Tying in G PhL 9.22.14

I was having trouble showing that $(\sigma_d/\varepsilon_d)C = G$ in the general case. I finally got it figured out after very many flailing hours, and it is all present in the new Section 4.11. Notes below are therefore no longer really needed, but I will archive them off anyway. [Review done 9/23/14]

I have a loose bolt in the logic sequence of Chapter 4. Consider this text I am proposing to install in Section 4.4 after 4.4.8: [but I never did it that way.]

The problem is that I have not shown that $(\sigma_d/\varepsilon_d)C = G$, the TL conductance. But here is how you can do it:

At 3 PM Mon 9/22/14 I am stuck on this trivial problem I solved 20 years ago.

My physical definition of the conductance would be something like this

$i_G(z) = [Gdz]\, V(z)$ // amps

where dz is a small length of TL. Here $i_G(z)$ is the cross current for this dz segment. I could define

$i_s(z) = j\omega[q_s dz] = j\omega[CV(z)\, dz]$ = the surface-charge charging current //amps

$i(z)$ = current in conductor // amps

Then I guess

d $i(z)$ = (change in $i_c(z)$ over dz) $= - i_G(z) - i_s(z)$ // everything is amps

Then

$di(z) = -i_G(z) - i_s(z) = -[Gdz]\, V(z) - j\omega[q_s dz]$

and then

$\partial_z i(z) = -GV(z) - j\omega q_s = -GV(z) - j\omega CV(z) = - V(z)[G + j\omega C]$

This I have just derived one of the TL equations. !! But I have *still* not shown that $G = (\sigma_d/\varepsilon_d)C$.

I do know that $[\partial_z i(z)]dz$ is the loss of current over distance dz due to capacitance and conductance. Thus, I know that $\partial_z i(z) = - V(z)[G + j\omega C]$. This part does seem clear.

I do know that

$q_s = C\ V(z)$ $\quad\quad$ $j\omega q_s = i_s$ = current doing into the surface charge

$q_c = C'\ V(z)$ $\quad\quad$ $j\omega q_c = i_G + i_s$ = current going into both

$C'/C = q_c/q_s = (\xi_d/\varepsilon_d) = 1 - j\sigma_d/(\omega\varepsilon_d)$

$C' = (\xi_d/\varepsilon_d) = C - j\sigma_d C/(\omega\varepsilon_d)$

$j\omega C' = C\ (\sigma_d/\varepsilon_d) + j\omega C$

Then how about

$d\ i(z) =$ (change in $i_c(z)$ over dz) $= - i_G(z) - i_s(z) = - [\ i_G(z) + i_s(z)] = - j\omega q_c dz$

$= - j\omega\ C'\ V(z)dz$

$\partial_z i(z) = - j\omega\ C'\ V(z)$

and finally I have the connection.

I need a picture!!! I can start with Fig 4.13

$[i_c\ dz] = [i_s\ dz] + [i_G\ dz]$

$i_C = i_s + i_G$

$i_C(z) = i_s(z) + i_G(z)$ $\quad\quad$ C_1 $\quad$ C_2

[I did draw a new picture]

Below is how I resolved all the confusion about G and C. I decided to make this a whole new Chapter 4 subsection before the transmission line equations section. It is all done and installed, but below is the original work on this upgrade.

4.11 Relations involving C and G and the charges and currents in a transmission line

Before continuing our development of the transmission line equations, we need to establish the connections between parameters C and G and the charges and currents in a transmission line Then we resume the development in Section 4.12.

Consider this picture showing a section of a transmission line of length dz :

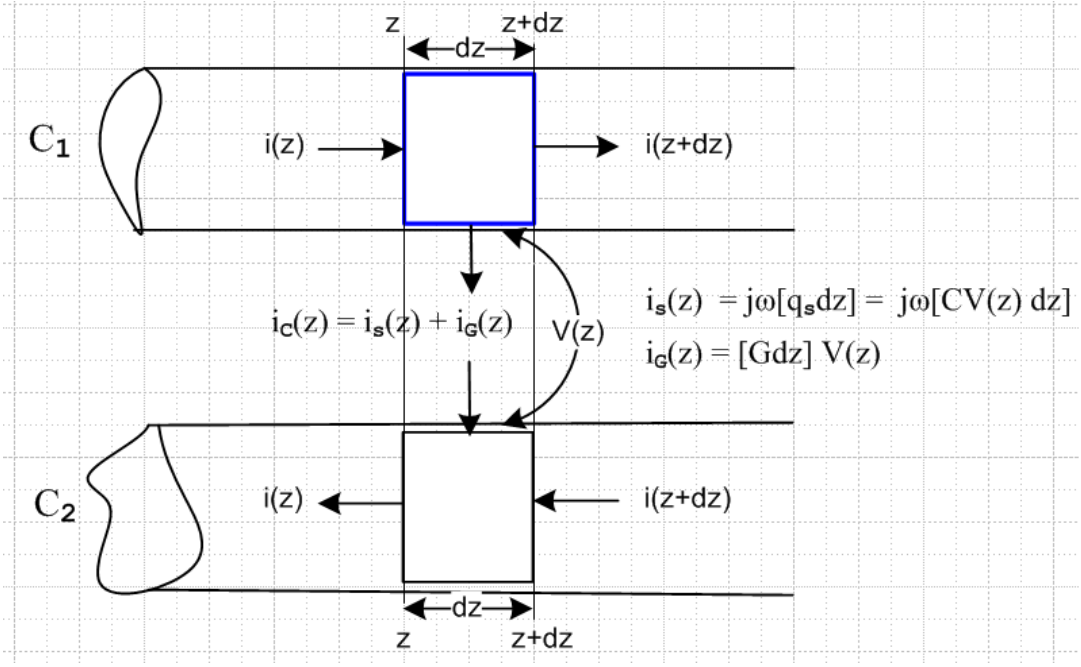


Fig 4.1

We focus on the upper conductor C_1. Over the distance dz, the current i(z) in this conductor is reduced by amount - di(z) = i(z) - i(z+dz) > 0 by the fact that current flows transversely to feed the surface charge q_s and to feed the conductance G.

Since the blue Gaussian box embedded just inside the upper conductor contains no free charge, we know from (1.1.35) that div **J** = 0 and $\int_S \mathbf{J} \bullet d\mathbf{S} = 0$. The latter means that the sum of all currents crossing the box boundary is 0. Thus,

$$- di(z) = i_G(z) + i_s(z) . \qquad \text{" current loss feeds capacitance and conductance"} \qquad (4.11.1)$$

The two currents may be written

$$i_G(z) = [Gdz]\, V(z) \qquad // \; i_G \text{ flows through the dielectric} \qquad (4.11.2)$$

$$i_s(z) = j\omega[q_s(z)dz] \qquad // \; i_s \text{ feeds the true surface charge per length } q_s \qquad (4.11.3)$$

The section of dielectric has some transverse resistance $R_t = 1/[Gdz]$ and then $i_G(z) = V(z)/R_t$. Quantity G is the conductance of the dielectric per unit length of the transmission line. The true total surface charge per length is q_s and i_s feeds this charge according to $i_s(z) = j\omega[q_s(z)dz]$ ($i_s = \partial_t[q_s dz]$ in the time domain). But we know for the dz-length capacitor that $q_s(z) = CV(z)$ where C is the capacitance per length. Then (4.11.1) can be written

$$- di(z) = i_G(z) + i_s(z) = [Gdz]\, V(z) + j\omega[CV(z)dz] = [\, G + j\omega C \,]\, V(z)dz \quad . \qquad (4.11.4)$$

Dividing by dz and taking dz→0 then gives,

$$\partial_z i(z) = -\ [\ G + j\omega C\ ]\ V(z) \qquad (4.11.5)$$

This is in fact one of the "two transmission line equations" we shall be deriving below.

Quantity $q_s(z)$ is the true surface charge (per length) at location z. Corresponding to this charge is the so-called transport charge (per unit length) $q_c(z)$, where

$$q_c(z) = (\xi_d/\varepsilon_d) q_s(z) \qquad (4.11.6)$$

as shown in (1.5.17). Here q_s and q_c are the integrals of the corresponding surface charge densities n_s and q_s over the surface of the C_1 conductor section shown in Fig **. The transport charge density q_c is larger than q_s because it accounts for both the surface charge and the charge lost due to leakage into the dielectric, as described in Section 1.5. The two charges are related to the real and complex capacitances in this manner

$$q_s(z) = C\ V(z) \qquad C = \text{real capacitance (per length)} \qquad (4.11.7)$$
$$q_c(z) = C'V(z) \qquad C' = \text{complex capacitance including effect of G (per length)} \qquad (4.11.8)$$

where the second line really defines the complex capacitance C' . Therefore using (4.11.6) and (1.5.1c) for ξ_d,

$$C'/C\ = q_c(z)/q_s(z) = (\xi_d/\varepsilon_d)\ = [\varepsilon_d - j\sigma_d/\omega]/\ \varepsilon_d = 1 + (1/j\omega)\ (\sigma_d/\varepsilon_d)$$

or

$$C' = C + (C/j\omega)\ (\sigma_d/\varepsilon_d)$$

or

$$j\omega C' = C(\sigma_d/\varepsilon_d) + j\omega C\ . \qquad (4.11.9)$$

The current feeding the transport charge in the transmission line section of length dz is given by

$$i_c(z) = i_s(z) + i_G(z) \qquad (4.11.10)$$

where

$$i_c(z) = j\omega\ [q_c(z)dz]\ . \qquad //\ i_c = \partial_t[q_c dz]\ \text{ in the time domain} \qquad (4.11.11)$$

Using (4.11.8) this says

$$i_c(z) = j\omega\ [C'V(z)dz]\ . \qquad (4.11.12)$$

Then (4.11.10) and (4.11.4) imply

$$j\omega\ [C'V(z)dz]\ \ =\ [\ G + j\omega C\ ]\ V(z)dz \qquad (4.11.13)$$

so that

$$j\omega C' = G + j\omega C \ . \qquad (4.11.14)$$

Comparing this with (4.11.9) shows that

$$G = C(\sigma_d/\varepsilon_d) \qquad (4.11.15)$$

which is an interesting relationship between G and C for a transmission line with arbitrary conductor shapes. We saw this relationship just below (1.5.20) for the special case of a parallel plate capacitor.

I then added a bit more after the above in lines doc.
Below are just leftover notes.

Therefore

$$j\omega\,[q_c(z)dz] \ = i_s(z) + i_G(z) \ = [\ G + j\omega C]V(z)dz$$

According to ** we can define the complex capacitance as

$$C' = V(z)/q_c(z) =$$

$$- di(z) = i(z)-i(z+dz) \ > 0$$

Since there is no free charge inside the box, continuity says that div $\mathbf{J} = 0$ and $\int_S \mathbf{J} \bullet d\mathbf{S} = 0$ in integral form. Thus then says that

$$i(z) - i(z+dz) \ = \text{loss of current in section of length } dz \ = i_s + i_G$$

The current loss is due to current flowing into the capacitance of the section and into the conductance of the dielectric.

Here I show the transverse currents

$$= - [$$

I also know that

$$V(z) = q_c(z)\frac{1}{C'} = q_s(z)\frac{1}{C}$$

Then can write

$$\partial_z\, i(z) = -\,V(z)[G + j\omega C] = -\,q_c(z)\frac{1}{C'}\,[G + j\omega C]$$

Can I say this:

$$i(z) = j\omega q_c \;??$$

Dimensions are wrong. Why can I say that $i_s(z) = j\omega q_s$ but I cannot say $i_c(z) = j\omega q_c$?
Answer: i(z) is the total current, some of it feeds q_s, some of it feeds i_G, but some of it continues through the section. Thus, you cannot say that $i(z) = \partial_t q_c(z)$.

The above is cleared up in the new lines doc Section 4.11.

Below is a useful comments about why the capacitor example is not quite right for the TL analysis.
But again, this is all made clear in that new Section 4.11.

Clarification:

1. In my conductor/dielectric boundary example of Fig 1,7, I show that $n_s = (\varepsilon_d/\xi_d)\, n_c$ and that there are in effect two different kinds of surface charge. n_s is the actual surface charge, whereas n_c is the what appears in $J_c = \partial_t n_c$ just below the surface. Thus current has to feed both the surface charge and the conduction of the dielectric. The ONLY result of this figure is that $n_s = (\varepsilon_d/\xi_d)\, n_c$.

2. I then do Fig 1.8 which is a parallel plate capacitor which embeds the previous figure. I use Q as Q_c. I define C' as $C' = Q_c/V$. I show that $j\omega C' = j\omega C + (1/R)$ where $1/R = G$ for this capacitor. But this is a special geometry and it is not obvious that things will work out this way in general. I also write $I = \partial_t Q_c$ and this is where the big difference lies! In the capacitor case, no current "continues through" as it does for a section of transmission line. That is why I cannot write $i(z) = \partial_t q_c(z)$ later on for the TL.

And we end with more scratch notes.

Support:

$$\xi_d = \varepsilon_d - j\sigma_d/\omega$$

$$\xi_d/\varepsilon_d = 1 - j\sigma_d/(\omega\varepsilon_d)$$

$$(\xi_d/\varepsilon_d) - 1 = -\,j\sigma_d/(\omega\varepsilon_d)$$

Then I have shown that

$$i_G(z) = j\omega q_s[- j\sigma_d/(\omega\varepsilon_d)] = q_s (\sigma_d/\varepsilon_d) = CV(z) (\sigma_d/\varepsilon_d)$$

Now above I show that

$$G = i_G(z)/V_z = CV(z) (\sigma_d/\varepsilon_d)/V_z = C (\sigma_d/\varepsilon_d)$$

and there is the result I want.

$= j\omega q_c[1 - (q_s/q_c)]$

$= j\omega q_c[1 - (\varepsilon_d/\xi_d)] = i(z) [1 - (\varepsilon_d/\xi_d)] = i(z) [\xi_d - \varepsilon_d]/\xi_d$

$= i(z) [(\varepsilon_d - j\sigma_d/\omega) - \varepsilon_d]/\xi_d = i(z) [(- j\sigma_d/\omega)]/\xi_d = (\sigma_d/\xi_d)(1/j\omega) i(z)$

$= (\sigma_d/\xi_d)(1/j\omega)j\omega q\, i(z)$

and then

$$1/ i_G(z) = (1/i(z)) j\omega (\xi_d/\varepsilon_d)$$

and then

$$1/G = V(z)/i_G(z) = [V(z)/i(z)] j\omega (\xi_d/\varepsilon_d)$$

$$= [V(z)/j\xi] j\omega (\xi_d/\varepsilon_d)$$

Then I get

$$(1/G) = V(z)/i_G(z)$$

so that this leakage is the difference of the other two.