2D Green's Function for Two Circles PhL 1.10.14

This is early work which is now all installed into Chapter 6, the guy from Perga and all that stuff. I think I just started over in Chapter 6 and ignored what is below.

1. First, I plan to use 2D bipolar coordinates, M&S picture of this is page 64 Fig 2.09. horizontal circles are labeled by u, vertical circles by v. This is extruded into a cylindrical coordinate system on page 81 system E4C where they are then called bi-cylindrical coordinates with that same Fig 2.09 mentioned. But here they write

$x = a\sinh\eta/(\cosh\eta-\cos\psi)$
$y = a\sin\psi/(\cosh\eta-\cos\psi)$
$z = z$

$g_{11} = g_{22} = a^2/(\cosh\eta-\cos\psi)^2 \quad g_{33} = 1$

$h_1 = h_2 = a/(\cosh\eta-\cos\psi) \qquad h_3 = 1$

Then on page 89 M&S give the differential operators including

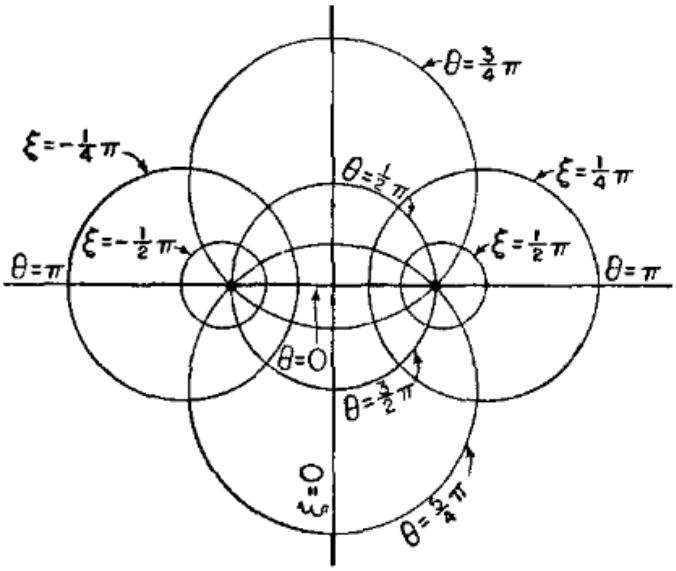
$$\nabla^2 = [(\cosh\eta-\cos\psi)^2/a^2]\,(\partial_\eta^2 + \partial_\psi^2) + \partial_z^2$$

From this I then know that

$$\nabla^2_{2D} = [(\cosh\eta-\cos\psi)^2/a^2]\,(\partial_\eta^2 + \partial_\psi^2)$$

Now I have to figure out how the various thing's are labeled! This really is the same as toroidal if you set $\varphi = 0$, and I have lots of data on that.

2. Using my "toroidal coordinates.doc" notes I will switch now to my usual coordinates,



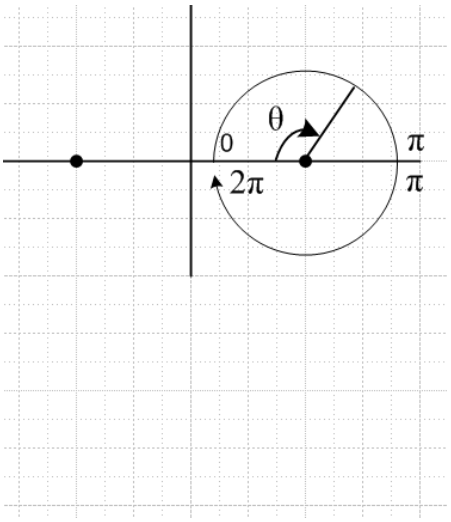


Fig. 10.10 Bipolar coordinates ξ and θ.

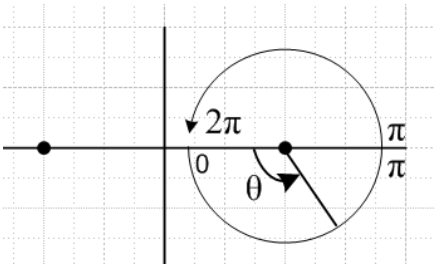
$$x = \frac{a \sinh \xi}{\cosh \xi + \cos \theta}; \quad y = \frac{a \sin \theta}{\cosh \xi + \cos \theta}; \quad h_\xi = h_\theta = \frac{a}{\cosh \xi + \cos \theta}$$

$$\xi = \tanh^{-1}\left[\frac{2ax}{a^2 + x^2 + y^2}\right]; \quad \theta = \tan^{-1}\left[\frac{2ay}{a^2 - x^2 - y^2}\right] \qquad (10.1.41)$$

$$r = a\sqrt{\frac{\sinh^2 \xi + \sin^2 \theta}{(\cosh \xi + \cos \theta)^2}}; \quad \phi = \tan^{-1}\left[\frac{\sin \theta}{\sinh \xi}\right]$$

I then go on to get the Cartesian equations of the circles.

I next change the meaning of θ to the following



and after this my equations become

$x = a\sinh\xi/(\cosh\xi+\cos\theta)$ $\quad h_\xi = h_\theta = a/(\cosh\xi+\cos\theta)$ $\quad -y/x = \sin\theta/\sinh\xi$

$-y = a\sin\theta/(\cosh\xi+\cos\theta)$

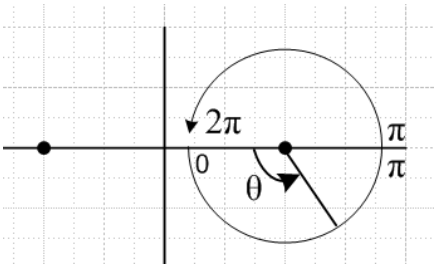
$\xi = \tanh^{-1}[2ax/(a^2+x^2+y^2)]$

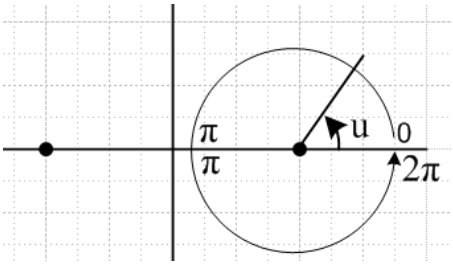
$\theta = \tan^{-1}[-2ay/(a^2-x^2-y^2)]$

$r^2 = a^2(\sinh^2\xi + \sin^2\theta)/(\cosh\xi+\cos\theta)^2$ $\quad \tan\varphi \equiv y/x = -\sin\theta/\sinh\xi$

$x^2 + (y - a\cot\theta)^2 = a^2/\sin^2\theta$ $\quad y_c = a\cot(\theta)$ $\quad R = a/|\sin\theta|$

I then change from θ to u like so





and here are the new equations.

$x = a\sinh\xi/(\cosh\xi - \cos u)$ $\quad h_\xi = h_u = a/(\cosh\xi - \cos u)$ $\quad y/x = \sin u/\sinh\xi$
$y = a\sin u/(\cosh\xi - \cos u)$

$\xi = \tanh^{-1}[2ax/(a^2 + x^2 + y^2)]$
$\tan u = [-2ay/(a^2 - x^2 - y^2)]$

$r^2 = a^2(\sinh^2\xi + \sin^2 u)/(\cosh\xi - \cos u)^2$ $\quad \tan\varphi \equiv y/x = \sin u/\sinh\xi$

$x^2 + (y - a\cot u)^2 = a^2/\sin^2 u$ $\quad y_c = a\cot(u)$ $\quad R = a/|\sin u|$
$(x - a\coth\xi)^2 + y^2 = a^2/\sinh^2\xi$ $\quad x_c = a\coth\xi$ $\quad R = a/|\sinh\xi|$

THESE equations then match those of M&S and then I presume that

$\nabla^2 = [(\cosh\xi - \cos u)^2/a^2](\partial_\xi^2 + \partial_u^2) + \partial_z^2$

$\nabla^2_{2D} = [(\cosh\xi - \cos u)^2/a^2](\partial_\xi^2 + \partial_u^2)$

The Apollonius horizontal circles are labeled by ξ (the toroid label). So my capacitor circles are going to be surfaces of constant ξ. Values are negative for left side circles, positive for right side. Then I guess my little lines doc Chapter 5 equation becomes

$[(\cosh\xi - \cos u)^2/a^2](\partial_\xi^2 + \partial_u^2)\varphi_t(\xi,u) = 0$ $\quad \varphi_t(\xi_1) = K/2$ $\quad \varphi_t(\xi_2) = -K/2$

Now here then is the Green's function problem

$[(\cosh\xi - \cos u)^2/a^2](\partial_\xi^2 + \partial_u^2)g(\xi,u|\xi',u') = \delta(\xi - \xi')\delta(u - u')(1/J)$ $\quad$??

Let's start in Cartesians

$-(\partial_x^2 + \partial_y)^2 g(x,y|x',y') = \delta(x - x')\delta(y - y')$

If we wanted to go to polar coordinates, we would have $h_1 = r$ and $h_2 = 1$ (think) and then

$$dV = rdrd\theta = J\, drd\theta = dxdy$$

$$x = r\cos\theta \qquad J = r$$
$$y = r\sin\theta \qquad \delta(x-x')\delta(y-y') = (1/r)\delta(r-r')\delta(\theta-\theta') = \delta(r-r')\delta(\theta-\theta')(1/J)$$

Now start again in Cartesians and go to bipolar coordinates. In this case

$$J = [a/(ch\xi - cosu)]^2 = h_1 h_2, \text{ and so we get}$$

$$-[(ch\xi-cosu)^2/a^2]\,(\partial_\xi^2 + \partial_u^2)g(\xi,u|\xi',u') = \delta(\xi-\xi')\delta(u-u')\,[(ch\xi-cosu)^2/a^2]$$

so the factor cancels on both sides and we end up with

$$-(\partial_\xi^2 + \partial_u^2)g(\xi,u|\xi',u') = \delta(\xi-\xi')\delta(u-u')$$

Do I believe this, it seems just too easy. It is identical to 2D Cartesian!

In lines Appendix I I show that in 2D,

$$-\nabla^2[\ln(1/R)/2\pi] = \delta(\mathbf{r}-\mathbf{r}') \qquad R = |\,\mathbf{r} - \mathbf{r}'\,| \qquad \text{(I.1.4)}$$

Now I suppose in bipolar coordinates this says

$$[(ch\xi-cosu)^2/a^2]\,(\partial_\xi^2 + \partial_u^2)\,\{ -\ln(1/R)/2\pi \} = \delta(\xi-\xi')\delta(u-u')\,[(ch\xi-cosu)^2/a^2]$$

or

$$-(\partial_\xi^2 + \partial_u^2)\,\{ \ln(1/R)/2\pi \} = \delta(\xi-\xi')\delta(u-u')$$

where you have to write our R in terms of the bipolar coordinates.

$$x = ash\xi/(ch\xi - cosu)$$
$$y = asinu/(ch\xi - cosu)$$

$$x' = ash\xi'/(ch\xi' - cosu')$$
$$y' = asinu'/(ch\xi' - cosu')$$

$$R^2 = (x-x')^2 + (y-y')^2 = [ash\xi/(ch\xi - cosu) - ash\xi'/(ch\xi' - cosu')]^2$$

$$+ [asinu/(ch\xi - cosu) - asinu'/(ch\xi' - cosu')]^2$$

$$= a^2(ch\xi - cosu)^{-2}(ch\xi' - cosu')^{-2}$$

$$\{ [sh\xi(ch\xi'–cosu') - sh\xi'(ch\xi–cosu)]^2 + [sinu(ch\xi'–cosu')-sinu'(ch\xi–cosu)]^2$$

Maple says this is a big mess. What am I doing wrong here? I looked on the web and found some very messy Green's Functions in the presence of two circles, so maybe I don't really want to go down this path.