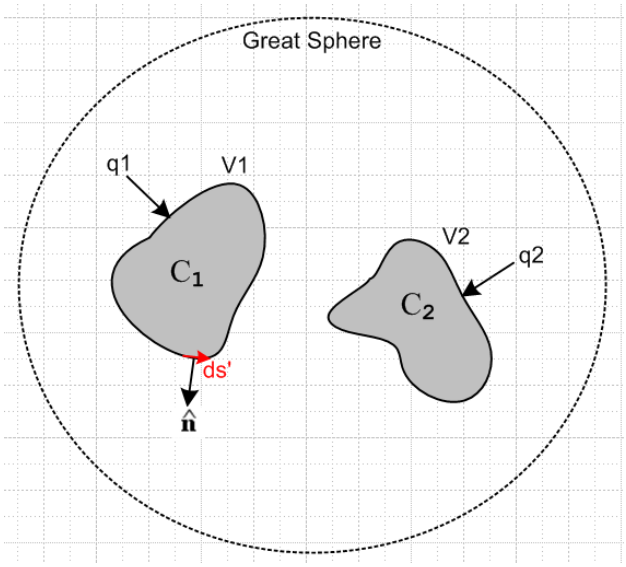
General Capacitor Problem PhL 1.13.14

Start with this picture in 3D space



Viewed from infinity, we see basically two point charges of q_1 and q_2. The potential of each charge decays as 1/r at so $\varphi(\infty) = 0$. Here is the Stakgold solution to this treated as a Dirichlet problem:

$$\varphi(\mathbf{x}) = -\int_\sigma dS_\xi\, f(\xi)\, \partial_{\xi n} g(x|\xi) \qquad \text{// Stakgold (6.81) .} \qquad (1.5.11)$$

$$= -\int_{C1} dS_\xi\, V_1\, \partial_{\xi n} g(x|\xi) - \int_{C2} dS_\xi\, V_2\, \partial_{\xi n} g(x|\xi) - 0$$

$$= V_1 F_1(\mathbf{x}) + V_2 F_2(\mathbf{x}) \ .$$

Then compute the E field at each conductor surface according to

$$E_n(\mathbf{x})|^{C1} = -\partial_n \varphi(\mathbf{x})|^{C1} = -\partial_n [V_1 F_{11}(\mathbf{x}) + V_2 F_{12}(\mathbf{x})] \equiv V_1 G_{11}(\mathbf{x}) + V_2 G_{12}(\mathbf{x})$$

$$E_n(\mathbf{x})|^{C2} = -\partial_n \varphi(\mathbf{x})|^{C2} = -\partial_n [V_1 F_{21}(\mathbf{x}) + V_2 F_{22}(\mathbf{x})] \equiv V_1 G_{21}(\mathbf{x}) + V_2 G_{22}(\mathbf{x})$$

where all functions are known. Then write

$$n_1(\mathbf{x}) = \varepsilon\, E_n(\mathbf{x})|^{C1} = \varepsilon V_1 G_{11}(\mathbf{x}) + \varepsilon V_2 G_{12}(\mathbf{x})$$

$$n_2(\mathbf{x}) = \varepsilon\, E_n(\mathbf{x})|^{C2} = \varepsilon V_1 G_{21}(\mathbf{x}) + \varepsilon V_2 G_{22}(\mathbf{x})$$

Then integrate to get

$$q_1 = \oint_{C1} ds'\, n_1(\mathbf{x}) = \varepsilon V_1 H_{11} + \varepsilon V_2 H_{12} = V_1 C_{11} + V_2 C_{12}$$

$$q_2 = \oint_{C2} ds'\, n_1(\mathbf{x}) = \varepsilon V_1 H_{21} + \varepsilon V_2 H_{22} = V_1 C_{21} + V_2 C_{22}$$

where suddenly Smythe's mutual capacitances appear. Then we have

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \qquad \text{or} \qquad \mathbf{q} = C\,\mathbf{V}$$

Invert to get

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = C^{-1} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = S \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

Now set $q_1 = q$ and $q_2 = -q$ and then

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = C^{-1} \begin{pmatrix} q \\ -q \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} q \\ -q \end{pmatrix} = q \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = q \begin{pmatrix} S_{11} - S_{12} \\ S_{21} - S_{22} \end{pmatrix}$$

so then

$$V_1 = q\,(S_{11} - S_{12})$$
$$V_2 = q\,(S_{21} - S_{22})$$

And then

$$V = V_1 - V_2 = q\,(S_{11} - S_{12}) - q\,(S_{21} - S_{22})$$

$$= q\,[\,(S_{11} - S_{12}) - (S_{21} - S_{22})]$$

$$= q\,[S_{11} - S_{12} - S_{21} + S_{22}]$$

$$= q\,[S_{11} + S_{22} - 2S_{12}]$$

so the capacitance is given by

$$C = 1/(S_{11} + S_{22} - 2S_{12})$$

Great, now I want to see someone verify this whole idea of mine.

By the way,

$$C = S^{-1} = \mathrm{cof}(C^{\mathrm{T}})/\det(C)$$

Now it happens that I know how to invert a 2x2 matrix. Suppose we have

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

Then

$$S = C^{-1} = \begin{pmatrix} C_{22} & -C_{12} \\ -C_{21} & C_{11} \end{pmatrix} / \det(C)$$

Then

$$S_{11} = C_{22} / \det(C)$$
$$S_{22} = C_{11} / \det(C)$$

$$C = 1/(S_{11} + S_{22} - 2S_{12})$$

$$C^{-1} = (S_{11} + S_{22} - 2S_{12})$$

$$= (C_{22} + C_{11} - 2[-C_{12}])/\det(C)$$

$$= \frac{C_{11} + C_{22} + 2C_{12}}{C_{11}C_{22} - C_{12}^2}$$

And then

$$C = \frac{C_{11}C_{22} - C_{12}^2}{C_{11} + C_{22} + 2C_{12}}$$

And here is a slight with 100% verification!! hurray!

Capacitance & Energy of Multi-Conductor Systems

The mutual capacitance of a two conductor system can also be expressed in terms of the capacity and induction coefficients c_{jk}. From Eq. (18) the electrostatic potential energy of a two conductor capacitor is given by $U_e = \frac{1}{2}Q^2(p_{11} + p_{22} - 2p_{12})$. Because $\mathcal{P} = \mathcal{C}^{-1}$, then

$$\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{pmatrix}^{-1} = \frac{1}{c_{11}c_{22} - c_{12}^2} \begin{pmatrix} c_{22} & -c_{12} \\ -c_{12} & c_{11} \end{pmatrix},$$

so that

$$U_e = \frac{1}{2}Q^2 \frac{1}{c_{11}c_{22} - c_{12}^2}(c_{11} + c_{22} + 2c_{12}).$$

Comparison of this expression with that given in Eq. (3) then yields

$$\boxed{C = \frac{c_{11}c_{22} - c_{12}^2}{c_{11} + c_{22} + 2c_{12}}} \tag{27}$$

for the mutual capacitance of a two conductor system.