(e) The relationship between $J_z(r,\theta)$ and $n(\theta)$ as a limit of (6.5.14)

Fact: In the extreme skin effect regime,

$$f_m(r) = -2j\,[\,(a/r)^{1/2}\, e^{-(a-r)/\delta}\;\; e^{j[(r-a)/\delta}]$$

$$f_m(a) = -2j \quad . \tag{6.5.22}$$

Proof: From NIST 10.7.8 we find that for large x argument,

$$J_m(x) \approx (2/\pi x)^{1/2} \cos(x - m\pi/2 - \pi/4) \quad . \qquad \text{NIST 10.7.8}$$

Then for $x = \beta r = (j-1)(r/\delta)$ from (6.5.9) and with $\delta << r \le a$ we keep only the larger exponential term so that

$$J_m(x) \approx (2/\pi x)^{1/2} \cos(x - m\pi/2 - \pi/4) \;\approx\; (2/\pi x)^{1/2}\;(1/2)\, e^{+r/\delta}\, e^{j[(r/\delta)+m\pi/2+\pi/4]}$$

$$= (1/\sqrt{2}\,\pi\beta r)^{1/2}\;\; e^{+r/\delta}\, e^{j[(r/\delta)+m\pi/2+\pi/4]} \quad . \tag{6.5.23}$$

From (6.5.14),

$$f_m(r) \;\equiv\; \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \quad . \tag{6.5.14}$$

We then examine the two terms

$$\frac{J_m(x)}{J_{m+1}(x_a)} = \frac{(1/\sqrt{2}\,\pi\beta r)^{1/2}\;\; e^{+r/\delta}\, e^{j[(r/\delta)+m\pi/2+\pi/4]}}{(1/\sqrt{2}\,\pi\beta a)^{1/2}\;\; e^{+a/\delta}\, e^{j[(a/\delta)+(m+1)\pi/2+\pi/4]}} = (a/r)^{1/2}\, e^{-(a-r)/\delta}\;\; e^{j[(r-a)/\delta-\pi/2]}$$

$$\frac{J_m(x)}{J_{m-1}(x_a)} = (a/r)^{1/2}\, e^{-(a-r)/\delta}\;\; e^{j[(r-a)/\delta+\pi/2]} \quad . \tag{6.5.24}$$

Then

$$f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] = (a/r)^{1/2}\, e^{-(a-r)/\delta}\;\; e^{j[(r-a)/\delta}\,[e^{-j\pi/2} - e^{+j\pi/2}]$$

$$= (a/r)^{1/2}\, e^{-(a-r)/\delta}\;\; e^{j[(r-a)/\delta}\,[-j - j]$$

$$= -2j\,[\,(a/r)^{1/2}\, e^{-(a-r)/\delta}\;\; e^{j[(r-a)/\delta}] \,.$$

which then verifies (6.5.22). QED

Recalling now (6.5.14),

$$J_z(r,\theta) = (I/\pi a^2)(\beta a/4)\,[\; f_0(r) + 2\,\Sigma_{m=1}^{\infty}\,(-1)^m\, e^{-m|\xi_1|}\, f_m(r)\cos(m\theta)\;] \qquad (6.5.14)$$

we immediately obtain this skin effect limit,

$$J_z(r,\theta) = (-2j)\,[(a/r)^{1/2}\, e^{-(a-r)/\delta}\; e^{j[(r-a)/\delta}\,](I/\pi a^2)(\beta a/4)\,* \\ [\; 1 + 2\,\Sigma_{m=1}^{\infty}\,(-1)^m\, e^{-m|\xi_1|}\cos(m\theta)\;] \quad . \qquad (6.5.25)$$

But the second object in square brackets also appears in (6.5.5),

$$n_1(\xi_1,\theta) = (q/2\pi)[\; 1 + 2\sum_{m\,=\,1}^{\infty}\,(-1)^m\, e^{-m|\xi_1|}\cos(m\theta)\;] \quad . \qquad \textit{Bipolar (A.13)} \qquad (6.5.5)$$

allowing us to write

$$J_z(r,\theta) = (-2j)\,[(a/r)^{1/2}\, e^{-(a-r)/\delta}\; e^{j[(r-a)/\delta}\,]\,(I/\pi a^2)(\beta a/4)\,(2\pi/q)\,n_1(\xi_1,\theta)$$

$$J_z(a,\theta) = (-2j)\,(I/\pi a^2)(\beta a/4)\,(2\pi/q)\,n_1(\xi_1,\theta) \quad . \qquad (6.5.26)$$

Before continuing with details, we see at once that both $J_z(r,\theta)$ and $J_z(a,\theta)$ have the same angular distribution as the surface charge density $n_1(\xi_1,\theta)$. Now recall that

$$I = q\,v_d \qquad (4.11.19a)$$

$$\beta = e^{j3\pi/4}\,(\sqrt{2}\,/\delta) \qquad (6.5.9)$$

where v_d is light speed in the transmission line dielectric. Then we find, setting r = a,

$$J_z(a,\theta) = (-2j)\,(I/\pi a^2)(\beta a/4)\,(2\pi/q)\,n_1(\xi_1,\theta)$$

$$= (-j)\,(I/a^2)(\beta a)\,(1/q)\,n_1(\xi_1,\theta) = (-j)\,(v_d/a^2)(\beta a)\,n_1(\xi_1,\theta)$$

$$= (-j)\,(v_d/a)([\; e^{j3\pi/4}\,(\sqrt{2}\,/\delta)]a)\,(1/a)\,n_1(\xi_1,\theta)$$

$$= e^{-j2\pi/4}\, e^{j3\pi/4}\,(v_d/a)\,\sqrt{2}\;(a/\delta)\,(1/a)\,n_1(\xi_1,\theta)$$

$$= e^{+j2\pi/4}\sqrt{2}\;(v_d/a)\,(a/\delta)\,(1/a)\,n_1(\xi_1,\theta)$$

$$= (1+j)\,(v_d/a)\,(a/\delta)\,(1/a)\,n_1(\xi_1,\theta)$$

$$= (v_d/a)\,(1+j)\,(a/\delta)\,(1/a)\,n_1(\xi_1,\theta) \qquad (6.5.27)$$

which agrees with the result (6.5.21) of the previous section. This seems to justify the assumptions made in the derivation of that result. Both derivations rely on the charge pumping boundary condition (D.2.24).