Rewrite of Chapter 5 PhL 1.8.14

Chapter 5: The Transverse Problem

5.1 Philosophy ?? maybe discard completely

5.2 The Helmholtz Equations and Separation of Variables

Since $\mathbf{J_a}$ and ρ_a no longer exist, as described above, we set $\mathbf{J_a} = 0$ in (1.5.1) and $\rho_a = 0$ in (1.5.2), so we have this pair of differential equations,

$$(\nabla^2 + \beta^2) \mathbf{A} = \mathbf{0} \qquad (5.2.1)$$

$$(\nabla^2 + \beta^2) \varphi = - (1/\varepsilon) \rho \qquad (5.2.2)$$

where

$$\beta \equiv \omega \sqrt{(\mu)\xi} \qquad (5.2.3)$$

with

$$\xi \equiv [\varepsilon + \sigma/(j\omega)] \qquad (5.2.4)$$

Since φ and $\mathbf{A}$ are the potentials arising from the presence of both conductors C_1 and C_2, the φ and $\mathbf{A}$ appearing in (5.2.1) and (5.2.2) correspond to φ_{12} and $\mathbf{A_{12}}$ of Chapter 4. The above equations apply at all points in space, both in the dielectric between the conductors, and inside the conductors as well.

Note: A differential equation of the form $(\nabla^2 + \beta^2) f = 0$ is usually referred to as the Helmholtz equation, in honor of Hermann von Helmholtz (1821-1894), an early electromagnetic researcher.

Let us first consider the equation (5.2.2) for φ. We shall do a separation of variables for both φ and ρ. For ρ, we perform a separation very similar to (4.1.2) through (4.1.4),

$$\rho(x,y,z) = \rho_t(x,y)\, q(z) \qquad (5.2.5)$$

where $q(z)$ is the charge per unit length on conductor C_1. This defines a transverse charge density $\rho_t(x,y)$. The integral of $\rho_t(x,y)$ over the surface of a C_1 slice is +1, and over the surface of a C_2 slice is -1.

For the potential, we make the following separation:

$$\varphi(x,y,z) = (1/2\pi\varepsilon)\, q(z)\, \varphi_t(x,y) \qquad (5.2.6)$$

We are of course free to set the scale factor arbitrarily, since changing the scale factor just changes the definition of $\varphi_t(x,y)$. Our motivation for the form shown in (5.2.6) is the work of Chapter 4, in particular equation (4.2.5)

In light of (4.1.13) and the related discussion, we realize that the separable form (5.2.6) is only possible if we are working in the "transmission line limit" where the wavelength λ along the transmission line is much larger than all transverse dimensions.

When (5.2.5) and (5.2.6) are inserted into (5.2.2), the result is

$$\left[\frac{\nabla_t^2 \varphi_t(x,y)}{\varphi_t(x,y)} + 2\pi \frac{\rho_t(x,y)}{\varphi_t(x,y)} \right] + \frac{\nabla_z^2 q(z)}{q(z)} = -\beta^2 \tag{5.2.7}$$

which has the general form,

$$[\, h(x,y) \,] + g(z) = -\beta^2 \tag{5.2.8}$$

The only way this can be true for all x,y,z in a region is if g(z) = some constant. For reasons that will be clear later, we write this constant as k_φ^2. Then we get

$$\frac{\nabla_z^2 q(z)}{q(z)} = k_\varphi^2 \qquad \left[\frac{\nabla_t^2 \varphi_t(x,y)}{\varphi_t(x,y)} + 2\pi \frac{\rho_t(x,y)}{\varphi_t(x,y)} \right] = -\beta^2 - k_\varphi^2 \tag{5.2.9}$$

We can rewrite these as

$$[\, \nabla_t^2 + (\beta^2 + k_\varphi^2)] \, \varphi_t(x,y) = -2\pi\rho_t(x,y) \tag{5.2.10}$$

$$[\, \nabla_z^2 - k_\varphi^2 \,] \, q(z) = 0 \tag{5.2.11}$$

We now repeat the process for the vector potential component A_z, which we know is the only significant component of concern. We make the following separation, similar to (5.2.6), which is motivated by (4.4.3),

$$A_z(x,y,z) = (\mu/2\pi)\, i(z)\, A_{zt}(x,y) \tag{5.2.12}$$

Putting (5.2.12) into (5.2.1) yields,

$$\left[\frac{\nabla_t^2 A_{zt}(x,y)}{A_{zt}(x,y)} \right] + \frac{\nabla_z^2 i(z)}{i(z)} = -\beta^2 \tag{5.2.13}$$

Again, this can only work if the second term is some constant, which we here call k_A^2.

$$\frac{\nabla_z^2 i(z)}{i(z)} = k_A^2 \qquad \left[\frac{\nabla_t^2 A_{zt}(x,y)}{A_{zt}(x,y)} \right] = -\beta^2 - k_A^2 \tag{5.2.14}$$

which leads to,

$$[\, \nabla_t^2 + (\beta^2 + k_A^2)] \, A_{zt}(x,y) = 0 \tag{5.2.15}$$

$$[\, \nabla_z^2 - k_A^2 \,] \, i(z) = 0 \tag{5.2.16}$$

Consider now equations (5.2.11) and (5.2.16), which we now apply to the full potentials by making use of (5.2.6) and (5.2.12),

$$\frac{\partial^2 A_z}{\partial z^2} - k_A^2 A_z = 0 \qquad \frac{\partial^2 \varphi}{\partial z^2} - k_\varphi^2 \varphi = 0 \tag{5.2.17}$$

We seek next to find the constants k_A and k_φ. Consider the differences that played such a major role in Chapter 4,

$$V(z) = \varphi(x_1, y_1, z) - \varphi(x_2, y_2, z)$$

$$W(z) = A_z(x_1, y_1, z) - A_z(x_2, y_2, z) \tag{5.2.18}$$

where $\mathbf{x_1}$ and $\mathbf{x_2}$ are points on the conductors at the same z. Applying (5.2.17) to these differences gives,

$$\frac{\partial^2 W}{\partial z^2} - k_A^2 W = 0 \qquad \frac{\partial^2 V}{\partial z^2} - k_\varphi^2 V = 0 \tag{5.2.19}$$

Now, application of $\partial/\partial z$ to the transmission line equations (4.5.9) gives,

$$\frac{\partial^2 i}{\partial z^2} - \mathbf{zy}\, i = 0 \qquad \frac{\partial^2 V}{\partial z^2} - \mathbf{zy}\, V = 0 \tag{5.2.20}$$

where constants **z** and **y** are the impedance and admittance of the transmission line. According to (4.5.1), $W(z) = L_e\, i(z)$, so we replace the first equation above with an identical one in W:

$$\frac{\partial^2 W}{\partial z^2} - \mathbf{zy}\, W = 0 \qquad \frac{\partial^2 V}{\partial z^2} - \mathbf{zy}\, V = 0 \tag{5.2.21}$$

Comparison of (5.2.21) with (5.2.19) gives us the result we seek,

$$k_A^2 = k_\varphi^2 = \mathbf{zy} \equiv k^2 \tag{5.2.22}$$

In retrospect, we are not surprised that the constants are equal in light of the Lorentz covariance discussion relating to equation (1.3.9).

From (4.5.11) and (4.5.12) we know that $\mathbf{z} = Z_i + j\omega L_e$, and $\mathbf{y} = j\beta_d^2/(\omega L_e)$, where Z_i is the total internal impedance of both conductors, and where β_d is β of the dielectric. From (4.5.15) we can also say $\mathbf{y} = j\omega C'$. Therefore

$$k^2 + \beta_d^2 = Z_i(j\omega C') = Z_i\, j\, \beta_d^2/(\omega L_e) \tag{5.2.23}$$

For perfect conductors, $Z_i = 0$. We might define a "low loss" transmission line as one where the right side of (5.2.23) is much smaller than β_d^2. Using $\beta^2 = \omega^2\mu\xi$ this condition becomes, with μ=1,

$$Z_i << \mu_0 f\, K = 4\pi K \times 10^{-7} f \approx K\, f(\text{MHz}) \text{ ohms/meter} \qquad (5.2.24)$$

where K is the dimensionless geometric integral defined in Chapter 4. Typically K is a number in the range 1-10, so the above statement is quite clear as a definition of "low loss". For a typical coax cable, Z_i is about 1 ohm/m at 1 GHz, K ≈ 3, so the above condition is well met since 1 << 3000.

Fact: For a "low loss" transmission line, as defined above, $k^2 \approx -\beta_d^2$.

There remains one important final connection to be made to our work of Chapter 4. The constants in the variable separations for φ and A_z given in (5.2.6) and (5.2.12) were carefully selected to yield the following result,

$$\varphi_t(\mathbf{x}_1) - \varphi_t(\mathbf{x}_2) = A_{zt}(\mathbf{x}_1) - A_{zt}(\mathbf{x}_2) = K \qquad (5.2.25)$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ are any points lying on C_1 and C_2 in the same z plane. Recall that K determines the three transmission line parameters G, C and L_e. The ingredients needed to show that (5.2.25) is true are (5.2.6) and (5.2.12) of this section, and (4.5.1) , (4.5.18) and (4.5.19).

We now summarize the key results of this section:

Separated Transmission Line Equations

Variable separations:

$\rho(x,y,z) = \rho_t(x,y)\, q(z)$

$\varphi(x,y,z) = (1/2\pi\varepsilon)\, q(z)\, \varphi_t(x,y)$

$A_z(x,y,z) = (\mu/2\pi)\, i(z)\, A_{zt}(x,y)$

Eigenvalue relation:

$k^2 = \mathbf{zy}$

$(\beta_d^2 + k^2) = j\omega Z_i C'$

≈ 0 "low loss" seems wrong

Transverse equations:

$[\nabla_t^2 + (\beta^2 + k^2)]\, \varphi_t(x,y) = -2\pi\rho_t(x,y)$

$[\nabla_t^2 + (\beta^2 + k^2)]\, A_{zt}(x,y) = 0$

Transverse boundary conditions:

$\varphi_t(\mathbf{x}_1) - \varphi_t(\mathbf{x}_2) = K$

$A_{zt}(\mathbf{x}_1) - A_{zt}(\mathbf{x}_2) = K$

Longitudinal equations:

$[\nabla_z^2 - k^2\ ]\, q(z) = 0$

$[\nabla_z^2 - k^2\ ]\, i(z) = 0$

$\beta^2 = \omega^2 \mu\xi$

$\beta_d^2 \approx \omega^2 \mu\varepsilon = \omega^2 / v^2$

$\beta_c^2 \approx -j(2/\delta^2)$, $\delta = \sqrt{2/\omega\mu\sigma}$

(5.2.26)