This is installed.

Chapter 6: Transmission Lines with Two Circular Conductors 1
6.1 A candidate transverse potential φ_t 1
6.2 Ancient Greece circa 230 BC 2
6.3 Back to the Future: Calculation of K 4
6.4 Summary of Results 11

Chapter 6: Transmission Lines with Two Circular Conductors

6.1 A candidate transverse potential φ_t

In the previous chapter (both Section 5.3 (b) and Section 5.4 (b)) we showed that the transverse potential of a 2-conductor balanced transmission line must have this form when viewed from far away,

$$\varphi_t(\mathbf{x}) \approx \ln(s_2^2/s_1^2) \quad \text{// limiting form as point } \mathbf{x} = (x,y) \text{ moves far from the conductors} \qquad (5.3.11)$$

$$s_1^2 = (x-x_1)^2 + (y-y_1)^2 = |\mathbf{x} - \mathbf{x}_1|^2$$
$$s_2^2 = (x-x_2)^2 + (y-y_2) = |\mathbf{x} - \mathbf{x}_2|^2 \qquad (6.1.1)$$

where the points $\mathbf{x}_1$ and $\mathbf{x}_2$ are the "center of charge" points for the C_1 and C_2 conductor cross sections.

Suppose now we take as a candidate dimensionless transverse potential φ_t *exactly* the above limiting expression. Our candidate φ_t is

$$\varphi_t(\mathbf{x}) = \ln(s_2^2/s_1^2) \ . \qquad \text{for } \textit{all} \text{ values of } \mathbf{r}\text{, close and far} \qquad (6.1.2)$$

where we specify that our center of charge points are $\mathbf{x}_1 = (d,0)$ and $\mathbf{x}_2 = (-d,0)$.

Certainly this meets our limiting form boundary condition (5.3.11)! We know also that this potential is a valid solution of the 2D Laplace equation, since $\ln(s_1)$ and $\ln(s_2)$ are each valid solutions. This fact was shown at the start of Section 5.4 (b). Since $-2\pi\ln s_1$ is the 2D free-space propagator, it follows that $-2\pi\ln s_1$ is a solution of $\nabla^2_{2D}(\varphi) = 0$ away from the point where $s_1 = 0$, and then so is $\ln s_1$. Then by superposition, $2\ln s_2 - 2\ln s_1$ is also a valid solution, and this is $\ln(s_2^2/s_1^2)$. Thus, our φ_t is a valid candidate for a lossless transmission line since for such a transmission line φ_t satisfies the 2D Laplace equation according to (5.4.3),

$$\nabla_t^2 \varphi_t(x,y) = 0 \qquad \varphi_t(C_1) = K_1 \quad \varphi_t(C_2) = K_2 \qquad K_1 - K_2 = K \ . \qquad (5.4.3)$$

The question then becomes: what are the surfaces C_i in 2D space on which this candidate φ_t is a constant? Such surfaces can then serve as possible conductor cross sections for a transmission line.

6.2 Ancient Greece circa 230 BC

Apollonius of Perga (262BC-190BC) [like Joe of Chicago] was a pretty smart guy as wiki explains. He did astronomy and therefore he did geometry. Besides giving conic sections their current names and writing eight books about them, he learned about what are now called the Apollonian Circles. These circles form the "level surfaces" for 2D bipolar (orthogonal) coordinates as shown in this picture

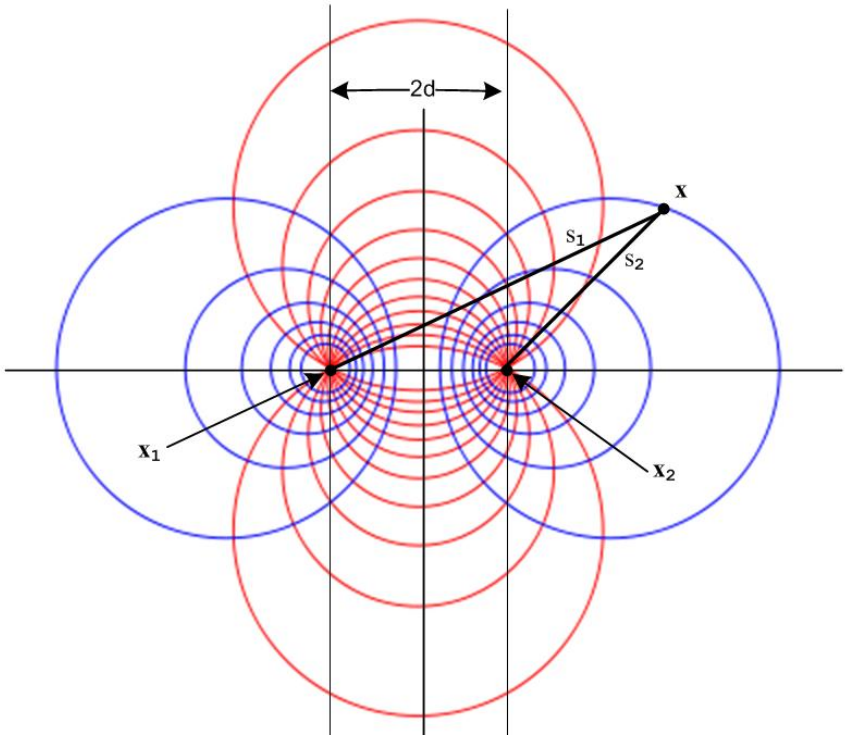


Fig 6.1

http://en.wikipedia.org/wiki/Apollonian_circles

When this picture is rotated around its vertical axis, the blue level circles become toroids and one then arrives at 3D toroidal coordinates, but that is another story. Our interest is in the 2D blue circles.

It turns out, as the reader may suspect, that the blue circles have the following simple property:

$$s_2/s_1 = \text{constant}$$

which we shall prove in a moment. Calling this constant e^{-B} we get

$$s_2/s_1 = e^{-B} \quad \Rightarrow \quad \ln(s_2/s_1) = -B \ . \tag{6.2.1}$$

Thus, since $\varphi_t(\mathbf{x}) = \ln(s_2^2/s_1^2) = 2\ln(s_2/s_1)$, the blue circles are candidate equipotential surfaces for our potential $\varphi_t(\mathbf{x})$!

To show that $s_2/s_1 = e^{-B}$ describes a circle, consider:

$$|\mathbf{x}-\mathbf{x}_2| \,/\, |\mathbf{x}-\mathbf{x}_1| = e^{-B}$$

$$|\mathbf{x}-\mathbf{x}_2|^2 = e^{-2B}\,|\mathbf{x}-\mathbf{x}_1|^2$$

$$(x-x_2)^2 + (y-y_2)^2 = e^{-2B}\,[(x-x_1)^2 + (y-y_1)^2\,] \ .$$

This equation has the following form

$$A(x^2 + y^2) + Bx + Cy + D = 0 \qquad\qquad A = (1-e^{-2B})$$

or

$$x^2 + y^2 + \alpha x + \beta y + \gamma = 0 \ .$$

One can then "complete the squares" to obtain the equation of a circle of radius r centered at (x_c,y_c) ,

$$(x - x_c)^2 + (y - y_c)^2 = r^2$$

$$\text{where} \qquad -2x_c = \alpha \quad -2y_c = \beta \qquad x_c^2 + y_c^2 - r^2 = \gamma \ . \tag{6.2.2}$$

For our particular locations of $\mathbf{x}_1$ and $\mathbf{x}_2$ shown in Fig 6.1, we have

$$x_1 = -d \qquad x_2 = d \qquad y_1 = y_2 = 0$$
$$s_1^2 = (x+d)^2 + y^2 \qquad s_2^2 = (x-d)^2 + y^2 \tag{6.2.3}$$

so

$$s_2/s_1 = e^{-B} \ \Rightarrow\ s_1/s_2 = e^{B} \ \Rightarrow\ e^{2B}s_2^2 = s_1^2 \ \Rightarrow\ (e^{B}/2)\, s_2^2 = (e^{-B}/2)\, s_1^2 \ \Rightarrow$$

$$(e^{B}/2)\,[x^2 - 2dx + d^2 + y^2] = (e^{-B}/2)\,[x^2 + 2dx + d^2 + y^2]$$

$$\text{shB}\,(x^2+d^2+y^2) + \text{chB}(-2dx) = 0 \qquad // \ \text{shB} = (e^{B}-e^{-B})/2,\ \text{chB} = (e^{B}+e^{-B})/2$$

$$(x^2+d^2+y^2) + \text{cothB}\,(-2dx) = 0$$

$$x^2 - 2d\,x\,\text{cothB} + y^2 = -d^2$$

$$x^2 - 2d\,x\,\text{cothB} + d^2\text{coth}^2\text{B} + y^2 = -d^2 + d^2\text{coth}^2\text{B} \qquad // \text{complete the square}$$

$$(x - d\,\text{cothB})^2 + y^2 = d^2\text{csch}^2\text{B} \ . \tag{6.2.4}$$

We conclude that our blue equipotential circles have this simple form

$$(x - x_c)^2 + y^2 = r^2 \qquad x_c = d\,\text{cothB} \qquad r = |d\,\text{cschB}| \tag{6.2.5}$$

Using d = 5, here is a plot of these circles for 10 different B values:

```
restart; with(plots):with(plottools):
xc := d*coth(B):
r := d*csch(B):
d := 5:
Bvals := [-3,-2,-1.5,-1,-.75,.75,1,1.5,2,3]:N := nops(Bvals):
for n from 1 to N do
   B := Bvals[n];
   c[n] := circle([xc,0],r);
od:
display(seq(c[n],n=1..N),color=blue,scaling = constrained,thickness=2);
```

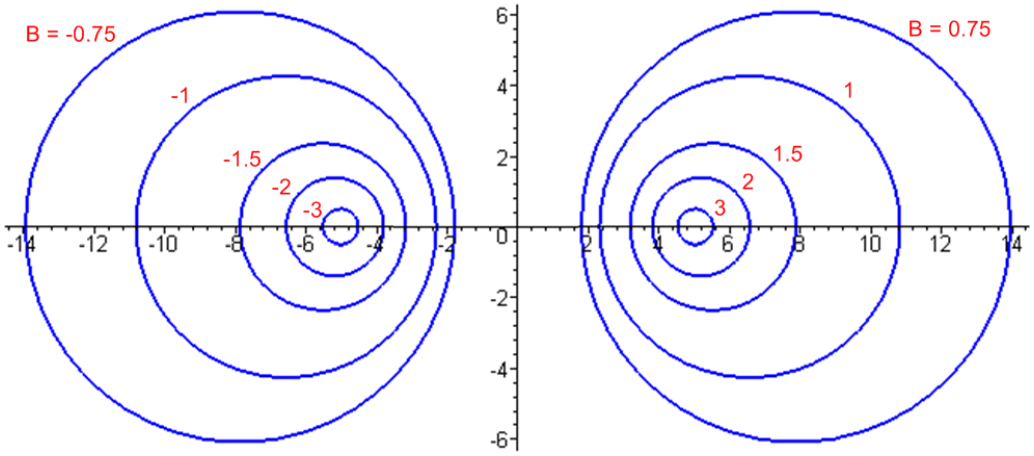


Fig 6.2

Since x_c = d cothB, the right side curves have B > 0 while the left side have B < 0. The value B = 0 corresponds to the vertical y axis, while B = ±∞ correspond to the two focal points at d = ± 5.

6.3 Back to the Future: Calculation of K

We select C_2 to be a circle on the right side, so that $B_2 > 0$.
For C_1 we select a second circle from either the left or the right, so B_1 can have either sign.

If we select C_1 from the left side, we have a two-wire transmission line(dielectric = gray),

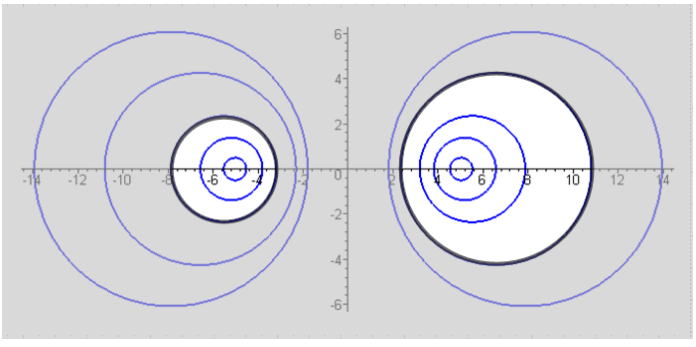


Fig 6.3

If we select C_2 from the right, we have an off-center coaxial transmission line.

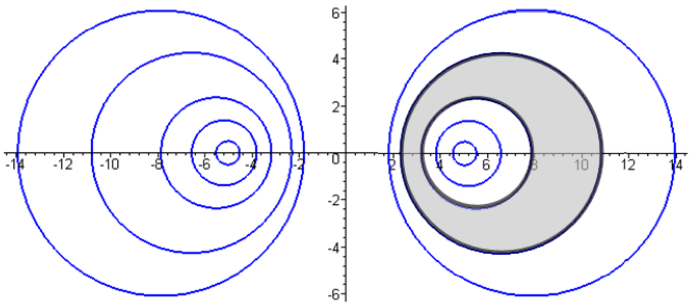


Fig 6.4

Fig 6.3 shows a transmission line cross section where the two conductors are round wires with unequal radii a_1 and a_2. Treated as a 2D capacitor, one's intuition at least suggests that the two focal points might be the conductor "centers of charge". The dielectric is of course outside the two conductors and it is possible to select a point in the dielectric that is "far away" from both conductors, so our limiting form discussion applies and the points $\mathbf{x}_1$ and $\mathbf{x}_2$ should be the centers of charge.

Figure 6.4 shows an off-center coaxial transmission line for which the dielectric is the region between the two black circles. In this case, one cannot take a point in the dielectric that is "far away" from both conductors, so the limiting form discussion does not apply. Here it appears that both conductors have the same center of charge located at $\mathbf{x}_2$.

We shall now determine K and therefore the 2D capacitance $C = 4\pi\varepsilon/K$ for the above cases.

Let $\sigma_1 = \text{sign}(B_1)$. We then have

$$\varphi_t(\mathbf{x}) = \ln(s_2^2/s_1^2) = 2\ln(s_2/s_1)$$

$$\begin{aligned}\varphi_t(C_1) &= 2\ln(s_2/s_1)|^{C1} = -2B_1 \\ \varphi_t(C_2) &= 2\ln(s_2/s_1)|^{C2} = -2B_2\ .\end{aligned} \qquad (6.3.1)$$

Recall from (5.1.3) that $\varphi_t(C_1) - \varphi_t(C_2) = K$. Therefore,

$$K = 2(B_2 - B_1) = 2(|B_2| - \sigma_1|B_1|). \qquad (6.3.2)$$

Once we know K, we know C, G and L_e for the transmission line from (4.11.30).

We must now do some slightly painful algebra. First, we know from (6.2.5) that

$$\begin{aligned}a_1 = d\,|\text{csch}B_1| &\Rightarrow (d/a_1) = \text{sh}(|B_1|) \Rightarrow |B_1| = \text{sh}^{-1}(d/a_1) \\ a_2 = d\,|\text{csch}B_2| &\Rightarrow (d/a_2) = \text{sh}(|B_2|) \Rightarrow |B_2| = \text{sh}^{-1}(d/a_2)\end{aligned} \qquad (6.3.3)$$

The separation of the centers of the two round wires is b, where

$$b = |x_{c2} - x_{c1}| = |d\,\text{coth}B_2 - d\text{coth}B_1| = d\,|\text{coth}B_2 - \text{coth}B_1| \qquad (6.3.4)$$

From (6.3.2) we write

$$ch(K/2) = ch\,[|B_2| - \sigma_1 |B_1|]$$

$$= ch|B_2|\, ch|B_1| \; - \sigma_1\, sh|B_2|\, sh|B_1|$$

$$= \sqrt{1+sh^2 B_2}\,\sqrt{1+sh^2 B_1} \; - \sigma_1\, sh|B_2|\, sh|B_1|$$

$$= \sqrt{1+(d/a_2)^2}\,\sqrt{1+(d/a_1)^2} \; - \sigma_1\,(d/a_2)\,(d/a_1) \; . \qquad (6.3.5)$$

Meanwhile,

$$b = \; d\,|coth B_2 - coth B_1| \; = |d\,[\,ch B_2/sh B_2 - ch B_1/sh B_1]\,| \; = \; |d\,[\,ch|B_2|/sh|B_2| - \sigma_1 ch|B_1|/sh|B_1|]\,|$$

$$= |\; d \;\; [ch|B_2|\, sh|B_1| - \sigma_1\, ch|B_1|\, sh|B_2|\,] \,/\, sh|B_1|\, sh|B_2| \;\;|$$

$$= |\;\; d\,[\sqrt{1+(d/a_2)^2}\,(d/a_1) - \sigma_1\,\sqrt{1+(d/a_1)^2} \;\; (d/a_2)\,] \,/\;\; (d/a_2)\,(d/a_1)\,|$$

$$= |\,[\sqrt{1+(d/a_2)^2}\,(1/a_1) - \sigma_1\,\sqrt{1+(d/a_1)^2} \;\; (1/a_2)\,] \,/\;\; (1/a_2)\,(1/a_1)\,|$$

$$= |\,[a_2\,\sqrt{1+(d/a_2)^2} - \sigma_1\, a_1\,\sqrt{1+(d/a_1)^2} \;\;]\,| \;\; . \qquad (6.3.6)$$

Square this to get

$$b^2 = a_2^2[1+(d/a_2)^2] \; + a_1^2[1+(d/a_1)^2] \; - 2\sigma_1 a_1 a_2 \sqrt{1+(d/a_2)^2}\,\sqrt{1+(d/a_1)^2}$$

so

$$2\,\sigma_1 a_1 a_2 \sqrt{1+(d/a_2)^2}\,\sqrt{1+(d/a_1)^2} \;\; = a_2^2[1+(d/a_2)^2] \; + a_1^2[1+(d/a_1)^2] - b^2$$

$$= a_2^2 + d^2 + a_1^2 + d^2 - b^2 \quad = \; a_1^2 + a_2^2 + 2d^2 - b^2 \; .$$

The purpose of doing this is to obtain the following expression for the radical product,

$$\sqrt{1+(d/a_2)^2}\,\sqrt{1+(d/a_1)^2} \quad = (a_1^2 + a_2^2 + 2d^2 - b^2) \,/\, (2\,\sigma_1 a_1 a_2\,) \;\; . \qquad (6.3.7)$$

We now install this into our expression (6.3.5) above for ch(K/2) to get

$$ch(K/2) = \sqrt{1+(d/a_2)^2}\,\sqrt{1+(d/a_1)^2} \; - \sigma_1\,(d/a_2)\,(d/a_1)$$

$$= (a_1^2 + a_2^2 + 2d^2 - b^2) \,/\, (2\,\sigma_1 a_1 a_2\,) - 2d^2/\,(2\sigma_1 a_2 a_1)$$

$$= (a_1^2 + a_2^2 - b^2) \,/\, (2\sigma_1 a_2 a_1) \;\; = \;\; \sigma_1\,(1/2)\,(a_1^2 + a_2^2 - b^2)/(a_1 a_2)$$

$$= \sigma_1 (1/2) [\ (a_1/a_2) + (a_2/a_1) - (b^2/a_1a_2)\ ] \qquad (6.3.8)$$

and the focal distance d has vanished from our expression. Therefore

$$K = 2\ ch^{-1}\ \{\ \sigma_1 (1/2) [\ (a_1/a_2) + (a_2/a_1) - (b^2/a_1a_2)\ ]\ \} \qquad (6.3.9)$$

Notice that the result is symmetric under $a_1 \leftrightarrow a_2$.

We now distinguish our two cases of interest. For the unequal twin-lead type transmission line of Fig 6.3 we know that $B_1 < 0$ since the C_1 circle is on the left, so $\sigma_1 = \text{sign}(B_1) = -1$ and then

$$K = 2\ ch^{-1}\ \{\ (1/2) [\ (b^2/a_1a_2) - (a_1/a_2) -\ (a_2/a_1)]\ \} \qquad \text{// Fig 6.3} \qquad (6.3.10)$$

which is an amazingly simple result. Recall from (4.4.16) that

$$Z_0 = (K/\sqrt{\varepsilon_{rel}})\ 30\Omega \qquad (4.4.16)$$

so then

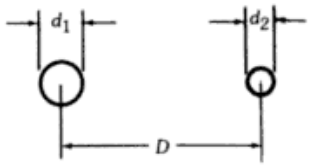
$$Z_0 = ch^{-1}\ \{\ (1/2) [\ (b^2/a_1a_2) - (a_1/a_2) -\ (a_2/a_1)]\ \}\ (1/\sqrt{\varepsilon_{rel}})\ 60\ \Omega\ . \qquad (6.3.11)$$

If the wires have diameters $d_1 = 2a_1$ and $d_2 = 2a_2$ this becomes

$$Z_0 = ch^{-1}\ \{\ (1/2) [\ (4b^2/d_1d_2) - (d_1/d_2) -\ (d_2/d_1)]\ \}\ (1/\sqrt{\varepsilon_{rel}})\ 60\ \Omega\ . \qquad (6.3.12)$$

For verification, we quote again from Reference RDE page 29-23,

N. **Balanced 2-wire—unequal diameters**



$$Z_0 = (60/\epsilon^{1/2}) \cosh^{-1} N$$

$$N = \tfrac{1}{2}[(4D^2/d_1d_2) - (d_1/d_2) - (d_2/d_1)]$$

where our b is called D.

On the other hand, if we are interested in an off-center coaxial transmission line as in Fig 6.4, we select C_1 from the right side of Fig 6.2 and then $\sigma_1 = \text{sign}(B_1) = +1$ and we find

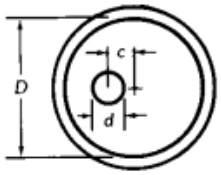
$$K = 2\ ch^{-1}\ \{\ (1/2) [\ (a_1/a_2) + (a_2/a_1) - (b^2/a_1a_2)\ ]\ \} \qquad \text{// Fig 6.4} \qquad (6.3.13)$$

$$Z_0 = \text{ch}^{-1} \{ (1/2) [(a_1/a_2) + (a_2/a_1) - (b^2/a_1a_2) \} (1/\sqrt{\varepsilon_{rel}}) \, 60\ \Omega \qquad (6.3.14)$$

$$Z_0 = \text{ch}^{-1} \{ (1/2) [(d_1/d_2) + (d_2/d_1) - (4b^2/d_1d_2) \} (1/\sqrt{\varepsilon_{rel}}) \, 60\ \Omega \qquad (6.3.15)$$

For verification, we quote again from Reference RDE page 29-24,

U. Eccentric line



$$Z_0 = (60/\epsilon^{1/2}) \cosh^{-1} U$$

$$U = \tfrac{1}{2} [(D/d) + (d/D) - (4c^2/dD)]$$

where we may take d = our d_1 and D = our d_2 and c = our b = the center-line separation.

There is one more case of interest that falls out from this analysis. If we take $B_1 = 0$ we have

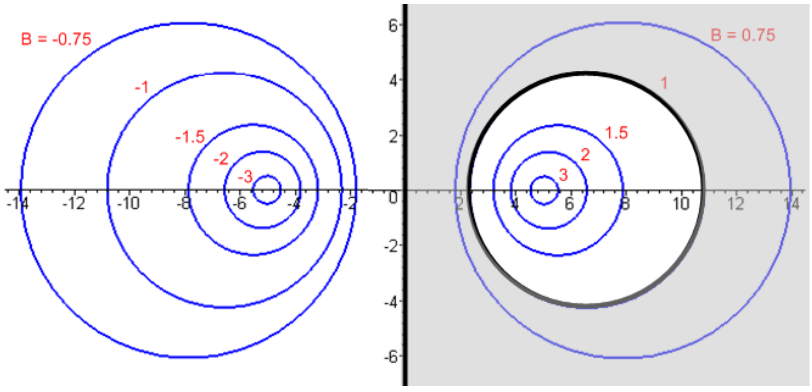


Fig 6.5

which is a transmission line consisting of a round wire above an infinite flat plane. This is a tricky limit of (6.3.10) where both $a_1 \to \infty$ and $b \to \infty$, so we ignore those expressions and work from scratch. Since $B_1 = 0$ we find from (6.3.2) that

$$K = 2B_2 \ . \qquad (6.3.16)$$

We know from (6.2.5) that

$$x_{2c} = d \coth B_2 = d \ \text{ch}B_2/\text{sh}B_2$$

$a_2 = d/sh(B_2)$. (6.3.17)

Therefore

$x_{2c}/a_2 = chB_2$ => $B_2 = ch^{-1}(x_{2c}/a_2)$ => $K = 2\ ch^{-1}(x_{2c}/a_2)$. (6.3.18)

Here x_{2c} is the distance from the wire center line to the ground plane. If we call this h and the wire radius a, we then have the following extremely simple and exact result,

$K = 2\ ch^{-1}(h/a)$ // wire radius a with center h over ground plane, exact
$Z_0 = (K/\sqrt{\varepsilon_{rel}})\ 30\Omega = ch^{-1}(h/a)\ (1/\sqrt{\varepsilon_{rel}})\ 60\ \Omega$ (6.3.19)

where of course we must have h > a to keep the wire from touching the ground plane. Using the identity $ch^{-1}x = \ln(x + \sqrt{x^2-1})$ for $x \geq 1$ we can write the above as

$K = 2 \ln [\ (h/a) + \sqrt{(h/a)^2-1}\ ]$ // wire radius a with center h over ground plane, exact
$Z_0 = \ln [\ (h/a) + \sqrt{(h/a)^2-1}\ ]\ (1/\sqrt{\varepsilon_{rel}})\ 60\ \Omega$ (6.3.20)

For h >> a this becomes ("thin wire")

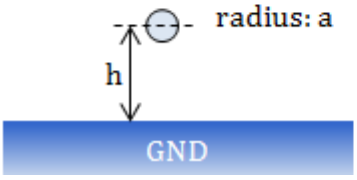
$K = 2 \ln(2h/a)$ // wire radius a center h over ground plane, h>> a
$Z_0 = \ln(2h/a)\ (1/\sqrt{\varepsilon_{rel}})\ 60\ \Omega$ (6.3.21)

For verification, we found the following web offering (where log means ln),

The characteristic impedance of a wire over a ground plane:

$$Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu}{\varepsilon}}\log\left(\frac{h}{a} + \sqrt{\left(\frac{h}{a}\right)^2 - 1}\right) \cong \frac{1}{2\pi}\sqrt{\frac{\mu}{\varepsilon}}\log\left(\frac{2h}{a}\right) \quad (a \ll h)$$

$$\frac{1}{2\pi}\sqrt{\frac{\mu}{\varepsilon}} \cong \frac{60[\Omega]}{\sqrt{\varepsilon_r}}$$



http://members3.jcom.home.ne.jp/zakii/tline_e/14_microstripline_z0.htm

which results are derived using an image method to handle the ground plane.

For some odd reason, our usual RDE source on this subject only gives the result for h >> a . Taking d to be the wire diameter,

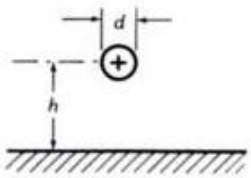
$$Z_0 = \ln(4h/d)\,(1/\sqrt{\varepsilon_{rel}})\,60\ \Omega$$

$$= \ln(10)\log(4h/d)\,(1/\sqrt{\varepsilon_{rel}})\,60\ \Omega$$

$$\approx \log(4h/d)\,(1/\sqrt{\varepsilon_{rel}})\,138.2\ \Omega \qquad (6.3.22)$$

which then compare to RDE p 29-22 ,

G. Single wire, near ground



For $d \ll h$

$$Z_0 = (138/\epsilon^{1/2})\log_{10}(4h/d)$$

Reader Exercise: Given $\varphi(\mathbf{x}) = \ln(s_2^2/s_1^2)$, compute $\mathbf{E} = -\nabla\varphi$, compute $E_n = \mathbf{E} \bullet \hat{\mathbf{n}}$ as the normal electric field at the surface of C_2, compute $n = \varepsilon E_n$ as the linear charge density on C_2, then using that n, find the "center of charge" <x> = [$\int_{C2}$ ds x n(**x**) ds / $\int_{C2}$ ds n(**x**)] and see if <x> = d. Decide whether or not it is worth while learning how to work in bipolar coordinates to carry out this exercise.

6.4 Summary of Results

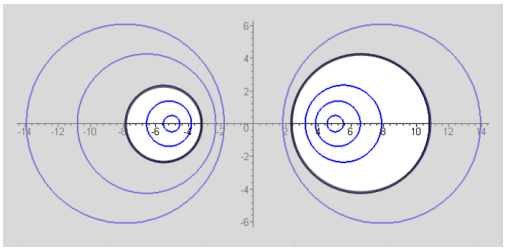
Summary for Transmission Line with Two Round Conductors (6.3.23)

Identities: $\text{ch}^{-1}x = \ln(x + \sqrt{x^2-1})$, $x \geq 1$ $\quad$ $\text{ch}^{-1}x \approx \ln(2x)$, $x >> 1$

$\text{ch}^{-1}\left[\frac{1}{2}\left(\frac{a}{b} + \frac{b}{a}\right)\right] = \text{sign}(b-a)\ln\frac{b}{a}$ $\quad$ $a > 0$ and $b > 0$ *(4.6.6)*

Line Properties: $C = 4\pi\varepsilon/K$ $\quad$ $G = 4\pi\sigma/K$ $\quad$ $L_e = \frac{\mu}{4\pi} K$ $\quad$ ε,σ,μ for dielectric

---



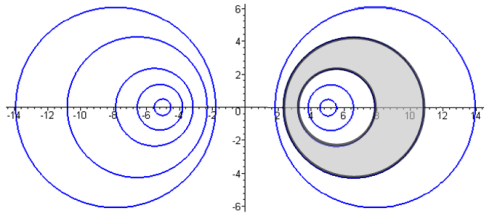
$K = 2\,\text{ch}^{-1}\{(1/2)[(b^2/a_1a_2) - (a_1/a_2) - (a_2/a_1)]\}$

a_i = radii $\quad$ b = center separation

Special case $a_1 = a_2 = a$: $\quad$ $K = 2\,\text{ch}^{-1}[(b^2/2a^2) - 1]$ $\quad$ (twin-lead)

Special case $b >> a_1, a_2$: $\quad$ $K = 4\ln(b/\sqrt{a_1a_2})$

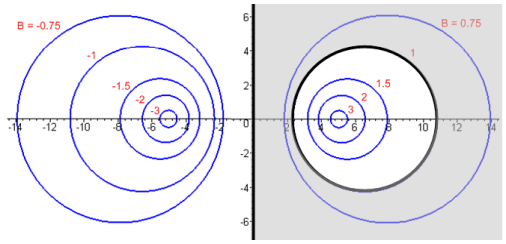
Special case $b >> a_1 = a_2 = a$: $\quad$ $K = 4\ln(b/a)$



$K = 2\,\text{ch}^{-1}\{(1/2)[(a_1/a_2) + (a_2/a_1) - (b^2/a_1a_2)]\}$

a_i = radii $\quad$ b = center separation

Special case $b = 0$ and $a_2 > a_1$: $\quad$ $K = 2\ln(a_2/a_1)$ $\quad$ (centered coaxial)



$K = 2\,\text{ch}^{-1}(h/a) = 2\ln[(h/a) + \sqrt{(h/a)^2-1}]$

a = radius $\quad$ h = height of center over plane

Special case $h >> a$: $\quad$ $K = 2\ln(2h/a)$ $\quad$ (thin wire)