Chapter 5 scratch work PhL 1.8.14

This is obs.

The φ Section

Let $\varphi \equiv \varphi_{12}(\mathbf{x})$ of Section 4.2. Then in the transmission line limit we found

$$\varphi(\mathbf{x}) = \frac{1}{4\pi\xi}\, q(z) \int_{-\infty}^{\infty} dz' \{ \int_{C_1} dx_1'\, dy_1'\, \alpha_1(x_1',y_1') \frac{1}{R_1} - \int_{C_2} dx_2'\, dy_2'\, \alpha_2(x_2',y_2') \frac{1}{R_2} \} \qquad (4.3.10)$$

which we write as

$$\varphi(x,y,z) = \frac{1}{4\pi\xi}\, q(z)\, \varphi_t(x,y) \qquad (5.2.6)$$

$$\varphi_t(x,y) = \int_{-\infty}^{\infty} dz' \{ \int_{C_1} dx_1'\, dy_1'\, \alpha_1(x_1',y_1') \frac{1}{R_1} - \int_{C_2} dx_2'\, dy_2'\, \alpha_2(x_2',y_2') \frac{1}{R_2} \}$$

We also had

$$\rho_1(x,y,z) = \alpha_1(x,y)\; q_1(z) \quad . \qquad (4.1.2)$$
$$\rho_2(x,y,z) = \alpha_2(x,y)\; q_2(z) \quad . \qquad (4.1.2)$$

$$\rho(x,y,z) = \rho_1(x,y,z) + \rho_2(x,y,z) = [\, \alpha_1(x,y) - \alpha_2(x,y)]\, q(z) \equiv \rho_t(x,y)\, q(z)$$

The Helmholtz equation for φ in the dielectric region is given by (1.5.3),

$$(\nabla^2 + \beta^2)\varphi(x,y,z) = -\,(1/\varepsilon)\, \rho(x,y,z) \qquad // (1.5.3)$$

Thus

$$(\nabla^2 + \beta^2) \frac{1}{4\pi\xi}\, q(z)\, \varphi_t(x,y) = -\,(1/\varepsilon)\, \rho_t(x,y)\, q(z)$$

or

$$(\nabla_t^2 + \partial_z^2 + \beta^2) \frac{1}{4\pi\xi}\, q(z)\, \varphi_t(x,y) = -\,(1/\varepsilon)\, \rho_t(x,y)\, q(z)$$

or

$$\nabla_t^2 \varphi_t(x,y)\, q(z) + \varphi_t(x,y)\, \partial_z^2 q(z) + \beta^2\, \varphi_t(x,y)\, q(z) = -\,(4\pi\xi/\varepsilon)\, \rho_t(x,y)\, q(z)$$

Now divide through by $\varphi_t(x,y)\, q(z)$ to get

$$\frac{\nabla_t^2 \varphi_t(x,y)}{\varphi_t(x,y)} + \frac{\partial_z^2\, q(z)}{q(z)} + \beta^2 = -\,(4\pi\xi/\varepsilon)\, \frac{\rho_t(x,y)}{\varphi_t(x,y)}$$

or

$$\left[\frac{\nabla_t^2 \varphi_t(x,y)}{\varphi_t(x,y)} + 4\pi(\xi/\varepsilon) \frac{\rho_t(x,y)}{\varphi_t(x,y)} \right] + \frac{\partial_z^2\, q(z)}{q(z)} = -\beta^2 \qquad \text{// original (5.2.7_}$$

which has the general form,

$$[\, h(x,y)\,] + g(z) = -\beta^2$$

The only way this can be true for all x,y,z in a region is if g(z) = some constant. For reasons that will be clear later, we write this constant as k_φ^2. Then we get

$$\frac{\nabla_z^2\, q(z)}{q(z)} = k_\varphi^2 \qquad \left[\frac{\nabla_t^2 \varphi_t(x,y)}{\varphi_t(x,y)} + 2\pi \frac{\rho_t(x,y)}{\varphi_t(x,y)} \right] = -\beta^2 - k_\varphi^2 \qquad (5.2.9)$$

We can rewrite these as

$$[\, \nabla_t^2 + (\beta^2 + k_\varphi^2)]\, \varphi_t(x,y) = -2\pi\rho_t(x,y) \qquad (5.2.10)$$

$$[\, \partial_z^2 - k_\varphi^2\,]\, q(z) = 0 \qquad (5.2.11)$$

The A_z Section

Let $A_z \equiv A_{z12}(\mathbf{x})$ of Section 4.9. Then in the transmission line limit we found

$$A_z(\mathbf{x}) = \frac{\mu}{4\pi}\, i(z) \int_{-\infty}^{\infty} dz' \{ \int_{C_1} dx_1'\, dy_1'\, b_1(x_1',y_1') \frac{1}{R_1} - \int_{C_2} dx_2'\, dy_2'\, b_2(x_2',y_2') \frac{1}{R_2} \} \qquad (4.9.1)$$

which we write as

$$A_z(x,y,z) = \frac{\mu}{4\pi}\, i(z)\, A_{zt}(x,y) \qquad 5.2.6)$$

$$A_{zt}(x,y) = \int_{-\infty}^{\infty} dz' \{ \int_{C_1} dx_1'\, dy_1'\, b_1(x_1',y_1') \frac{1}{R_1} - \int_{C_2} dx_2'\, dy_2'\, b_2(x_2',y_2') \frac{1}{R_2} \}$$

We also had

$$J_{z1}(x,y,z) = b_1(x,y)\, i_1(z) \quad . \qquad (4.7.3)$$
$$J_{z2}(x,y,z) = b_2(x,y)\, i_2(z) \quad . \qquad (4.1.2)$$

$$J_z(x,y,z) = J_{z1}(x,y,z) + J_{z2}(x,y,z) = [\, b_1(x,y) - b_2(x,y)]\, i(z) \equiv J_{zt}(x,y)\, i(z)$$

The Helmholtz equation for A_z in the dielectric region is given by (1.5.4),

$$(\nabla^2 + \beta^2) A_z(x,y,z) = 0 \qquad \text{// (1.5.4)}$$

Thus

$$(\nabla^2 + \beta^2)\,\frac{\mu}{4\pi}\; i(z)\, A_{zt}(x,y) = 0$$

or

$$(\nabla_t^2 + \partial_z^2 + \beta^2)\,\frac{\mu}{4\pi}\; i(z)\, A_{zt}(x,y) = 0$$

or

$$\nabla_t^2 A_{zt}(x,y)\, i(z) + A_{zt}(x,y)\partial_z^2 i(z) \; + \beta^2\, A_{zt}(x,y)\, i(z) \; = 0$$

Now divide through by $A_{zt}(x,y)\, i(z)$ to get

$$\frac{\nabla_t^2\, A_{zt}(x,y)}{A_{zt}(x,y)} \; + \frac{\partial_z^2\, i(z)}{i(z)} \; + \beta^2 \; = 0$$

or

$$\frac{\nabla_t^2\, A_{zt}(x,y)}{A_{zt}(x,y)} \; + \frac{\partial_z^2\, i(z)}{i(z)} \; = -\beta^2$$

which has the general form,

$$[\, h(x,y)\,] \; + \; g(z) \; = -\,\beta^2$$

The only way this can be true for all x,y,z in a region is if g(z) = some constant. For reasons that will be clear later, we write this constant as k_A^2. Then we get

$$\frac{\partial_z^2\, i(z)}{i(z)} \; = \; k_A^2 \qquad\qquad \frac{\nabla_t^2\, A_{zt}(x,y)}{A_{zt}(x,y)} \; = -\,\beta^2 \; - k_A^2 \qquad\qquad (5.2.9)$$

We can rewrite these as

$$[\, \nabla_t^2 + (\beta^2 + k_A^2)]\, A_{zt}(x,y) \; = 0 \qquad\qquad (5.2.10)$$

$$[\, \partial_z^2 - k_A^2 \;]\, i(z) = 0 \qquad\qquad (5.2.11)$$

which agrees with original (5.2.16) .

That which comes next

Gathering from above we have

$$[\, \partial_z^2 - k_\varphi^2 \;]\, q(z) = 0$$
$$[\, \partial_z^2 - k_A^2 \;]\, i(z) = 0$$

But,

$$\varphi(x,y,z) = \frac{1}{4\pi\xi}\, q(z)\, \varphi_t(x,y) \qquad (5.2.6)$$

$$A_z(x,y,z) = \frac{\mu}{4\pi}\, i(z)\, A_{zt}(x,y) \qquad 5.2.6)$$

And therefore

$$[\, \partial_z^2 - k_\varphi^2 \,]\, \varphi(x,y,z) = 0$$
$$[\, \partial_z^2 - k_A^2 \,]\, A_z(x,y,z) = 0$$

Recall that

$$V(z) = \varphi(x_1, y_1, z) - \varphi(x_2, y_2, z)$$
$$W(z) = A_z(x_1, y_1, z) - A_z(x_2, y_2, z) \qquad (5.2.18)$$

where $\mathbf{x_1}$ and $\mathbf{x_2}$ are points on the conductors at the same z. Applying (***to these differences gives,

$$[\, \partial_z^2 - k_\varphi^2 \,]\, V(z) = 0$$
$$[\, \partial_z^2 - k_A^2 \,]\, W(z) = 0$$

Now, application of ∂_z to the transmission line equations

$$\partial_z V(z) = -\mathbf{z}\, i(z) \qquad \partial_z i(z) = -\mathbf{y}\, V(z)$$

gives

$$\partial^2_z V(z) = -\mathbf{z}\partial_z\, i(z) \qquad \partial_z^2 i(z) = -\mathbf{y}\, \partial_z V(z)$$

or combining the last two pairs of equations,

$$\partial^2_z V(z) - \mathbf{zy}\, V(z) = 0$$
$$\partial^2_z W(z) - \mathbf{zy}\, W(z) = 0$$

huh?? where did W(z) come from?

or

$$[\, \partial_z^2 - \mathbf{zy} \,]\, V(z) = 0$$
$$[\, \partial_z^2 - \mathbf{zy} \,]\, W(z) = 0$$

Comparison *** shows that

$$k_A^2 = k_\varphi^2 = \mathbf{zy} \equiv k^2 \qquad (5.2.22)$$

The resulting transverse equations are then

$$[\, \nabla_t^2 + (\beta^2 + k^2)\,]\, A_{zt}(x,y) = 0 \qquad (5.2.10)$$

$$[\nabla_t^2 + (\beta^2 + k^2)]\,\varphi_t(x,y) = -2\pi\rho_t(x,y) \qquad (5.2.10)$$

If I show that $k^2 \approx -\beta_d^2$ then I get

$$[\nabla_t^2 + (\beta^2 - \beta_d^2)]\,A_{zt}(x,y) = 0 \qquad (5.2.10)$$

$$[\nabla_t^2 + (\beta^2 - \beta_d^2)]\,\varphi_t(x,y) = -2\pi\rho_t(x,y) \qquad (5.2.10)$$

If I wander off now to Appendix D, I see there this typical equation

$$[r^2\partial_r^2 + r\,\partial_r - m^2 + r^2(\beta^2 - \beta_d^2)]\,E_z(r,m) = 0\ . \qquad (D.1.15)$$

This is of course in a partial wave expansion business for a round wire, but you see $(\beta^2 - \beta_d^2)$ appearing there just as it is now appearing in Chapter 5.

What do I know about problems like this:

$$[\nabla_{2D}^2 + (\beta^2 + k^2)]\,A_{zt}(x,y) = 0 \qquad A_{zt}(\text{boundary}) = \text{known} \qquad (5.2.10)$$

$$[\nabla_{2D}^2 + (\beta^2 + k^2)]\,\varphi_t(x,y) = -2\pi\rho_t(x,y) \qquad (5.2.10)$$