.2.4}$$

Since φ and $\mathbf{A}$ are the potentials arising from the presence of both conductors C_1 and C_2, the φ and $\mathbf{A}$ appearing in (5.2.1) and (5.2.2) correspond to φ_{12} and $\mathbf{A}_{12}$ of Chapter 4. The above equations apply at all points in space, both in the dielectric between the conductors, and inside the conductors as well.
[well this really depends on your gauge selection. In the King gauge there are sources in (5.2.1) in the conductor regions.]

Let us first consider the equation (5.2.2) for φ. We shall do a separation of variables for both φ and ρ. For ρ, we perform a separation very similar to (4.1.2) through (4.1.4),

$$\rho(x,y,z) = \rho_t(x,y)\, q(z) \tag{5.2.5}$$

where $q(z)$ is the charge per unit length on conductor C_1. This defines a transverse charge density $\rho_t(x,y)$. The integral of $\rho_t(x,y)$ over the surface of a C_1 slice is +1, and over the surface of a C_2 slice is -1.

For the potential, we make the following separation:

$$\varphi(x,y,z) = (1/2\pi\varepsilon)\, q(z)\, \varphi_t(x,y) \tag{5.2.6}$$

We are of course free to set the scale factor arbitrarily, since changing the scale factor just changes the definition of $\varphi_t(x,y)$. Our motivation for the form shown in (5.2.6) is the work of Chapter 4, in particular equation (4.2.5)

In light of (4.1.13) and the related discussion, we realize that the separable form (5.2.6) is only possible if we are working in the "transmission line limit" where the wavelength λ along the transmission line is much larger than all transverse dimensions.

When (5.2.5) and (5.2.6) are inserted into (5.2.2), the result is

$$\left[\frac{\nabla_t^2 \varphi_t(x,y)}{\varphi_t(x,y)} + 2\pi \frac{\rho_t(x,y)}{\varphi_t(x,y)} \right] + \frac{\nabla_z^2\, q(z)}{q(z)} = -\beta^2 \tag{5.2.7}$$

which has the general form,

$$[\, h(x,y)\,] + g(z) = -\beta^2 \tag{5.2.8}$$

The only way this can be true for all x,y,z in a region is if g(z) = some constant. For reasons that will be clear later, we write this constant as k_φ^2. Then we get

$$\frac{\nabla_z^2\, q(z)}{q(z)} = k_\varphi^2 \qquad \left[\frac{\nabla_t^2 \varphi_t(x,y)}{\varphi_t(x,y)} + 2\pi \frac{\rho_t(x,y)}{\varphi_t(x,y)} \right] = -\beta^2 - k_\varphi^2 \tag{5.2.9}$$

We can rewrite these as

$$[\, \nabla_t^2 + (\beta^2 + k_\varphi^2)]\, \varphi_t(x,y) = -2\pi\rho_t(x,y) \tag{5.2.10}$$

$$[\, \nabla_z^2 - k_\varphi^2\,]\, q(z) = 0 \tag{5.2.11}$$

We now repeat the process for the vector potential component A_z, which we know is the only significant component of concern. We make the following separation, similar to (5.2.6), which is motivated by (4.4.3),

$$A_z(x,y,z) = (\mu/2\pi)\, i(z)\, A_{zt}(x,y) \tag{5.2.12}$$

Putting (5.2.12) into (5.2.1) yields,

$$\left[\frac{\nabla_t^2\, A_{zt}(x,y)}{A_{zt}(x,y)} \right] + \frac{\nabla_z^2\, i(z)}{i(z)} = -\beta^2 \tag{5.2.13}$$

Again, this can only work if the second term is some constant, which we here call k_A^2.

$$\frac{\nabla_z^2\, i(z)}{i(z)} = k_A^2 \qquad \left[\frac{\nabla_t^2\, A_{zt}(x,y)}{A_{zt}(x,y)}\right] = -\beta^2 - k_A^2 \tag{5.2.14}$$

which leads to,

$$[\,\nabla_t^2 + (\beta^2 + k_A^2\,)]\, A_{zt}(x,y) = 0 \tag{5.2.15}$$

$$[\,\nabla_z^2 - k_A^2\,]\, i(z) = 0 \tag{5.2.16}$$

Consider now equations (5.2.11) and (5.2.16), which we now apply to the full potentials by making use of (5.2.6) and (5.2.12),

$$\frac{\partial^2 A_z}{\partial z^2} - k_A^2\, A_z = 0 \qquad \frac{\partial^2 \varphi}{\partial z^2} - k_\varphi^2\, \varphi = 0 \tag{5.2.17}$$

We seek next to find the constants k_A and k_φ. Consider the differences that played such a major role in Chapter 4,

$$V(z) = \varphi(\, x_1, y_1, z\,) - \varphi(\, x_2, y_2, z\,)$$

$$W(z) = A_z(\, x_1, y_1, z\,) - A_z(\, x_2, y_2, z\,) \tag{5.2.18}$$

where $\mathbf{x_1}$ and $\mathbf{x_2}$ are points on the conductors at the same z. Applying (5.2.17) to these differences gives,

$$\frac{\partial^2 W}{\partial z^2} - k_A^2\, W = 0 \qquad \frac{\partial^2 V}{\partial z^2} - k_\varphi^2\, V = 0 \tag{5.2.19}$$

Now, application of $\partial/\partial z$ to the transmission line equations (4.5.9) gives,

$$\frac{\partial^2 i}{\partial z^2} - \mathbf{zy}\, i = 0 \qquad \frac{\partial^2 V}{\partial z^2} - \mathbf{zy}\, V = 0 \tag{5.2.20}$$

where constants **z** and **y** are the impedance and admittance of the transmission line. According to (4.5.1), $W(z) = L_e\, i(z)$, so we replace the first equation above with an identical one in W:

$$\frac{\partial^2 W}{\partial z^2} - \mathbf{zy}\, W = 0 \qquad \frac{\partial^2 V}{\partial z^2} - \mathbf{zy}\, V = 0 \tag{5.2.21}$$

Comparison of (5.2.21) with (5.2.19) gives us the result we seek,

$$k_A^2 = k_\varphi^2 = \mathbf{zy} \equiv k^2 \tag{5.2.22}$$

In retrospect, we are not surprised that the constants are equal in light of the Lorentz covariance discussion relating to equation (1.3.9).

From (4.5.11) and (4.5.12) we know that $\mathbf{z} = Z_i + j\omega L_e$ [ok], and $\mathbf{y} = j\beta_d^2/(\omega L_e)$ [ok], where Z_i is the total internal impedance of both conductors, and where β_d is β of the dielectric. From (4.5.15) we can also say $\mathbf{y} = j\omega C'$. Therefore

$$k^2 + \beta^2 = Z_i(j\omega C') = Z_i\, j\, \beta_d^2/(\omega L_e) \qquad (5.2.23)$$

Where is this left equality coming from. The two right expressions are just $Z_i\mathbf{y}$.

For perfect conductors, $Z_i = 0$. We might define a "low loss" transmission line as one where the right side of (5.2.23) is much smaller than β_d^2. Using $\beta^2 = \omega^2\mu\xi$ this condition becomes, with μ=1,

$$Z_i << \mu_0 f\, K = 4\pi K \times 10^{-7} f \approx K\, f(\text{MHz}) \text{ ohms/meter} \qquad (5.2.24)$$

where K is the dimensionless geometric integral defined in Chapter 4. Typically K is a number in the range 1-10, so the above statement is quite clear as a definition of "low loss". For a typical coax cable, Z_i is about 1 ohm/m at 1 GHz, K ≈ 3, so the above condition is well met since 1 << 3000.

Fact: For a "low loss" transmission line, as defined above, $k^2 \approx -\beta_d^2$.

There remains one important final connection to be made to our work of Chapter 4. The constants in the variable separations for φ and A_z given in (5.2.6) and (5.2.12) were carefully selected to yield the following result,

$$\varphi_t(\mathbf{x}_1) - \varphi_t(\mathbf{x}_2) = A_{zt}(\mathbf{x}_1) - A_{zt}(\mathbf{x}_2) = K \qquad (5.2.25)$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ are any points lying on C_1 and C_2 in the same z plane. Recall that K determines the three transmission line parameters G, C and L_e. The ingredients needed to show that (5.2.25) is true are (5.2.6) and (5.2.12) of this section, and (4.5.1) , (4.5.18) and (4.5.19).

We now summarize the key results of this section:

Separated Transmission Line Equations

Variable separations:

$\rho(x,y,z) = \rho_t(x,y)\, q(z)$

$\varphi(x,y,z) = (1/2\pi\varepsilon)\, q(z)\, \varphi_t(x,y)$

$A_z(x,y,z) = (\mu/2\pi)\, i(z)\, A_{zt}(x,y)$

Eigenvalue relation:

$k^2 = \mathbf{zy}$

$(\beta^2 + k^2) = j\omega Z_i C'$

≈ 0 "low loss"

Transverse equations:

$[\nabla_t^2 + (\beta^2 + k^2)]\, \varphi_t(x,y) = -2\pi\rho_t(x,y)$

$[\nabla_t^2 + (\beta^2 + k^2)]\, A_{zt}(x,y) = 0$

Transverse boundary conditions:

$\varphi_t(\mathbf{x}_1) - \varphi_t(\mathbf{x}_2) = K$

$A_{zt}(\mathbf{x}_1) - A_{zt}(\mathbf{x}_2) = K$

Longitudinal equations:

$$[\ \nabla_z^2 - k^2\ ]\ q(z) = 0$$

$$[\ \nabla_z^2 - k^2\ ]\ i(z) = 0$$

$$\beta^2 = \omega^2\ \mu\xi$$

$$\beta_d^2 \approx \omega^2\ \mu\varepsilon\ = \omega^2/\ v^2$$

$$\beta_c^2 \approx -j(2/\delta^2)\ ,\ \delta = \sqrt{2/\omega\mu\sigma}$$

(5.2.26)

Comments: All of the above separation stuff is now in the Chapter 5 rewrite. I changed k^2 to $-k^2$, and ρ_t continues to be α. I seem be distinquishing β and β_d here as both being dielectric properties which I don't think I have done elsewhere. I do think my new presentation is clearer, but I basically had it right in the original doc.

5.3 A Formal Solution to the Transverse Problem

The eigenvalue problem.

Wait a minute. We really have a *Dirichlet* problem that I can solve with a Green's Function. Where is this eigenvalue thing coming from?

In either the dielectric or the conductor, both φ and all Cartesian components of **A** satisfy the same transverse Helmholtz equation,

$$[\ \frac{\partial^2}{\partial x^2}\ +\ \frac{\partial^2}{\partial y^2}\ +(\beta^2+ k^2)\ ]\ f = 0 \qquad (5.3.1)$$

The fact that there exists a surface charge affects the φ equation only at the surface. We know that for a reasonably low loss transmission line, $k^2 \approx -\beta_d^2$ ($k \approx +j\beta_d$) which is a "small" quantity, especially compared to β^2 in the conductors. The amount by which k^2 differs from $-\beta_d^2$ is what we seek to find. We will have an eigenvalue problem for k^2. [that is in fact what Matick does in his stripline thing]

The longitudinal equation tells us that (wave travels in +z direction)

$$\frac{\partial f}{\partial z}\ = -\ kf. \qquad (5.3.2)$$

It frequently happens in this kind of problem that things contrive to make solutions f exist only for certain specific values of the eigenvalue k^2. In each region, the solution must have a certain type of behavior, and then at the boundaries, various quantities must be continuous. These conditions are so stringent that only specific values of k^2 permit any solution at all.

[When I wrote the above, I was thinking of particle in a 1D box in QM and quantization of energies.

The parameter β is vastly different in these two media, as shown in the box (5.2.26). [true]

In the dielectric, β^2 is mostly real, with a slight imaginary part due to the small conductivity σ of the dielectric. The solution f in this region will therefore tend to be oscillatory, since then second derivatives

create negative contributions to cancel the positive β^2 contribution in (5.3.1). The solution for the lowest eigenvalue will have an extremely mild oscillatory behavior with no nodes. In fact, the oscillation is so mild that the main fields are approximately constant across the dielectric.

I seem to be examining the larger problem of dielectric + conductors as if this were a Saxon 1D QM wavefunction problem for particle in a box where we have oscillation in the box and expo in the tails, requiring that φ and $\partial_x\varphi$ be continuous does in that case create eigenvalues. But I don't think this analysis is necessary....

Inside the conductor, σ is huge, so β^2 becomes large and negative imaginary,

$$\beta^2 = -j\omega\mu\,\sigma = -j(2/\delta^2) \tag{5.3.3}$$

In a 1-dimensional problem if one has an equation of the form $[\partial^2/\partial x^2 - j(2/\delta^2)]f = 0$, the solution must look like,

$$\exp(\pm\sqrt{j}(\sqrt{2}/\delta)\,x) = \exp[\pm(1+j)\,x/\delta] \tag{5.3.4}$$

If we interpret x as going into the surface of the metal, we see that in addition to a strong oscillation, there is a strong exponential growth or decay. Exponential growth is unphysical, so we must select the minus sign. This is the well-known skin effect which we have already encountered several times. [true]

So, we are solving for $f = \varphi$ and $\mathbf{A}$ and we know that in the dielectric f is doing some reasonable pattern, and in the conductors f is decaying exponentially to 0. The boundary conditions can be expressed in several ways, one of which is to say that various components of the **E** and **B** fields which derive from φ and **A** must be *continuous* at each dielectric-conductor interface. In particular, E_z and H_φ must be continuous at each boundary.

In terms of our two main potentials, these continuity requirements are,

$$\begin{aligned} E_z(x,y,z) &= -k\varphi - j\omega A_z \\ &= -k(1/2\pi\varepsilon)\,q(z)\,\varphi_t(x,y) - j\omega(\mu/2\pi)\,i(z)\,A_{zt}(x,y) = \text{continuous} \end{aligned}$$

$$\begin{aligned} H_\varphi(x,y,z) &= (1/\mu)\,B_\varphi \\ &= (1/\mu)(-\partial_r A_z) = (i(z)/2\pi)(-\partial_r A_{zt}(x,y)) = \text{continuous} \quad [\text{ ok I guess}] \end{aligned} \tag{5.3.5}$$

The first line is satisfied if $\varphi_t(x,y)/\varepsilon$ and $A_{zt}(x,y)/\mu$ are continuous, while the second line requires that $\partial_r A_{zt}(x,y)$ be continuous, where ∂_r means the normal derivative at the surface. Here we have continued to neglect other components of **A**, and other components of the E and B fields. Note that $\partial_r\varphi$ is definitely not continuous due to the charge density on the surface.

In any event, as noted, when the dust settles, the eigenvalues for k^2 emerge. [and how is that?]

The lowest eigenvalue will have the smoothest behavior of A_{zt} between the conductors with no nodes, and this is the behavior that describes our TEM mode. Higher eigenvalues have increasing numbers of

nodes (that is, more waves) in the transverse direction between the conductors. These are the waveguide modes discussed in Appendix 3.2. [an interesting claim, I doubt it is true.]

When this lowest eigenvalue is found for k^2 by the above analysis, we then know $\mathbf{zy} = k^2$. Since we are in the frequency domain, we expect that $k^2 = \mathbf{zy}$ will be a function of ω. Knowledge of **zy** then allows an analysis of longitudinal waves down the transmission line. [well, OK...]

The interested reader will find in Matick (Sections 4.5 and 4.8) samples of the above eigenvalue method applied to stripline, the simplest of all possible geometries. Since the eigenvalue k^2 is directly related to the internal impedance Z_i of the transmission line, according to (5.2.23), these discussions are also discussions of the internal impedance problem. [yes, it is all true, Matick does it this way, I just checked]

Determination of the line parameters .

When the eigenvalue problem noted above is solved, we end up with a number for $k^2 = \mathbf{zy}$, and a functional form for $A_{zt}(x,y)$ and $\varphi_t(x,y)$. By evaluating either of these between the two surfaces, we learn, according to (5.2.25), the magic number K, and this in turn determines the three line parameters according to,

$$L_e = (\mu/2\pi)\,K \qquad C = 2\pi\varepsilon/K \qquad G = 2\pi\sigma/K \tag{5.3.6}$$

where μ, ε and σ refer to properties of the dielectric. [ok, you could do it this way...]

Equality of potentials in the dielectric.

In the dielectric we notice that $A_{zt}(x,y)$ and $\varphi_t(x,y)$ obey the *same* transverse Helmholtz equation, and they have the *same* boundary conditions, namely, constant on each conductor surface, and the difference between the two surfaces must be K. Therefore we may conclude that:

$$\varphi_t(x,y) = A_{zt}(x,y) \quad \text{in the dielectric} \qquad [\text{ maybe true }] \tag{5.3.7}$$

This is true to the extent that A_z really is constant on the boundaries. We know from the various "proofs" of A_z = constant given in Section 3.8 for Fact 7 that certain approximations are involved, such as the neglect of terms associated with other components of **A**, or such as the neglect of B_z. In fact, A_z deviates a small amount from being constant on the boundary of each conductor cross section. This becomes clear when one evaluates the electric field at the boundary:

$$E_z(x,y,z) = -k\varphi(x,y,z) - j\omega A_z(x,y,z) \qquad x,y \text{ on boundary} \tag{5.3.8}$$

For a perfect conductor, $E_z = 0$ on the boundary, and the two terms cancel. For a real conductor, the two terms do not cancel, and we get a small difference $E_z(x,y,z)$. We know that $E_z(x,y,z)$ can have some variation on the conductor boundary because $E_z = J_z/\sigma$ and J_z has variation due to the non-uniformity of the current distribution in the conductors, most noticeable when they are fat and closely spaced. Since φ = constant on the boundary, the only way we can have $E_z(x,y,z)$ vary on the boundary is if $A_z(x,y,z)$ varies.

This variation of $E_z(x,y,z)$ corresponds to a variation of the surface impedance of the conductor at different points on the surface. [sounds like BS]

Although the variation of $A_z(x,y,z)$ on the boundary is miniscule, we have in (5.3.8) that E_z is the difference between two relatively large quantities which very nearly cancel out. Thus, the very small variation in $A_z(x,y,z)$ on a boundary can become a large variation in E_z..

Comparison of potentials inside the conductors.

Although $\varphi_t(x,y)$ and $A_{zt}(x,y)$ are nearly identical in the dielectric, they are grossly different inside the conductors. The reason is that the surface charge density $\rho_t(x,y)$ affects $\varphi_t(x,y)$ but not $A_{zt}(x,y)$, see the transverse equations in box (5.2.26) above. We have seen already that $\partial_r A_{zt}(x,y)$ is continuous through the boundary, whereas $\partial_r \varphi_t(x,y)$ has a large discontinuity due to the surface charge.

As noted in Section 3.2, the surface charge on the conductors is for all practical purposes infinitely thin. Table 3 of Section 3.6 shows that outside this charge layer, the radial electric fields can be very large. Inside the charge layer, there is some very small radial electric field E_r which goes with the radial current J_r which supplies the charge layer. The potential $\varphi_t(x,y)$ has the same value on both sides of the surface charge layer and is constant over the surface, Section 3.8 Fact 6. [I am just flinging words here.]

Rubber sheeting of fields and potentials inside the conductors

Inside the conductors, the general behavior of all potentials and fields depends on the frequency ω, which is to say, the behavior depends on the size of the skin depth δ relative to the cross sectional dimensions of the conductors.

For high frequency and small skin depth, all potentials and fields have the same characteristic behavior. They all "rubber-sheet" down to zero as you move away from the boundary toward the interior, and they do so exponentially right in the skin depth layer. The reason for this is that all fields and potentials satisfy the Helmholtz equation with $\beta^2 = -j(2/\delta^2)$.

As one moves out of this limit to lower frequencies, things change and the pattern of fields and potentials is frequency dependent. At very low frequency, the field E_z exists everywhere inside the conductors, although it can be non-uniform. The azimuthal and radial components of **A**, which were neglected in the dielectric, now become important. This fact becomes clear when one considers what happens to div**A** = $-j(\beta^2/\omega)\varphi$ as one passes through a boundary. Whereas A_z and φ are continuous, β^2 undergoes a dramatic change, so the other terms in div**A** must be significant. [probably all true, but seems irrelevant]

Nature of the surface charge density.

The radial electric field E_r is much larger outside the conductor surface than inside. In fact, in Section 3.6 Table 3 we argued that the exterior E_r field is typically 1 million times larger than the interior field. By placing a gaussian box at the surface, using Maxwell (1.1.3), and ignoring the interior E_r field, one finds this expression for the surface charge density n:

$$n(x,y,z) = \varepsilon E_r = -\varepsilon \partial_r \varphi = -(1/2\pi)\, q(z)\, \partial_r \varphi_t(x,y) \qquad (5.3.9)$$

Thus, the surface charge density is completely determined by the radial electric field at the surface, which in turn is controlled by the radial gradient of the transverse potential at the surface. We expect that the surface charge density so obtained can be highly non-uniform over the conductor surfaces. The charge should be larger where the conductor surfaces have their closest approach, and the E fields are largest. If the conductors are large and very closely spaced, the non-uniformity of n over the surface should be quite dramatic. This leads to the non-uniformity in J_z which has been noted several times. [irrelevant]

Self Consistent Currents.

We have noted above the fact that the potentials penetrate into the surfaces of non-perfect conductors and create $E_z(z)$ and $J_z(z)$ in the conductors. As noted in Section 5.1 above, the currents in the wires are not externally applied, they just fall out as part of the self-consistent solution. If the skin depth is small compared to the conductor thickness (diameter), the current in the conductors only flows in a small region near the surface of the conductor. This result was not obvious from the approach taken in Chapter 4, where we thought of the current as more or less uniformly spread over the cross section of a conductor, since we imagined that the "external driving source" made the current this way. Here we see that current is really due to the potentials and fields, and is thus a surface effect.

This, then, is the formal solution of the transverse problem. We shall not attempt this program, but the point should be clear that the problem and the solution are well defined. With a computer, one could solve an arbitrary cross section transmission line in this manner, to any degree of accuracy desired.

[this is all horrible! I am just doing due diligence reading it on 1.9.14]

Caveat on Accuracy

We should not let this presumed perfect accuracy make us lose sight of the "transmission line limit" of Chapter 4 which is always operative. As suggested by the series in (4.1.13), there are really higher order terms in the transmission line equations (4.5.9) which we ignore because we assume β is small. By doing a "perfect" solution of the transverse problem, we get "perfect" values for the first order coefficients (line parameters) which appear in (4.5.9). But we still have to assume small β if we want to ignore the higher order terms such as one proportional to i"(z).

The small-β corrections ($\beta = 2\pi/\lambda$) are of order β^2, as claimed in Section 4.2, and as shown in Appendix 4.1. Thus, for example, we expect our overall accuracy to be on the order of $(d/\lambda)^2$, where d is the largest transverse dimension of our transmission line. For example, if d = 0.5 cm for some coaxial line, and λ = 1 meter, we expect our accuracy to be roughly 1 part in 10^5, which is pretty good. Note that λ is the wavelength in the dielectric, and so is affected by the dielectric constant. [ok ok]

5.4 An approximate solution to the transverse problem.

In Section 5.3 we described a "formal solution" of a transmission line which required the solution of the transverse Helmholtz potential equations. Once the lowest eigenvalue k^2 is found, along with the

associated solutions for the potentials, we could compute everything, including the surface impedance of the transmission line.

Since this eigenvalue problem is hard to solve, we describe here an approximation method that gives reasonable results. Only the dielectric part of the problem is treated, and the conductors are at first ignored, then later their influence is approximately included through estimates of their surface impedance.

Notice that the factor $(\beta^2 + k^2)$ appears in the transverse equations for $A_{zt}(x,y)$ and $\varphi_t(x,y)$. From 5.2 (23) we know that

$$k^2 + \beta^2 = j\omega Z_i C' \qquad (5.4.1)$$

where Z_i is the total surface impedance of both conductors. If we are willing to assume that these conductors are "perfect", then $Z_i \approx 0$, so $(\beta^2 + k^2) \approx 0$ as well. The degree of "perfection" can be measured using our "low loss" condition (5.2.24) which says Z_i must be less than $\mu_0 f K$.

Then we have the following approximated transverse equations from (5.2.26),

$$\nabla_t^2 \varphi_t(x,y) = -2\pi\rho_t(x,y) \qquad (5.4.2)$$
$$\nabla_t^2 A_{zt}(x,y) = 0$$

[I am now approximating Helmholtz equation as a Laplace equation]

and we know that surface charge $\rho_t(x,y)$ exists only on the conductor surfaces. In the dielectric, both equations have the same form, the same boundary conditions, and in fact the same solution, as already noted in Section 5.3. Thus, we set,

$$f(x,y) = \varphi_t(x,y) = A_{zt}(x,y) \qquad \text{/ in the dielectric} \qquad (5.4.3)$$

Then f(x,y) must satisfy the following equation and boundary conditions:

$$\nabla_t^2 f(x,y) = 0 \qquad \text{(2D Laplace)} \qquad (5.4.4)$$

$$f(x,y) = \text{constant on each conductor surface slice} \qquad (5.4.5)$$

The plan is then to solve the 2D Laplace equation for f(x,y) subject to the condition of constancy on the two conductor cross sectional surfaces. When this has been done, we can evaluate the geometric factor K from,

$$f(\mathbf{x}_1) - f(\mathbf{x}_2) = K \qquad \text{[I will have to check this]} \qquad (5.4.6)$$

and we then have good estimates for the line parameters L_e, C and G which are functions of K.

What we do *not* get by this approximation method is an estimate for the surface impedance, since we assumed it was zero. We must therefore estimate Z_i by some independent method. For round wires, a method was presented in Chapter 2. For other standard conductor shapes, see Matick Chapter 4.

An estimate of error in the above approximation.

Once we have an estimate for Z_i, we can go back and see how good (or bad) our approximation was for the potential f(x,y) and hence for K. Plugging this Z_i into (5.4.1), we get an estimate for the size of the quantity (β^2+k^2) which we assumed was zero in our first solution (in the dielectric).

Basically this involves doing perturbation theory on the solution of a differential equation. One needs to identify a dimensionless smallness parameter s and expand the solution as a power series of functions weighted with powers of s. The solution is highly convergent if s is small. In fact, one can use s itself as a measure of the size of the correction terms to the basic solution we have found above.

In our case, the dimensionless smallness parameter is

$$s = \left| \frac{\beta^2 + k^2}{\beta^2} \right| = \frac{2\pi|Z_i|}{\mu\omega K} = \frac{|Z_i|}{\omega L_e} = \pi\sigma\delta^2 |Z_i|/K \qquad (5.4.7)$$

Beware: since β^2 appearing above is in the dielectric, the symbols σ and δ refer to the dielectric.

As an example of how small s might be, consider our two-wire transmission line solution given in the box (4.6.13), with $a_1 = a_2 = a$. Using the round wire estimates, we found in the skin depth limit that,

$$|Z_i| = \sqrt{2}\ R_i = \sqrt{2}\ / (\pi\sigma'\delta'a) \qquad (5.4.8)$$

$$K = 2 \ln(b/a)$$

where primed quantities refer to parameters of the conductor. Thus we get,

$$s = (\sigma/\sigma')\ (\delta/\delta')^2\ (\delta'/a)(\sqrt{2}\ / K) \approx (\delta'/a)(\sqrt{2}/K) \qquad (5.4.9)$$

where we assume that $\mu \approx \mu'$. Since we have already assumed the skin depth limit $(\delta'/a) \ll 1$, we have $s \ll 1$, and our approximation is a good one. We expect any fractional errors to be on the order of s .

Next, we consider the low-frequency limit. In this case we have,

$$|Z_i| \approx R_i = (2/\pi\sigma'a^2) \qquad (5.4.10)$$

so that,

$$s = (2/K)(\sigma/\sigma')(\delta/a)^2 \approx (2/K)\ (\delta'/a)^2 = (\delta'/a)^2\ / \ln(b/a) \qquad (5.4.11)$$

This suggests that at very low frequencies, such that s ~ 1, our approximation is no good. One way to say this is that a very low frequencies, the inductive reactance ωL_e of the transmission line per unit length is no longer much larger than the resistive losses R_i in the conductors.

When we set $(\beta^2 + k^2) \approx 0$ in the transverse equation for f(x,y), the equation becomes the Laplace equation which has a very smooth solution which just meets the boundary conditions in the smoothest possible fashion. When $(\beta^2 + k^2)$ becomes larger, then the transverse solution of **** starts becoming oscillatory in the dielectric. The solution must acquire a higher transverse curvature to cancel out the $(\beta^2 + k^2)f$ term. The solutions for the potentials and fields then no longer match our intuitive notion of a simple transmission line, and in fact become more like those of a waveguide.

It is a characteristic of the Helmholtz equation that when things become oscillatory in x and y, they become exponential in z. Thus, one is not surprised to find that in the low [high!?] frequency limit, the transmission line starts developing a severe attenuation per wavelength. As shown in (5.2.23), we have:

$$k^2 = \beta^2 \left[\frac{jZ_i}{\omega L_e} - 1 \right] \qquad \text{huh?} \qquad (5.4.12)$$

and waves propagate down the transmission line as e^{-kx}. When $Z_i \approx R_i$ at low frequencies, and $R_i/\omega L_e$ is large, we get:

$$k \approx (2\pi/\lambda) \sqrt{R_i/\omega L_e} \quad (1 + j) / \sqrt{2} \qquad (5.4.13)$$

In this case, waves are attenuated over a distance of $\lambda / \sqrt{R_i/\omega L_e}$. If $\sqrt{R_i/\omega L_e} \sim 1$, then our wave goes about one wavelength on the transmission line and dies.

Power transmission lines approach this low frequency regime. As shown in Section 2.2, the skin depth in copper is about 0.85 cm at 60 Hz, 1.09 cm for aluminum. For wires of diameter 1 cm, separated a distance of 1 m, (11) becomes

$$s \approx (\delta'/a)^2 / \ln(b/a) \approx 4/ \ln(200) = 0.75$$

Thus, we would expect significant attenuation over one wavelength of such a power transmission line. Since a wavelength is about 5000 km at 60 Hz, this is presumably acceptable.

A similar result obtains for Belden 8281 coaxial cable in the range of 10-100 KHz.

[OK, I have read it all, and it is now going to get tossed out!]

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Chapter 6: An Example

In this Chapter we use the method outlined in Section 5.4 to solve for the parameters of a transmission line consisting of two round wires of radius a_1 and a_2 whose center lines are separated by a distance b.

The geometry is as shown in Section 4.6 Fig 1. The big difference here is that we put no restrictions on the size of b relative to the two radii. The methods of Section 4.6 cannot be used in this case, since the charge distribution becomes non-symmetric on each wire.

To review, our problem then is to solve this equation,

$\nabla_t^2 f(x,y) = 0$ (2D Laplace)

such that

f(x,y) = constant on each circular conductor slice

When this has been done, we can evaluate the geometric factor K from,

$f(\mathbf{x_1}) - f(\mathbf{x_2}) = K$

and we then have good estimates for the line parameters L_e, C and G which are functions of K.

[K in this old version differs from my new K by a factor of 2.]

First we will show that logarithmic functions are the natural solutions of the 2D Laplace equation, and then we will find combinations of such logarithms which have circles as equipotentials. The final step will be to line up two of these circles with the boundaries of our round wires.

6.1 Why logarithms?

In 3D space, it is well known that the potential of a point charge is $(1/4\pi\varepsilon)\, q/r$. This potential must satisfy the 3D Laplace equation. That it does so is very easily shown in 3D spherical coordinates centered at the point charge. Since the potential has no dependence on θ and φ, only the radial portion of the Laplacian matters. In this case we have,

$\nabla^2 f(r) = (1/r^2)\, \partial_r [r^2\ \partial_r f(r)]$

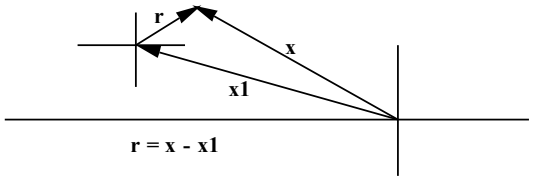
With f(r) = 1/r, the inner bracket [..] becomes -1, and $\partial_r[-1] = 0$, so $\nabla^2(1/r) = 0$ for $r \neq 0$.

In 2D space, a similar thing happens, except in 2D, the potential of a point charge behaves as ln(r) instead of (1/r). In a manner similar to the above, we can prove quickly that ln(r) solves the 2D Laplace equation. This time, we use the transverse Laplacian in cylindrical coordinates, and we ignore the longitudinal z coordinate. Since ln(r) does not depend on θ, only the radial part of the Laplacian survives, so we get

$\nabla_t^2 f(r) = (1/r)\, \partial_r [\, r\, \partial_r f(r)]$

With f(r) = ln(r), the inner bracket [..] becomes 1, and $\partial_r[1] = 0$, so $\nabla_t^2 \ln(r) = 0$ for $r \neq 0$.

If we now shift the origin of our cylindrical transverse coordinate system by some arbitrary 2D vector $\mathbf{x}_1$, then the function that was ln(r) becomes ln $|\mathbf{x} - \mathbf{x}_1|$. Here is a drawing to support this claim. Things are viewed along the z axis, the vectors lie in the plane of paper.



The original coordinate system is on the upper left, the new one is on the lower right. Changing the origin of a coordinate system cannot change the fact that a function satisfies the Laplace equation, so we may conclude without further ado that:

Fact: The function f(x,y) = ln s_1 where $s_1 = |\mathbf{x} - \mathbf{x}_1|$ is a solution of $\nabla_t^2 f(x,y) = 0$. This is true for any 2D vector $\mathbf{x}_1$. Of course any linear combination of such functions like A ln s_1 + B ln s_2 also solves the 2D Laplace equation, where $s_1 = |\mathbf{x} - \mathbf{x}_1|$ and $s_2 = |\mathbf{x} - \mathbf{x}_2|$. If A = -B, then we find that the function B ln(s_2/s_1) is a solution.

Comment: The above discussion hopefully gives some insight as to why the ln(s) factors keep appearing in our Chapter 4 equations, such as (4.2.5). The integral expression for V(z) came from the difference of the potential φ between two conductors. As noted in equation (5.2.10), in the transmission line limit, this φ can be written as an integral involving ln(s) ,

$$\varphi_{12}(x,y,z) = \frac{2}{4\pi\varepsilon}\ q(z)\ \{\ \int_{C_1} dx'\ dy'\ a_1(x',y')\ \ln(s) - \int_{C_2} dx'\ dy'\ a_2(x',y')\ \ln(s)\ \}$$

where the dependence on x,y is contained in $s = \sqrt{(x-x')^2 + (y-y')^2}$. This potential must satisfy the transverse Laplace equation. Application of the 2D Laplacian to both sides gives,

$$\nabla_t^2\ \varphi_{12}(x,y,z) = \frac{2}{4\pi\varepsilon}\ q(z)\ \{\ \int_{C_1} dx'\ dy'\ a_1(x',y')\ \nabla_t^2 \ln(s) - \int_{C_2} dx'\ dy'\ a_2(x',y')\ \nabla_t^2\ \ln(s)\ \} = 0$$

The result is 0 since $\nabla_t^2\ \ln(s) = 0$, as noted in the above Fact.

6.2 The equipotentials of ln[s_2/s_1] are circles.

In light of Section 6.1, we know that the following function satisfies the 2D Laplace equation:

$$f(x,y) = \ln[s_2/s_1] \qquad (6.2.1)$$

where $s_1 = |\mathbf{x} - \mathbf{x}_1|$ and $s_2 = |\mathbf{x} - \mathbf{x}_2|$, and $\mathbf{x}_1$ and $\mathbf{x}_2$ are arbitrary points in 2D space. Consider now the surface defined by

$$\ln[s_2/s_1] = B, \qquad (6.2.2)$$

where B is a constant. This means that $s_2/s_1 = e^B$ or $(s_2)^2 = e^{2B} (s_1)^2$. Thus,

$$[(x-x_2)^2 + (y-y_2)^2] = e^{2B} [(x-x_1)^2 + (y-y_1)^2] \quad (6.2.3)$$

Since this is a quadratic form in which the coefficient of x^2 is the same as the coefficient of y^2, this must represent a circle. Assuming then the form,

$$(x - x_c)^2 + (y - y_c)^2 = r^2 \quad (6.2.4)$$

where (x_c, y_c) is the location of the circle center, and r is the radius, we find these results by multiplying out the terms in (6.2.3) and completing the two squares,

$$x_c = (x_2 - kx_1) / (1-k) \quad (6.2.5)$$

$$y_c = (y_2 - ky_1) / (1-k) \quad (6.2.6)$$

$$r^2 = [k(x_1^2 + y_1^2) - (x_2^2 + y_2^2)]/(1-k) + x_c^2 + y_c^2 \quad (6.2.7)$$

where we have defined,

$$k = e^{2B} \quad (6.2.8)$$

6.3 Aligning the circles.

Since the equipotential surfaces of $f(x,y) = \ln[s_2/s_1]$ are circles, the problem remains to cause two of these circles to align with the boundaries of our two wires. If we put the centers of the two wires at y=0, we see from (6.2.6) that we should select $y_1 = y_2 = 0$. This causes all equipotential circles to be centered at y=0.

What values should be used for x_1 and x_2? Without loss of generality, we write

$$x_1 = D + d \qquad x_2 = D - d \quad (6.3.1)$$

Equations (6.2.5) and (6.2.7) for the circle center and radius then become

$$x_c = D + d\,\frac{k+1}{k-1} = D + d\coth(B) \quad (6.3.2)$$

$$r = d\sqrt{\left(\frac{k+1}{k-1}\right)^2 - 1} = d\sqrt{\coth^2(B) - 1} = d\,|\mathrm{csch}(B)| \quad (6.3.3)$$

The hyperbolic function expressions in terms of B follow directly from (8). Here are some crude plots of these two hyperbolics:

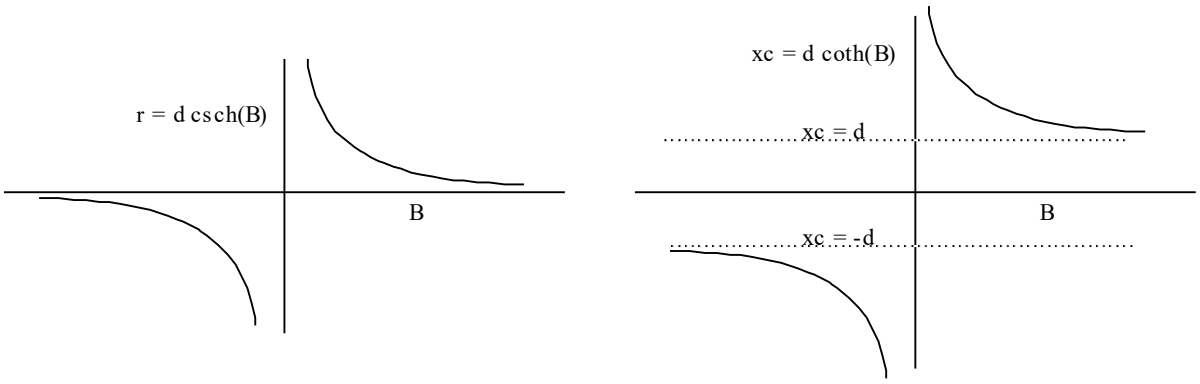


Fig 1: Plots of the radius and x-center of equipotential circles versus B.

It is now extremely helpful to make a plot of the family of circles that arises as the constant B is varied. We have already shown in Section 6.2 that each B corresponds to a circle, so we will now learn where all the circles lie.

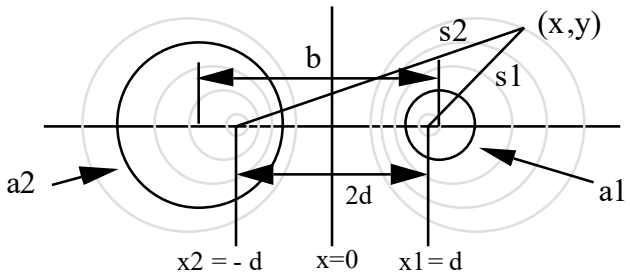


Fig 2: Plots of equipotential circles for various values of B (drawn for D=0).

The circles on the right correspond to positive B. As $B \to \infty$, Fig 1 shows that $r \to 0$ and $x_c \to d$. Thus, the circles shrink down around the point x=d on the right. For negative B, the same thing happens on the left. For B = 0, the circle is the plane at x=0. Apollonius?

In Fig 2 we have indicated in heavy ink where our two wires might lie. The one on the left has radius a_2 and that on the right has radius a_1. Notice that the centers of the circles move out from x = |d| as the circles get larger. The distance between the centers of our wires is b. As the figure shows, $b \geq 2d$.

We are now ready to require an alignment of the equipotential circles shown in Fig 2 with the cross sections of our two wires. The constraints are as follows:

$a_1 = d\ \mathrm{csch}(B_1)$ / radius of right circle is a_1 ($B_1 > 0$) (6.3.4)

$a_2 = d\ \mathrm{csch}(-B_2)$ / radius of left circle is a_2 ($B_2 < 0$) (6.3.5)

$b = d\ [\ \coth(B_1) - \coth(B_2)\ ]$ / distance between centers is b (6.3.6)

Notice that the constant D which appears in (6.3.1) and (6.3.2) does not appear in the above three equations. It cancels out in (6.3.6), and has absolutely no bearing on the problem. We therefore set D = 0. This corresponds to the x=0 plane being as drawn in Fig 2 above.

So, we now have three equations in three unknowns, B_1, B_2, and d. Inserting (6.3.4) and (6.3.5) into (6.3.6) and squaring twice gives us an expression for d as a function of a_1, a_2, b:

$$d = (1/2b)\sqrt{b^4 + (a_1^2 - a_2^2)^2 - 2b^2(a_1^2 + a_2^2)} \qquad (6.3.7)$$

[I never verified this result in my new version because it was not needed.]

The potentials on the two wire surfaces are given by

$$B_1 = f(\mathbf{x}_1) = \operatorname{csch}^{-1}(a_1/d) = \ln\left[(d/a_1) + \sqrt{(d/a_1)^2 + 1}\right] \qquad (6.3.8)$$

$$B_2 = f(\mathbf{x}_2) = -\operatorname{csch}^{-1}(a_2/d) = -\ln\left[(d/a_2) + \sqrt{(d/a_2)^2 + 1}\right] \qquad (6.3.9)$$

And the potential f(x,y) at all points in space between the conductors is given by,

$$f(x,y) = \ln(s_2/s_1) = \ln\left[\frac{(x+d)^2 + y^2}{(x-d)^2 + y^2}\right] \qquad (6.3.10)$$

Putting (6.3.4) and (6.3.5) into (6.3.2), we get these useful formulas for the circle center locations:

$$x_{c1} = d\coth(B_1) = d\sqrt{1 + (a_1/d)^2}$$

$$x_{c2} = d\coth(B_2) = -d\sqrt{1 + (a_2/d)^2} \qquad (6.3.11)$$

Finally, we arrive at our evaluation for K ,

$$K = f(\mathbf{x}_1) - f(\mathbf{x}_2) = B_1 - B_2 = \ln\left[(d/a_1) + \sqrt{(d/a_1)^2 + 1}\right] + \ln\left[(d/a_2) + \sqrt{(d/a_2)^2 + 1}\right] \qquad (6.3.12)$$

If we are interested in a transmission with one conductor contained inside the other,

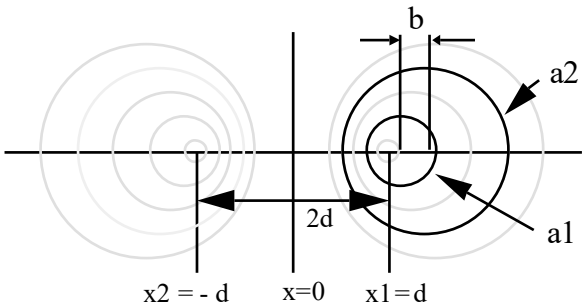


Fig 3: Situation when conductors are concentric.

then the above analysis still applies, except in this case B_2 is positive, so there is *no* minus sign in (6.3.9), and there *is* a minus sign before the second term in (6.2.12).

Here then is a summary box for our two wire transmission line.

General solution of two-wire transmission line :

$\varphi(x,y,z) = (1/2\pi\varepsilon)\, q(z)\, f(x,y)$ $\qquad$ $L_e = (\mu/2\pi)K$

$A_z(x,y,z) = (\mu/2\pi)\, i(z)\, f(x,y)$ $\qquad$ $C = 2\pi\varepsilon/K$

$G = 2\pi\sigma/K$

$$f(x,y) = \ln\left[\frac{(x+d)^2 + y^2}{(x-d)^2 + y^2}\right]$$

$$K = \ln\left[(d/a_1) + \sqrt{(d/a_1)^2 + 1}\right] \pm \ln\left[(d/a_2) + \sqrt{(d/a_2)^2 + 1}\right]$$

+ sign for conductors as in Figure 2 (separated)
– sign for conductors as in Figure 3 (concentric)

$$d = (1/2b)\sqrt{b^4 + (a_1^2 - a_2^2)^2 - 2b^2(a_1^2 + a_2^2)} \qquad (6.3.13)$$

6.4 Reduction to special cases

(a) Two wire line: $b >> a_1, a_2$

In the limit that $b >> a_1, a_2$ we find from the box (6.3.13) that,

$$d \approx b/2 \qquad (6.4.1)$$

$$K \approx \ln[\, b/a_1] + \ln[\, b/a_2] = \ln[\, b^2/(a_1a_2)] = 2\ln[b/\sqrt{a_1a_2}\,] \qquad (6.4.2)$$

and we are much relieved to find that we have duplicated the result (4.6.4). If we further assume that $a_1 = a_2 = a$, the result becomes,

$$K = 2\ln(b/a) \qquad (6.4.3)$$

Two wire line, $b>>a_1, a_2$

$K = 2\ln[\, b/\sqrt{a_1a_2}\,]$ $\qquad$ [new result is $K = 4\ln(b/\sqrt{a_1a_2}\,)$ so $K_{new} = 2K_{old}$]

$$L_e = (\mu/2\pi)K \qquad C = 2\pi\varepsilon/K \qquad G = 2\pi\sigma/K \tag{6.4.4}$$

(b) Twin Lead: $a_1 = a_2 = a$

In the case that $a_1 = a_2 = a$ with arbitrary b, we find that

$$d/a = (b/2a)\sqrt{1 - (2a/b)^2} \tag{6.4.5}$$

$$K = 2 \ln [\ (d/a) + \sqrt{(d/a)^2 + 1}\ ] \tag{6.4.6}$$

When (6.4.5) is inserted into (6.4.6), we find

$$K = 2 \ln (b'/a) \quad b' = b\,\frac{1 + \sqrt{1 - (2a/b)^2}}{2} \tag{6.4.7}$$

Comparison of (6.4.7) with (6.4.3) shows that the large b result applies at arbitrary b if b is replaced by the b' shown. Notice that b' < b. One can interpret this as saying that, when the conductors are close together, the charges and currents are offset in each conductor toward the other conductor, so the effective separation is less than the nominal b.

Two wire line, $a_1 = a_2 = a$

$$K = 2 \ln (b'/a) \qquad b' = b\,\frac{1 + \sqrt{1 - (2a/b)^2}}{2} \quad [\text{ probably OK for } K_{old}\ ]$$

$$L_e = (\mu/2\pi)K \qquad C = 2\pi\varepsilon/K \qquad G = 2\pi\sigma/K \tag{6.4.8}$$

(c) Off center coax: $b \ll a_2$

Look at Figure 3 in the previous section. Here we are talking about a coaxial cable where a_2 refers to the radius of the outer conductor, and b is the offset between the conductor center lines. Normally b=0, but we are interested here to see how a manufacturing irregularity might affect the coaxial line properties. Let us define these dimensionless quantities:

$$\alpha = (b/a_2) \qquad \beta = (a_2/a_1) > 1 \tag{6.4.9}$$

From our general formula for d given in (6.3.13), we may write

$$d = a_1 f \tag{6.4.10}$$

where

$$2f \equiv (1/\alpha\beta)\sqrt{\alpha^4\beta^4 + (1 - \beta^2)^2 - 2\alpha^2\beta^2(1 + \beta^2)} \tag{6.4.11}$$

Then we get

$$(d/a_1) = f \qquad (d/a_2) = (f/\beta) \tag{6.4.12}$$

Inserting these into the formula (6.3.13) for K, we get

$$K = \ln(\beta) + \ln\left[\frac{f + \sqrt{f^2 + 1}}{f + \sqrt{f^2 + \beta^2}}\right] \tag{6.4.13}$$

Notice that $\ln(\beta) = \ln(a_2/a_1)$ is the usual result for a coaxial cable that is perfectly centered, as we obtained in (4.7.6), so the second term is the result of being off-centered.

So far we have been exact, now we make an approximation. We assume that $\alpha << 1$. In this case, looking at (6.4.11) above, we see that $f >> 1$. Then we can approximate the second term in (6.4.13) using standard methods, and we get this result

$$K \approx \ln(\beta) - \frac{(\beta^2 - 1)}{(2f)^2} \tag{6.4.14}$$

Again, from (6.4.11) we keep only the largest term inside the radical to get

$$2f \approx (1 - \beta^2)/\alpha\beta \tag{6.4.15}$$

Putting this into (6.4.14) gives

$$K \approx \ln(\beta) - \frac{\alpha^2}{[1 - \beta^{-2}]} = \ln(a_2/a_1) - \frac{(b/a_2)^2}{[1 - (a_1/a_2)^2]} \tag{6.4.16}$$

Example: Consider Belden 8281 coaxial cable. Here are the basic numbers

$$a_2 = 2510\mu \qquad a_1 = 394\mu \qquad \beta = a_2/a_1 = 6.37 \qquad \ln\beta = 1.85 \tag{6.4.17}$$

For these numbers, the denominator in the second term in (16) is essentially 1, so we get

$$K = 1.85 - (b/a_2)^2 \tag{6.4.18}$$

Suppose that the coaxial cable is off center by 20%, which seems a large amount, but an amount that could conceivably occur in very poor manufacturing. In this case $\alpha = b/a_2 = 0.2$, $\alpha^2 = .04$, so we get

$$K = 1.85 - 0.04 = 1.81$$

Therefore, even such a large 20% defect results in a change in K that is a mere 2%. With a 10% defect we get (1/2)% error in K. We conclude that coaxial cables are fairly immune to eccentricity variations!

By way of interpretation, notice that the correction term is negative. As the center wire moves off center, the capacitance of the cable increases, since K decreases. In the limit that the inner conductor almost touches the outer one, we have $a_1 + b = a_2$ which says $\alpha\beta = \beta - 1$. When this is put into (11) for f, we find that f = 0. Looking then at (13), we get the result that $K = \ln(\beta) + \ln(1/\beta) = 0$, so the capacitance is infinite.

Off center coaxial cable

b = distance between center lines $a_2 > a_1$

$$K = n(a_2/a_1) - \frac{(b/a_2)^2}{[1 - (a_1/a_2)^2]}$$ // this is an approximate result assuming various things

$$L_e = (\mu/2\pi)K \qquad C = 2\pi\varepsilon/K \qquad G = 2\pi\sigma/K \qquad (6.4.19)$$

(d) Wire of radius a a distance h above a ground plane.

This limit arises by taking $a_2 = \infty$ in Figure 3 so that circle a2 lines up with the plane x=0. In this case, x_{c1} = h, the height of the a_1 wire center above the ground plane. From 6.3 (11) we have then

$$h = d\sqrt{1 + (a_1/d)^2} \qquad (20)$$

or

$$h^2 = d^2 + a_1^2 \quad \text{and} \quad (d/a_1)^2 = (h/a_1)^2 - 1 \qquad (6.4.21)$$

From 6.3 (5), if $a_2 = \infty$, then $B_2 = 0$. Therefore the second logarithm in 6.3 (12) is zero, and we get

$$K = \ln\left[(d/a_1) + \sqrt{(d/a_1)^2 + 1} \right] \qquad (6.4.22)$$

Inserting (21) into (22) gives

$$K = \ln(h'/a_1), \quad h' = h\left(1 + \sqrt{1 - (a_1/h)^2} \right) \qquad (6.4.23)$$

This result has an easy interpretation. It represents one half of the twin lead situation discussed in (b) above. If we set h = b/2, then the result (23) represents 1/2 of the result shown in (8). Since K is half, the capacitance of a wire over a ground plane is twice that of the corresponding twin lead situation. It should

be clear that to the right of x=0, the potential and field lines in both cases are identical. And in the limit of a thin wire, the above becomes $K = \ln(h/a_1)$.

Wire over ground plane

h = height of center line over plane a = radius of wire

$K = \ln(h'/a)$, $h' = h\left(1 + \sqrt{1 - (a/h)^2}\right)$

$L_e = (\mu/2\pi)K$ $C = 2\pi\varepsilon/K$ $G = 2\pi\sigma/K$ (6.4.24)

// new version is MUCH better.