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Chapter 6: Transmission Lines with Circular Conductors

6.1 A candidate transverse potential φ_t

In the previous chapter (both Section 5.3 (b) and Section 5.4 (b)) we showed that the transverse potential of a 2-conductor balanced transmission line must have this form when viewed from far away,

$$\varphi_t(\mathbf{x}) \approx \ln(s_2^2/s_1^2) \quad \text{// limiting form as point } \mathbf{x} = (x,y) \text{ moves far from the conductors} \qquad (5.3.11)$$

$$s_1^2 = (x-x_1')^2 + (y-y_1')^2 = |\mathbf{x} - \mathbf{x}_1'|^2$$
$$s_2^2 = (x-x_2')^2 + (y-y_2') = |\mathbf{x} - \mathbf{x}_2'|^2$$

where $\mathbf{x}_1 = (x_1',y_1')$ is any point on C_1 and $\mathbf{x}_2 = (x_2',y_2')$ is any point on C_2 . In Section 5.4.(b) we took these points to be the "center of charge" points, but any points will do when $\mathbf{x}$ is very far away.

Suppose now we take as a candidate dimensionless transverse potential φ_t exactly the above limiting expression. Our candidate φ_t is then

$$\varphi_t(\mathbf{x}) = \ln(s_2^2/s_1^2) \qquad \text{for } \textit{all} \text{ values of } \mathbf{r}\text{, close and far}$$

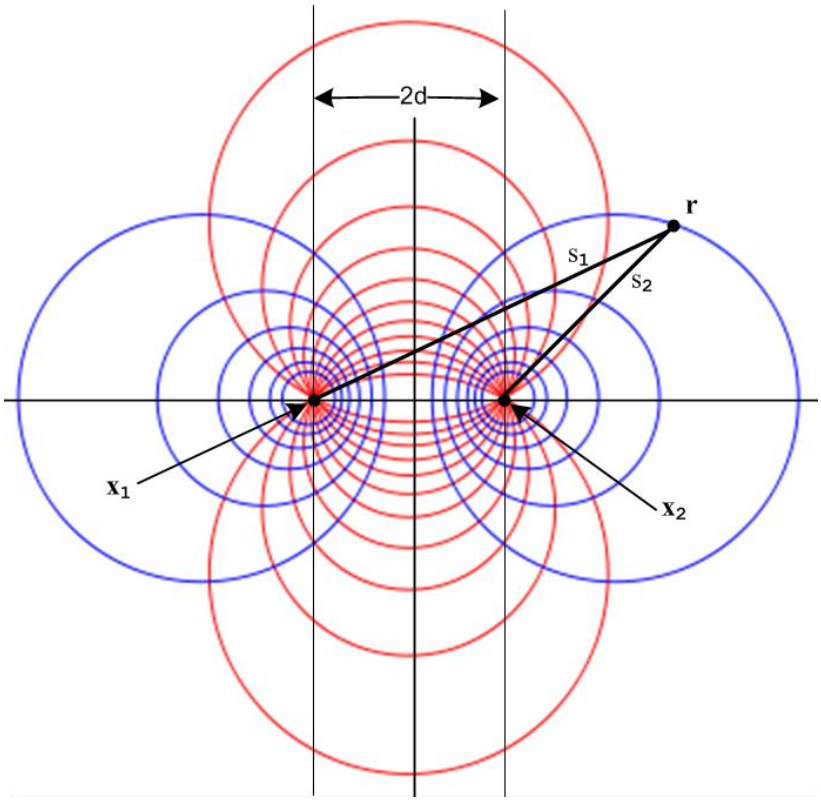
Certainly this meets our limiting form boundary condition (5.3.11)! We know also that this potential is a valid solution of the 2D Laplace equation, since $\ln(s_1)$ and $\ln(s_2)$ are each valid solutions. This fact was shown at the start of Section 5.4 (b). Since $-2\pi\ln R_1$ is the 2D free-space propagator, it follows that $-2\pi\ln R_1$ is a solution of $\nabla^2_{2D}(\varphi) = 0$ away from the point where $R_1 = 0$, and then so is $\ln R_1 = \ln s_1$. Then by superposition, $2\ln s_2 - 2\ln s_1$ is also a valid solution, and this is $\ln(s_2^2/s_1^2)$. Thus, our φ_t is a valid candidate for a lossless transmission line, since for such a transmission line φ_t satisfies the 2D Laplace equation according to (5.4.3)

$$\nabla_t^2 \varphi_t(x,y) = 0 \qquad \varphi_t(C_1) = K_1 \quad \varphi_t(C_2) = K_2 \qquad K_1 - K_2 = K \qquad (5.4.3)$$

The question then becomes: what are the surfaces C_i in 2D space on which this candidate φ_t is a constant? Such surfaces can then serve as possible conductor cross sections for a transmission line.

6.2 Ancient Greece circa 230 BC

Apollonius of Perga (262BC-190BC) [like Joe of Chicago] was a pretty smart guy as wiki explains. He did astronomy and therefore he did geometry. Besides giving conic sections their current names and writing eight books about them, he learned about what are now called the Apollonian Circles. These circles form the "level surfaces" for 2D bipolar orthogonal coordinates as shown in this picture



http://en.wikipedia.org/wiki/Apollonian_circles

When this picture is rotated around its vertical axis, the blue level circles become toroids and one then arrives at 3D toroidal coordinates, but that is another story. Our interest is in the 2D blue circles.

It turns out, as the reader no doubt suspects, that the blue circles have the following simple property:

s_2/s_1 = constant

which we shall prove in a moment. Calling this constant e^{-B} we get

$s_2/s_1 = e^{-B}$ => $\ln(s_2/s_1) = -B$

Thus, since $\varphi_t(\mathbf{x}) = \ln(s_2^2/s_1^2) = 2\ln(s_2/s_1)$, the blue circles are candidate equipotential surfaces for our potential $\varphi_t(\mathbf{x})$!

To show that $s_1/s_2 = e^B$ describes a circle, consider:

$|\mathbf{r}-\mathbf{r}_2| / |\mathbf{r}-\mathbf{r}_1| = e^{-B}$

$|\mathbf{r}-\mathbf{r}_2|^2 = e^{-2B} |\mathbf{r}-\mathbf{r}_1|^2$

$(x-x_1)^2 + (y-y_1)^2 = e^{-2B} [(x-x_2)^2 + (y-y_2)^2]$

This equation has the following form

$A(x^2 + y^2) + Bx + Cy + D = 0 \qquad A = (1-e^{-2B})$

or

$x^2 + y^2 + \alpha x + \beta y + \gamma = 0$.

One can then "complete the squares" to obtain the equation of a circle of radius r centered at (x_c,y_c) ,

$(x - x_c)^2 + (y - y_c)^2 = r^2$

where $-2x_c = \alpha \quad -2y_c = \beta \qquad x_c^2 + y_c^2 - r^2 = \gamma$.

For our particular locations of $\mathbf{r}_1$ and $\mathbf{r}_2$ shown in the above figure, we have

$x_1 = d \qquad x_2 = -d \qquad y_1 = y_2 = 0$

$s_1^2 = (x_1+d)^2 + y^2 \qquad s_2^2 = (x_1-d)^2 + y^2$

so

$s_2/s_1 = e^{-B} \Rightarrow s_1/s_2 = e^{B} \Rightarrow e^{2B}s_2^2 = s_1^2 \Rightarrow (e^{B}/2)\, s_2^2 = (e^{-B}/2)\, s_1^2 \Rightarrow$

$(e^{B}/2)\,[x^2 - 2dx + d^2 + y^2] = (e^{-B}/2)\,[x^2 + 2dx + d^2 + y^2]$

$shB\,(x^2+d^2+y^2) + chB(-2dx) = 0 \qquad //\ shB = (e^{B}-e^{-B})/2,\ chB = (e^{B}+e^{-B})/2$

$(x^2+d^2+y^2) + cothB\,(-2dx) = 0$

$x^2 - 2dx\ cothB + y^2 = -d^2$

$x^2 - 2dx\ cothB + d^2coth^2B + y^2 = -d^2 + d^2coth^2B \qquad$ // complete the square

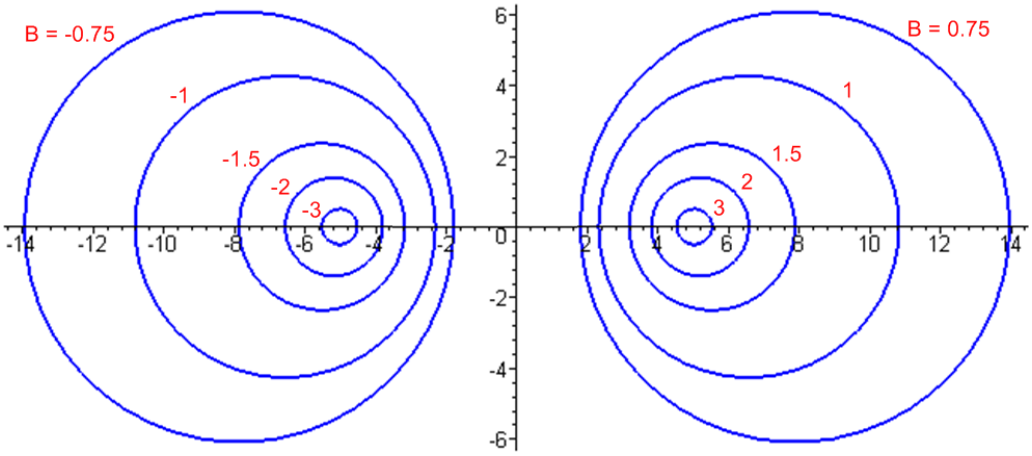
$(x - dcothB)^2 + y^2 = d^2csch^2B$.

We conclude that our blue equipotential circles have this simple form

$(x - x_c)^2 + y^2 = r^2 \qquad x_c = d\ cothB \qquad r = |d\ cschB|$

Using d = 5, here is a plot of these circles for 10 different B values:

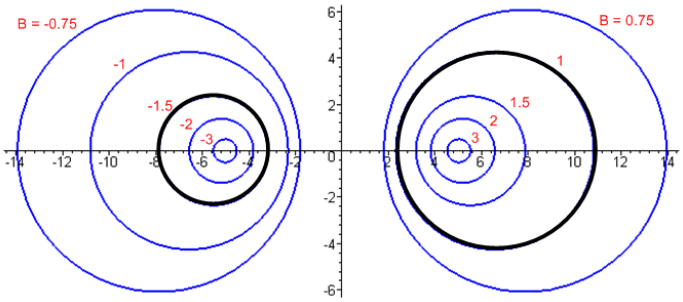
```
restart; with(plots):with(plottools):
xc := d*coth(B):
r := d*csch(B):
d := 5:
Bvals := [-3,-2,-1.5,-1,-.75,.75,1,1.5,2,3]:N := nops(Bvals):
for n from 1 to N do
   B := Bvals[n];
   c[n] := circle([xc,0],r);
od:
display(seq(c[n],n=1..N),color=blue,scaling = constrained,thickness=2);
```



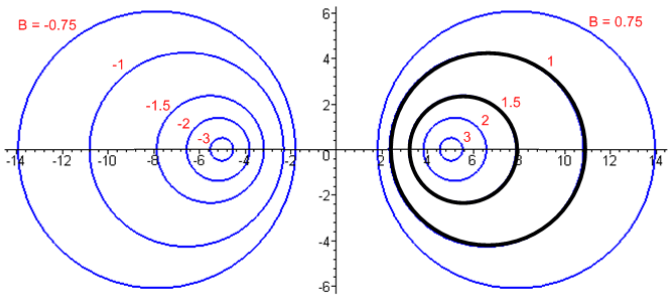
Since x_c = d cothB, the right side curves have B > 0 while the left side have B < 0. The value B = 0 corresponds to the vertical y axis, while B = ±∞ correspond to the two focal points at d = ± 5.

6.3 Back to the Future: Reading off the Capacitance

We shall now first select C_2 to be a circle on the right side, so that $B_2 > 0$.
For C_1 we select a second circle from either the left or the right, so B_1 can have either sign.
If we select C_1 from the left side, we have a two-wire transmission line.



If we select C_2 from the right, we have an off-center coaxial transmission line.



Let $\sigma_1 = \text{sign}(B_1)$. We then have

$$\varphi_t(\mathbf{x}) = \ln(s_2^2/s_1^2) = 2\ln(s_2/s_1)$$

$$\varphi_t(C_1) = 2\ln(s_2/s_1)|^{c1} = -2B_1$$
$$\varphi_t(C_2) = 2\ln(s_2/s_1)|^{c2} = -2B_2 .$$

Recall from (5.1.3) that $\varphi_t(C_1) - \varphi_t(C_2) = K$. Therefore,

$$K = 2(B_2 - B_1) = 2(|B_2| - \sigma_1|B_1|).$$

Once we know K, we know C, G and L_e for the transmission line from ****.

We must now do some slightly painful algebra.

First, we know from **** above that

$$a_1 = d\,|\text{csch}B_1| \Rightarrow (d/a_1) = \text{sh}(|B_1|) \Rightarrow |B_1| = \text{sh}^{-1}(d/a_1)$$
$$a_2 = d\,|\text{csch}B_2| \Rightarrow (d/a_2) = \text{sh}(|B_2|) \Rightarrow |B_2| = \text{sh}^{-1}(d/a_2)$$

$$b = |x_{c2} - x_{c1}| = |d\,\text{coth}B_2 - d\text{coth}B_1| = d\,|\text{coth}B_2 - \text{coth}B_1|$$

From ** we write

$$\text{ch}(K/2) = \text{ch}\,[|B_2| - \sigma_1|B_1|]$$

$$= \text{ch}|B_2|\,\text{ch}|B_1| - \sigma_1\,\text{sh}|B_2|\,\text{sh}|B_1|$$

$$= \sqrt{1+\text{sh}^2B_2}\,\sqrt{1+\text{sh}^2B_1} - \sigma_1\,\text{sh}|B_2|\,\text{sh}|B_1|$$

$$= \sqrt{1+(d/a_2)^2}\,\sqrt{1+(d/a_1)^2} - \sigma_1\,(d/a_2)\,(d/a_1) .$$

Meanwhile,

$$b = d\,|\coth B_2 - \coth B_1| = |d\,[\,\mathrm{ch}B_2/\mathrm{sh}B_2 - \mathrm{ch}B_1/\mathrm{sh}B_1]\,| = |d\,[\,\mathrm{ch}|B_2|/\mathrm{sh}|B_2| - \sigma_1 \mathrm{ch}|B_1|/\mathrm{sh}|B_1|]\,|$$

$$= |\,d\,\,[\mathrm{ch}|B_2|\,\mathrm{sh}|B_1| - \sigma_1\,\mathrm{ch}|B_1|\,\mathrm{sh}|B_2|\,]\,/\,\mathrm{sh}|B_1|\,\mathrm{sh}|B_2|\,\,|$$

$$= |\,\,d\,[\sqrt{1+(d/a_2)^2}\,(d/a_1) - \sigma_1\sqrt{1+(d/a_1)^2}\,\,(d/a_2)\,]\,/\,\,(d/a_2)\,(d/a_1)\,|$$

$$= |\,[\sqrt{1+(d/a_2)^2}\,(1/a_1) - \sigma_1\sqrt{1+(d/a_1)^2}\,\,(1/a_2)\,]\,/\,\,(1/a_2)\,(1/a_1)\,|$$

$$= |\,[a_2\sqrt{1+(d/a_2)^2} - \sigma_1\,a_1\sqrt{1+(d/a_1)^2}\,\,]\,|\quad .$$

Square this to get

$$b^2 = a_2^2[1+(d/a_2)^2]\,\, + a_1^2[1+(d/a_1)^2]\,\, - 2\sigma_1 a_1 a_2\sqrt{1+(d/a_2)^2}\,\sqrt{1+(d/a_1)^2}$$

so

$$2\,\sigma_1 a_1 a_2\sqrt{1+(d/a_2)^2}\,\sqrt{1+(d/a_1)^2}\,\, = a_2^2[1+(d/a_2)^2]\,\, + a_1^2[1+(d/a_1)^2] - b^2$$

$$= a_2^2 + d^2 + a_1^2 + d^2 - b^2 \quad = a_1^2 + a_2^2 + 2d^2 - b^2\ .$$

The purpose of doing this is to obtain the following expression for the radical product

$$\sqrt{1+(d/a_2)^2}\,\sqrt{1+(d/a_1)^2} \quad = (a_1^2 + a_2^2 + 2d^2 - b^2)\,/\,(2\,\sigma_1 a_1 a_2\,)\ .$$

We now install this into our expression above for ch(K/2) to get

$$\mathrm{ch}(K/2) = \sqrt{1+(d/a_2)^2}\,\sqrt{1+(d/a_1)^2}\,\, - \sigma_1\,(d/a_2)\,(d/a_1)$$

$$= (a_1^2 + a_2^2 + 2d^2 - b^2)\,/\,(2\,\sigma_1 a_1 a_2\,) - 2d^2/\,(2\sigma_1 a_2 a_1)$$

$$= (a_1^2 + a_2^2 - b^2)\,/\,(2\sigma_1 a_2 a_1) \;=\; \sigma_1\,(1/2)\,(a_1^2 + a_2^2 - b^2)/(a_1 a_2)$$

$$= \sigma_1\,(1/2)\,[\;\,(a_1/a_2) + (a_2/a_1) - (b^2/a_1 a_2)\,]$$

and the annoying constant d has vanished from our expression. Therefore

$$K = 2\,\mathrm{ch}^{-1}\,\{\,\sigma_1\,(1/2)\,[\;\,(a_1/a_2) + (a_2/a_1) - (b^2/a_1 a_2)\,]\;\}$$

Notice that the result is symmetric under $a_1 \leftrightarrow a_2$.

We now distinguish our two cases of interest. For a twin-lead type transmission line (unequal lead diameters) we know that $B_1 < 0$ since the C_1 circle is on the left, so $\sigma_1 = \mathrm{sign}(B_1) = -1$ and then

$$K = 2\,\mathrm{ch}^{-1}\,\{\,(1/2)\,[\;\,(b^2/a_1 a_2) - (a_1/a_2) - \,(a_2/a_1)]\;\}$$

which is an amazingly simple result. Recall from *** that

$$Z_0 = (K/\sqrt{\varepsilon_{rel}})\ 30\Omega$$

so then

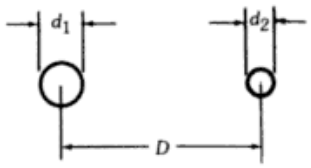
$$Z_0 = \text{ch}^{-1}\ \{ (1/2) [\ (b^2/a_1a_2) - (a_1/a_2) - (a_2/a_1)]\ \} (1/\sqrt{\varepsilon_{rel}})\ 60\ \Omega$$

If the wires have diameters $d_1 = 2a_1$ and $d_2 = 2a_2$ this becomes

$$Z_0 = \text{ch}^{-1}\ \{ (1/2) [\ (4b^2/d_1d_2) - (d_1/d_2) - (d_2/d_1)]\ \} (1/\sqrt{\varepsilon_{rel}})\ 60\ \Omega$$

For verification, we quote again from Reference RDE page 29-23,

N. **Balanced 2-wire—unequal diameters**



$$Z_0 = (60/\epsilon^{1/2}) \cosh^{-1} N$$

$$N = \tfrac{1}{2}[(4D^2/d_1d_2) - (d_1/d_2) - (d_2/d_1)]$$

where our b is called D.

On the other hand, if we are interested in an off-center coaxial transmission line, we select C_1 from the right side of Fig *** and then $\sigma_1 = \text{sign}(B_1) = +$ and we find

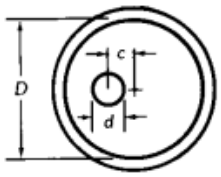
$$K = 2\ \text{ch}^{-1}\ \{ (1/2) [\ (a_1/a_2) + (a_2/a_1) - (b^2/a_1a_2)\ ]\ \}$$

$$Z_0 = \text{ch}^{-1}\ \{ (1/2) [\ (a_1/a_2) + (a_2/a_1) - (b^2/a_1a_2)\ \} (1/\sqrt{\varepsilon_{rel}})\ 60\ \Omega$$

$$= \text{ch}^{-1}\ \{ (1/2) [\ (d_1/d_2) + (d_2/d_1) - (4b^2/d_1d_2)\ \} (1/\sqrt{\varepsilon_{rel}})\ 60\ \Omega$$

For verification, we quote again from Reference RDE page 29-24,

U. Eccentric line

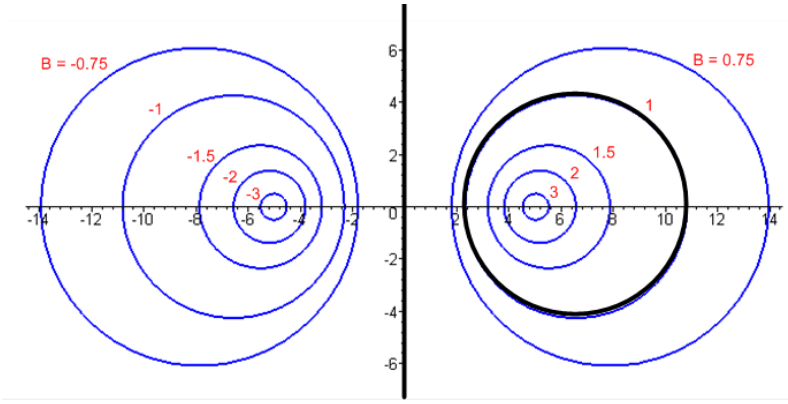


$$Z_0 = (60/\epsilon^{1/2}) \cosh^{-1} U$$

$$U = \tfrac{1}{2}\,[(D/d) + (d/D) - (4c^2/dD)]$$

where we may take d = our d_1 and D = our d_2 and c = our b = the center-line separation.

There is one more case of interest that falls out from this analysis. If we take $B_1 = 0$ we have



which is a transmission line consisting of a round wire above an infinite flat plane. Since $B_1 = 0$ we find from *** that

$$K = 2B_2$$

We know that

$$x_{2c} = d \coth B_2 = d\ \ chB_2/shB_2$$
$$a_2 = d/sh(B_2)$$

Therefore

$$x_{2c}/a_2 = chB_2 \quad \Rightarrow \quad B_2 = ch^{-1}(x_{2c}/a_2) \quad \Rightarrow \quad K = 2\ ch^{-1}(x_{2c}/a_2)$$

Here x_{2c} is the distance from the wire center line to the ground plane. If we call this h and the wire radius a, we then have the following extremely simple and exact result,

$K = 2\ ch^{-1}(h/a)$ // wire radius a with center h over ground plane, exact

$Z_0 = (K/\sqrt{\varepsilon_{rel}})\ 30\Omega = ch^{-1}(h/a)\ (1/\sqrt{\varepsilon_{rel}})\ 60\ \Omega$

where of course we must have h > a to keep the wire from touching the ground plane. Using the identity $ch^{-1}x = \ln(x + \sqrt{x^2-1})$ for $x \geq 1$ we can write the above as

$K = 2 \ln [\ (h/a) + \sqrt{(h/a)^2-1}\ ]$ // wire radius a with center h over ground plane, exact

$Z_0 = \ln [\ (h/a) + \sqrt{(h/a)^2-1}\ ]\ (1/\sqrt{\varepsilon_{rel}})\ 60\ \Omega$

For h >> a this becomes

$K = 2 \ln(2h/a)$ // wire radius a center h over ground plane, h>> a

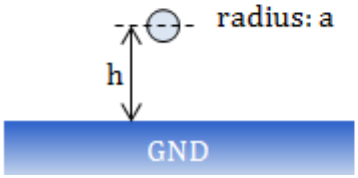
$Z_0 = \ln(2h/a)\ (1/\sqrt{\varepsilon_{rel}})\ 60\ \Omega$

For verification, we found the following obscure web offering,

The characteristic impedance of a wire over a ground plane:

$$Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu}{\varepsilon}}\log\left(\frac{h}{a} + \sqrt{\left(\frac{h}{a}\right)^2 - 1}\right) \cong \frac{1}{2\pi}\sqrt{\frac{\mu}{\varepsilon}}\log\left(\frac{2h}{a}\right) \quad (a \ll h)$$

$$\frac{1}{2\pi}\sqrt{\frac{\mu}{\varepsilon}} \cong \frac{60[\Omega]}{\sqrt{\varepsilon_r}}$$



http://members3.jcom.home.ne.jp/zakii/tline_e/14_microstripline_z0.htm

which results are derived using an image method to handle the ground plane.

For some odd reason, our usual RDE source on this subject only gives the result for h >> a . Taking d to be the wire diameter,

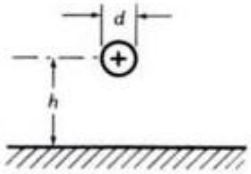
$Z_0 = \ln(4h/d)\ (1/\sqrt{\varepsilon_{rel}})\ 60\ \Omega$

$= \ln(10) \log (4h/d)\ (1/\sqrt{\varepsilon_{rel}})\ 60\ \Omega$

$\approx \log (4h/d)\ (1/\sqrt{\varepsilon_{rel}})\ 138.2\ \Omega$

which then compare to RDE p 29-22

G. Single wire, near ground



For $d \ll h$

$$Z_0 = (138/\epsilon^{1/2}) \log_{10}(4h/d)$$