Chapter 5 rewrite #2 PhL 1.8.14

Chapter 5: The Transverse Problem

5.1 Separation of φ

Let $\varphi \equiv \varphi_{12}(\mathbf{x})$ of Section 4.2. Then in the transmission line limit we found

$$\varphi(\mathbf{x}) = \frac{1}{4\pi\xi}\, q(z) \int_{-\infty}^{\infty} dz' \{ \int_{C_1} dx_1' dy_1' \alpha_1(x_1',y_1') \frac{1}{R_1} - \int_{C_2} dx_2' dy_2' \alpha_2(x_2',y_2') \frac{1}{R_2} \} . \qquad (4.3.10)$$

We now rewrite this as

$$\varphi(x,y,z) = \frac{1}{4\pi\xi}\, q(z)\, \varphi_t(x,y) \qquad (5.1.1)$$

$$\varphi_t(x,y) = \int_{-\infty}^{\infty} dz' \{ \int_{C_1} dx_1' dy_1' \alpha_1(x_1',y_1') \frac{1}{R_1} - \int_{C_2} dx_2' dy_2' \alpha_2(x_2',y_2') \frac{1}{R_2} \} \qquad (5.1.2)$$

where $R_1 = |\mathbf{x} - \mathbf{x}_1'|$, $R_2 = |\mathbf{x} - \mathbf{x}_2'|$, and $\mathbf{x}$ is a point in the dielectric. We thus identify φ_t as a certain "transverse potential" associated with the full potential φ. We also had

$$\rho_1(x,y,z) = \alpha_1(x,y)\ q_1(z) \quad . \qquad (4.1.2)$$

$$\rho_2(x,y,z) = \alpha_2(x,y)\ q_2(z) \quad . \qquad (4.1.2)$$

so

$$\rho(x,y,z) = \rho_1(x,y,z) + \rho_2(x,y,z) = [\alpha_1(x,y) - \alpha_2(x,y)]\, q(z) \equiv \rho_t(x,y)\, q(z)$$

so that

$$\rho(x,y,z) = \rho_t(x,y)\, q(z) \qquad \rho_t \equiv \alpha_1(x,y) - \alpha_2(x,y) \qquad (5.1.3)$$

where ρ_t is a "transverse charge distribution" associated with ρ.

The Helmholtz equation for φ in the dielectric region is given by (1.5.3),

$$(\nabla^2 + \beta^2)\varphi(x,y,z) = -(1/\varepsilon)\, \rho(x,y,z) \qquad (1.5.3) \qquad (5.1.4)$$

where ρ(x,y,z) exists on the boundary of the dielectric region (ie, on the conductor surfaces). Combining equations (5.1.1,3,4),

$$(\nabla^2 + \beta^2) \frac{1}{4\pi\xi}\, q(z)\, \varphi_t(x,y) = -(1/\varepsilon)\, \rho_t(x,y)\, q(z)$$

or

$$(\nabla_t^2 + \partial_z^2 + \beta^2) \frac{1}{4\pi\xi}\, q(z)\, \varphi_t(x,y) = -(1/\varepsilon)\, \rho_t(x,y)\, q(z) \qquad // \ \nabla_t^2 = \nabla_{2D}^2 = \nabla^2 - \partial_z^2$$

or

$$\nabla_t^2\varphi_t(x,y)\, q(z) + \varphi_t(x,y)\, \partial_z^2 q(z) + \beta^2\, \varphi_t(x,y)\, q(z) = -(4\pi\xi/\varepsilon)\, \rho_t(x,y)\, q(z) .$$

Now divide through by $\varphi_t(x,y)\, q(z)$ to get

$$\frac{\nabla_t^2\varphi_t(x,y)}{\varphi_t(x,y)} + \frac{\partial_z^2\, q(z)}{q(z)} + \beta^2 = -(4\pi\xi/\varepsilon)\, \frac{\rho_t(x,y)}{\varphi_t(x,y)}$$

or

$$\left[\frac{\nabla_t^2\varphi_t(x,y)}{\varphi_t(x,y)} + 4\pi(\xi/\varepsilon)\, \frac{\rho_t(x,y)}{\varphi_t(x,y)}\right] + \frac{\partial_z^2\, q(z)}{q(z)} = -\beta^2 \qquad (5.1.5)$$

which has the general form,

$$[\, h(x,y)\,] + g(z) = -\beta^2 .$$

The only way this can be true for all x,y,z in a region is if g(z) = some constant, which call k_φ^2. Then,

$$\frac{\partial_z^2\, q(z)}{q(z)} = k_\varphi^2 \qquad \left[\frac{\nabla_t^2\varphi_t(x,y)}{\varphi_t(x,y)} + 2\pi\, \frac{\rho_t(x,y)}{\varphi_t(x,y)}\right] = -\beta^2 - k_\varphi^2 . \qquad (5.1.6)$$

We can rewrite these equations as

$$[\, \nabla_t^2 + (\beta^2 + k_\varphi^2)]\, \varphi_t(x,y) = -2\pi\rho_t(x,y) \qquad (5.1.7)$$

$$[\, \partial_z^2 - k_\varphi^2\,]\, q(z) = 0 . \qquad (5.1.8)$$

The second equation has the following solution

$$q(z) = q(0)\, e^{-jk_\varphi z} \qquad \Rightarrow \qquad q(z,t) = q(0)\, e^{j(\omega t - k_\varphi z)} \qquad (5.1.9)$$

and we find that q(z) has the form of a wave traveling down the transmission line with wavenumber k_φ. The reader of Chapter 2 or of Appendix D will recognize this as the form assumed for the electric field in (2.1.1) or (D.1.1) where it was assumed as an ansatz without much *a priori* justification.

5.2 Separation of A_z

Let $A_z \equiv A_{z12}(\mathbf{x})$ of Section 4.9. Then in the transmission line limit we found

$$A_z(\mathbf{x}) = \frac{\mu}{4\pi}\, i(z) \int_{-\infty}^{\infty} dz' \left\{ \int_{C_1} dx_1'\, dy_1'\, b_1(x_1',y_1') \frac{1}{R_1} - \int_{C_2} dx_2'\, dy_2'\, b_2(x_2',y_2') \frac{1}{R_2} \right\} . \qquad (4.9.1)$$

We now rewrite this as

$$A_z(x,y,z) = \frac{\mu}{4\pi}\, i(z)\, A_{zt}(x,y) \tag{5.2.1}$$

$$A_{zt}(x,y) = \int_{-\infty}^{\infty} dz' \{ \int_{C_1} dx_1' \, dy_1' \, b_1(x_1',y_1') \frac{1}{R_1} - \int_{C_2} dx_2' \, dy_2' \, b_2(x_2',y_2') \frac{1}{R_2} \} \tag{5.2.2}$$

where $R_1 = |\mathbf{x} - \mathbf{x_1}'|$, $R_2 = |\mathbf{x} - \mathbf{x_2}'|$, and $\mathbf{x}$ is a point in the dielectric. We thus identify A_{zt} as a certain "transverse vector potential" associated with the full vector potential A_z.

The Helmholtz equation for A_z is given by (1.5.4) for a region including dielectric and conductors,

$$(\nabla^2 + \beta^2)A_z(x,y,z) = - \Sigma_{i=2}^{N} \mu_i \mathbf{J}_i \quad . \qquad \textit{(1.5.4)} \tag{5.2.3}$$

These $\mathbf{J}_i$ are currents inside the conductors. Although there is small conduction current in the dielectric, it has been absorbed into β^2 as shown in (1.3.21) in the time domain with the use of the King gauge. If we take our region of interest to be the dielectric alone, we then have

$$(\nabla^2 + \beta^2)A_z(x,y,z) = 0 \quad . \tag{5.2.4}$$

Note: there are no free surface currents on the conductor surfaces which would correspond to the free surface charge appearing in (5.1.4).

Inserting (5.2.1) into (5.2.4) yields,

$$(\nabla^2 + \beta^2) \frac{\mu}{4\pi}\, i(z)\, A_{zt}(x,y) = 0$$

or

$$(\nabla_t^2 + \partial_z^2 + \beta^2) \frac{\mu}{4\pi}\, i(z)\, A_{zt}(x,y) = 0$$

or

$$\nabla_t^2 A_{zt}(x,y)\, i(z) + A_{zt}(x,y) \partial_z^2 i(z) + \beta^2 A_{zt}(x,y)\, i(z) = 0 \, .$$

Now divide through by $A_{zt}(x,y)\, i(z)$ to get

$$\frac{\nabla_t^2 A_{zt}(x,y)}{A_{zt}(x,y)} + \frac{\partial_z^2 i(z)}{i(z)} + \beta^2 = 0 \tag{5.2.5}$$

which has the general form,

$$[\, h(x,y) \,] + g(z) = - \beta^2 \quad .$$

The only way this can be true for all x,y,z in a region is if g(z) = some constant, which call k_A^2. Then,

$$\frac{\partial_z^2\, i(z)}{i(z)} = k_A^2 \qquad \frac{\nabla_t^2\, A_{zt}(x,y)}{A_{zt}(x,y)} = -\beta^2 - k_A^2 \ . \tag{5.2.6}$$

We can rewrite these equations as

$$[\, \nabla_t^2 + (\beta^2 + k_A^2)]\, A_{zt}(x,y) = 0 \tag{5.2.7}$$

$$[\, \partial_z^2 - k_A^2 \,]\, i(z) = 0 \ . \tag{5.2.8}$$

The second equation has the following solution

$$i(z) = i(0)\, e^{-jk_A z} \qquad \Rightarrow \qquad i(z,t) = i(0)\, e^{j(\omega t - k_A z)} \tag{5.2.9}$$

and we find that i(z) has the form of a wave traveling down the transmission line with wavenumber k_A.

Comparing (5.2.9) with (5.1.9), it would certainly seem odd if q(z) and i(z) had the form of travelling waves with different wavenumbers $k_\varphi \neq k_A$. We will formally show in the next section that $k_\varphi = k_A$.

5.3 Statement of the transverse problem

Gathering facts from above we have

$$[\, \partial_z^2 - k_\varphi^2 \,]\, q(z) = 0 \qquad \textit{(5.1.8)}$$
$$[\, \partial_z^2 - k_A^2 \,]\, i(z) = 0 \ . \qquad \textit{(5.2.8)}$$

But,

$$\varphi(x,y,z) = \frac{1}{4\pi\xi}\, q(z)\, \varphi_t(x,y) \qquad \textit{(5.1.1)}$$

$$A_z(x,y,z) = \frac{\mu}{4\pi}\, i(z)\, A_{zt}(x,y) \ . \qquad \textit{(5.2.1)}$$

Therefore,

$$[\, \partial_z^2 - k_\varphi^2 \,]\, \varphi(x,y,z) = 0$$
$$[\, \partial_z^2 - k_A^2 \,]\, A_z(x,y,z) = 0 \quad . \tag{5.3.1}$$

Recall that $\mathbf{x_1}$ and $\mathbf{x_2}$ are points on the surfaces of conductors C_1 and C_2. If we write equations (5.3.1) first at $\mathbf{x_1}$ and then at $\mathbf{x_2}$ and then subtract, we get

$$[\, \partial_z^2 - k_\varphi^2 \,]\, V(z) = 0 \qquad // \ V(z) = \varphi(\mathbf{x_1}) - \varphi(\mathbf{x_2})$$
$$[\, \partial_z^2 - k_A^2 \,]\, W(z) = 0 \qquad // \ W(z) = A_z(\mathbf{x_1}) - A_z(\mathbf{x_2}) \tag{5.3.2}$$

where we have used the definitions V(z) and W(z) from (4.4.1) and (4.10.1). Recall now the transmission line equations (4.11.11),

$\partial_z V(z) = -\mathbf{z}\, i(z)$ $\qquad$ $\partial_z i(z) = -\mathbf{y} V(z)$. *(4.11.11)*

Apply ∂_z to each equation and move the right side to the left side,

$\partial_z^2 V(z) + \mathbf{z}\, \partial_z i(z) = 0$ $\qquad$ $\partial_z^2 i(z) + \mathbf{y}\, \partial_z V(z) = 0$.

Now reuse (4.11.11) to replace $\partial_z i(z) = -\mathbf{y} V(z)$ and $\partial_z V(z) = -\mathbf{z}\, i(z)$

$[\partial_z^2 - \mathbf{zy}\,] V(z) = 0$ $\qquad$ $[\partial_z^2 - \mathbf{zy}\,] i(z) = 0$

Finally, use (4.10.8) to replace $i(z) = W(z)/L_e$ and then our equations become

$$[\partial_z^2 - \mathbf{zy}\,] V(z) = 0$$
$$[\partial_z^2 - \mathbf{zy}\,] W(z) = 0 \ . \qquad (5.3.3)$$

Comparison of (5.3.3) with (5.3.2) shows that

$$k_\varphi^2 = k_A^2 = \mathbf{zy} \equiv k^2 \qquad (5.3.4)$$

which fulfills the claim made earlier that $k_\varphi = k_A$.

The resulting transverse equations (5.1.7) and (5.2.7) are then,

$$[\nabla_t^2 + (\beta^2 + k^2)]\, \varphi_t(x,y) = -2\pi\rho_t(x,y) \qquad (5.3.5)$$

$$[\nabla_t^2 + (\beta^2 + k^2)]\, A_{zt}(x,y) = 0 \qquad (5.3.6)$$

$k^2 = \mathbf{z}\, \mathbf{y}$.

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$$[\nabla_t^2 + (\beta^2 + k^2)]\, A_{zt}(x,y) = 0 \qquad (5.3.6)$$

$k^2 = \mathbf{z}\, \mathbf{y}$.