This text was installed into lines doc, and then edits were done there for the newer forms of the E and B fields which are general and handle G = 0 and G>0 at the same time. But the basic outline is as shown below. This is then just an ARCHIVE file.

Chapter 7: The Low Frequency Limit of the Theory 1
7.1 A Review of Appendix D 1
(a) The Four Equations 1
(b) The Two Boundary Conditions 2
(c) The e^{-jkz} Ansatz 2
(d) The Appendix D E-field Solutions 4
7.2 First Sign of Trouble: J_z Asymmetry at DC 4
7.3 A proof that J_z must be uniform at DC 7
7.4 Second Sign of Trouble: Infinite B fields as ω→0 13
7.5 What is the cause of the Trouble as ω→ 0 ? 17

Chapter 7: The Low Frequency Limit of the Theory

By "the theory" we mean a combination of the developments of Chapters 1 through 6 on transmission lines, and the development of Appendix D which describes the E fields inside a round wire. It would seem to be a simple process to take the various results of this theory and evaluate expressions in the limit ω→ 0 (the DC limit). However, carrying out this task is like trying to run lengthwise through a Pyracantha hedge in one's birthday suit. The theory has many moving parts, and each attempted step forward seems to result in a newfound pain. In this chapter we shall explore this low-frequency issue in as systematic a fashion as possible. We shall show that our entire theory is invalid as ω→0, and shall show why that is the case.

7.1 A Review of Appendix D

Since the reader has likely not read this very long appendix, we summarize it here.

The appendix considers a round wire of radius a which is one conductor of an infinite transmission line. The other conductor(s) may or may not be round in cross section.

A cylindrical coordinate system is set up in the round conductor with the conductor center line as the z axis. The azimuthal angle θ is replaced by an integer m in a complex Fourier Series transform, for example $f(\theta) = \Sigma_{m=-\infty}^{\infty} e^{jm\theta} f(m)$. This has the benefit of allowing the replacement of ∂_θ in "θ space" by jm in "m space": $\partial_\theta f(\theta) = \Sigma_{m=-\infty}^{\infty} e^{jm\theta}$ [jmf(m)].

Time dependence of everything is taken to be $e^{j\omega t}$.

We then consider a set of four equations, two boundary conditions, and a special ansatz.

(a) The Four Equations

One of the four equations is div **E** = 0, which we presume is true inside our conducting round wire. We ignore the miniscule deviation due to the radial Hall effect described in Appendix N.7 which causes a tiny charge density ρ to exist inside the conductor. The other three equations are $(\nabla^2 + \beta^2)$ **E** = 0 which is a vector Helmholtz equation treated in cylindrical coordinates. This vector equation is of course three separate equations, two of which cross-couple field components. Parameter β is the wavenumber inside the conductor and for all practical purposes we know from (1.5.1d) that that $\beta^2 = -j\omega\mu\sigma$ where μ and σ are the magnetic permeability and electrical conductivity of the conductor, which we often imagine to be made of copper which has $\mu = \mu_0$.

(b) The Two Boundary Conditions

The first boundary condition is charge conservation at the wire surface r = a. If the dielectric is a vacuum, this boundary condition takes the form $E_r(r=a,\theta) = (j\omega/\sigma)\, n(\theta)$ in θ-space, and $E_r(r=a,m) = (j\omega/\sigma)\, N_m$ in m-space, where N_m are the Fourier Series projections of the surface charge n(θ). Location a really means a-ε, a point just below the surface. The idea is that the charge on the surface is fed by the radial current just below the surface, so $J_r(r=a-\varepsilon,\theta) = j\omega\, n(\theta)$, which would say $J_r = \partial_t n(\theta)$ in the time domain. One might fairly wonder about the validity of this condition, imagining that n(θ) could also be fed by surface currents flowing in either the θ or z direction. By surface currents, we do not mean the skin effect current sheath which exists at large ω; we refer instead to the current of the actual free surface charge on the conductor surface moving around, and we refer to such currents as Debye surface currents, since the surface charge has a tiny thickness on the order of the Debye length (Appendix E). We have convinced ourselves that such surface currents are not large enough compared to the bulk conductor currents to alter the boundary condition, but it is not a slam dunk argument, and the reader is referred to our fretting in Section D.9. In Section D.9 (d) this boundary condition is generalized for a conducting dielectric.

The second boundary condition is that the azimuthal tangential electric field E_θ must vanish at the round wire surface. This is written $E_\theta(r=a,\theta) = 0$ in θ-space and $E_\theta(r=a,m) = 0$ in m-space. The main argument here is that up to quite a large frequency one has a quasi-electrostatic situation in this θ dimension of the problem, and any surface E_θ field that might appear would be instantly neutralized by an adjustment of the free surface charges on the conductor surface. Again, the argument is not a slam dunk, and the reader is referred to our further fretting in Section D.8.

(c) The e^{-jkz} Ansatz

In addition to our four equations and two boundary conditions, we make a rather brutal assumption at the start of Appendix D which is this: the z dependence of all three E field components is the same and is given by e^{-jkz} where k is some complex number. This is our "traveling wave ansatz" since it then implies that the field components in θ-space have the form $E_i(r,\theta,z,t) = e^{j(\omega t - kz)}\, E_i(r,\theta)$, and we just replace θ by m for m-space. Here (as usual) we use the overloaded notation of Section 1.6 (f).

We select e^{-jkz} instead of e^{+jkz} to have a wave that travels in the +z direction for Re(k) > 0.

This then injects an unspecified constant k into the machinery of Appendix D. We imagine that in the dielectric there exists a moving field pattern of strong E and B fields which slides down the transmission line. In order to match fields at the conductor boundary, these external fields must also have the traveling wave form $e^{j(\omega t - kz)}$ with the same constant k, a point we return to momentarily.

Within the context of the round wire and ignoring what is going on outside, any value of k is viable in solving our set of 4 equations, 2 boundary conditions, and the e^{-jkz} ansatz. Since the four equations are linear, new solutions can be formed by superposing solutions with different values of k.

In fact, the E_z equation is just a scalar Helmholtz equation $(\nabla^2 + \beta^2)E_z = 0$ (since E_z is a Cartesian coordinate). This equation is "separable" and any solution must be writable as a linear combination of the equation's atomic forms (harmonics). In this case those atoms are $[\ e^{jm\theta}\ ]\ [\ J_m(\sqrt{\beta^2-k^2}\ r)]\ [e^{-jkz}\ ]$. We then recognize m and k as the two "separation constants". Since the geometry requires all θ dependence to be periodic in θ with period 2π, constant m must be an integer, and we identify this with our m-space "m" already discussed. Similarly, we identify separation constant k with the k in our ansatz e^{-jkz}. To really be complete, we let m take all integral values and k take all real values. $Y_m(\sqrt{\beta^2-k^2}\ r)$ is rejected since it blows up at r = 0 which is the center of the physical wire.

So in general, whatever $E_z(r,\theta.z)$ is inside the round wire, one must be able to write it this way

$$E_z(r,\theta.z) = \Sigma_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dk\ A_{mk}\ e^{jm\theta}\ J_m(\sqrt{\beta^2-k^2}\ r)\ e^{-jkz}$$

where A_{mk} are appropriate coefficients. In Appendix D, we take a particular value of k so we don't have this general $\int dk$ integration appearing, but we must allow in a general sense that it could be required to produce a viable solution for the transmission line.

Given then this discussion of the atomic forms of the scalar Helmholtz equation, our ansatz that the z dependence is e^{-jkz} seems slightly less "brutal", and slightly more justifiable. The idea that a single value of k might suffice for the problem solution (finding the E fields inside the conductor) is then based on the idea that this internal k must be the same as the external k which arises in the transmission line dielectric region. Either from the physics model of Chapters 4 and 5, or from the electrical engineering network model of Appendix K, we have this familiar form for k

$$k = -j\sqrt{\mathbf{zy}} = -j\sqrt{(R+j\omega L)(G+j\omega C)}$$

and to get the internal/external boundary condition to match, *this* is the k we use for the k of the interior solution. One might note that the network model with its lumped components says nothing about E fields inside conductors. It is a model for the action in the dielectric.

Where does the above expression for k come from, the reader might ask. In both the physics model and the network model, one obtains certain "transmission line equations" of first and second order,

$$\frac{dV(z)}{dz} = -\mathbf{z}\, i(z) \qquad \left(\frac{d}{dz^2} - \mathbf{zy}\right) V(z) = 0 \qquad \mathbf{z} = R + j\omega L \qquad \text{transmission line equations}$$

$$\frac{di(z)}{dz} = -\mathbf{y} V(z) \qquad \left(\frac{d}{dz^2} - \mathbf{zy}\right) i(z) = 0 \qquad \mathbf{y} = G + j\omega C \qquad \textit{(4.11.14), (4.11.15)}$$

In Appendix K these appear as (K.5) and (K.6). Looking at the second order equations, one sees that both V(z) and i(z) must have the form e^{-jkz} where $k^2 = -\mathbf{zy}$, and this basically forces all E and B fields to have this same form. This then dovetails perfectly with our e^{-jkz} ansatz and furthermore provides a specific formula for k. It all seems so nice!

(d) The Appendix D E-field Solutions

Grinding through Appendix D, one obtains the following exact E field component expressions which solve the above stated problem :

Second summary of the E field solutions : $R_{dc} = \frac{1}{\sigma\pi a^2}$ $\beta'^2 = \beta^2 - k^2$ *(D.2.33)*

$E_z(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (a\beta')\, f_m$ $f_m = f_m(x,x_a) = [\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}]$ $x = \beta' r$

$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\, (ak)\, g_m$ $g_m = g_m(x,x_a) = [\frac{J_{m+1}(x)}{J_{m+1}(x_a)} + \frac{J_{m-1}(x)}{J_{m-1}(x_a)}]$ $x_a = \beta' a$

$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\, (ak)\, h_m$ $h_m = h_m(x,x_a) = [\frac{J_{m+1}(x)}{J_{m+1}(x_a)} - \frac{J_{m-1}(x)}{J_{m-1}(x_a)}]$ $\eta_m \equiv N_m/N_0$

Note the definitions $\beta' = \sqrt{\beta^2-k^2}$, $x = \beta' r$, $x_a = \beta' a$, as well as $R_{dc} = 1/(\sigma\pi a^2)$ (DC resistance per unit length of the round wire) and $\eta_m = N_m/N_0$ (the relative surface charge moment). The symbol I is the total current in the round conductor, an integral of $J_z(r,\theta) = \sigma E_z(r,\theta)$ over the cross section. It is related to other parameters in this way,

$$I = 2\pi a\, (\omega/k)\, N_0 \,. \qquad \textit{(D.2.31)}$$

The quantity $2\pi a\, (\omega/k)\, N_0$ is what really appears in (D.2.33) and we just replace it by I for short. The symbol N_0 can be written

$$N_0 = \langle n(\theta) \rangle = q/(2\pi a) = CV/(2\pi a) \quad \Rightarrow \quad I = CV\,(\omega/k) \qquad \textit{(D.1.8)}$$

where q is the total surface charge per unit length of the round wire (at z = 0), C is the capacitance of the transmission line per unit length, and V is the voltage applied to the line at z = 0.

7.2 First Sign of Trouble: J_z Asymmetry at DC

As a simple "check" on our theory, it seems reasonable to casually take $\omega \to 0$ and make sure the obvious DC results are replicated. As we now show, something is wrong with this limit.

Using E_z from the above box, consider this ratio, where we have canceled all common factors,

$$\frac{E_z(r,m)}{E_z(r,0)} = \eta_m \frac{f_m(x,x_a)}{f_0(x,x_a)} \qquad f_m(x,x_a) = [\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}] \,. \qquad (7.2.1)$$

Let us assume that as $\omega \to 0$, $\beta' \to \beta'_0$, some finite value. One then finds that

$$\text{Lim}_{\omega\to 0}\frac{E_z(r,m)}{E_z(r,0)} = \eta_m \frac{f_m(\beta'_0 r, \beta'_0 a)}{f_0(\beta'_0 r, \beta'_0 a)} \qquad f_m(x,x_a) = [\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}] \ . \qquad (7.2.2)$$

This ratio does not vanish as $\omega\to 0$! It is some complicated function of β'_0, m, r and a. The moments η_m do not vanish for $m \neq 0$, see for example (6.5.4) for two round wires.

Suppose β'_0 is very small such that $|\beta'_0 a| << 1$. Then we can use these limits of the f_m functions for small argument,

$$f_m \to (r/a)^{|m|}\,(|m|+1)\,(2/\beta' a) \qquad f_0 \to 4/(a\beta') \qquad \textit{(D.11.6)}$$

to find that

$$\text{Lim}_{\omega\to 0}\frac{E_z(r,m)}{E_z(r,0)} \approx \eta_m \frac{(r/a)^{|m|}\,(|m|+1)}{2} \qquad |\beta'_0 a| << 1 \qquad (7.2.3)$$

where the fact that the ratio is non-zero is quite explicit and dramatic.

This ratio is the same even if $\beta'_0 = 0$.

If at DC we have $E_z(r,m) \neq 0$, there must be some $\cos(m\theta)$ component in $J_z = \sigma E_z$ and therefore the current density J_z is not uniform over the round wire cross section. It happens that the current $I \to 0$ for a transmission line having conductance G = 0 (since $Z_0\to\infty$) , but we would expect that *very close to* $\omega = 0$ we should see J_z at least *approaching* uniformity on the cross section, but that is not happening.

Our conclusion is that, according to our theory, regardless of the value of $\beta' = \sqrt{\beta^2-k^2}$ as $\omega\to 0$, the above ratio is non-zero. The implication is that E_z and therefore the current distribution $J_z = \sigma E_z$ is not uniform over the round wire cross section at DC!

We know that this conclusion is incorrect and that in fact J_z *is* uniform across a wire cross section at DC. We know this from eddy current arguments as in Appendix P, and from the proof which follows in the next section. Therefore, something is wrong with our theory as $\omega\to 0$, the first sign of trouble.

Superposition does not help.

The above conclusion is unchanged even if we allow a superposition of k values for E_z (and thereby don't invoke any connection between k and $k(\omega) = -j\sqrt{(R+j\omega L)(G+j\omega C)}$). Such a superposition really makes no sense since we really want to match interior and exterior boundary conditions and have $k = k(\omega)$, but we discuss it anyway just to show that, even if it could somehow make sense, it still results in an asymmetric J_z at $\omega = 0$.

In order to find the nature of a possible superposed solution, we require it to satisfy these two boundary conditions,

$$E_r(r=a-\varepsilon,\theta,z) = (j\omega/\sigma)\, n(\theta,z,\omega) = (j\omega/\sigma)\, n(\theta)\, c(z,\omega)$$

$$E^{r}(r=a-\varepsilon,\theta,z) = 0 \qquad (7.2.4)$$

where the new feature is that $c(z,\omega)$ describes how the surface charge density varies with z. For the single-k solution we of course have $c(z,\omega) = e^{-jkz}$. It is useful to define $C(k,\omega)$ as the Fourier Transform of $c(z,\omega)$ with the convention of (1.6.8),

$$C(k,\omega) = (1/2\pi) \int dz\, e^{jkz}\, c(z,\omega)\ . \qquad (7.2.5)$$

We shall now skip a few details and just outline the development. We first assume that

$$E_z(r,m,z) = \int dk\, d_m(k,\omega)\, e^{-jkz}\, J_m(\beta' r) \qquad (7.2.6)$$

which is the most general form noted above, and we then determine $d_m(k,\omega)$ using the Helmholtz equations and div E = 0 and the two boundary conditions noted above. We find that $d_m(k,\omega) = C_{zm}$, a constant appearing in (D.2.4) which we also identify with $C_{zm} = (1/2j)(\beta'/k)K_m$ as in (D.2.9). We end up with unknown constants a_m and K_m just as we do in (D.2.21). It turns out, however, that these a_m and K_m have an extra factor of $C(k,\omega)$ [the Fourier Transform of $c(z,\omega)$ above] relative to the coefficients stated in (D.2.28). The continuous superposition solution which solves this problem is then the following

$$E_z(r,m,z) = \int dk\ C(k,\omega)\, e^{-jkz}\, E_z(r,m;\ \text{single-k}) \qquad (7.2.7)$$

where $E_z(r,m;$ single-k$)$ is what appears in (D.2.33),

$$E_z(r,m;\ \text{single-k}) = (1/4)\,\eta_m\, I\, R_{dc}\ (a\beta')\, f_m \qquad \textit{(D.2.33)} \qquad (7.2.8)$$

where

$$f_m = \left[\frac{J_m(x)}{J_{m+1}(x_a)} - \frac{J_m(x)}{J_{m-1}(x_a)}\right] \quad , \quad x = \beta' r \ , \quad x_a = \beta' a \ , \quad \beta' = \sqrt{\beta^2 - k^2}\ .$$

We now replace I by its original value $I = 2\pi a\, (\omega/k)\, N_0$ as just above (D.2.31), taking note that I is a function of k which is the integration variable. The result is

$$E_z(r,m;\ \text{single-k}) = (1/4)\,\eta_m\,[2\pi a\,(\omega/k)\,N_0]\,R_{dc}\ (a\beta')\, f_m = (1/2)\,\pi a^2\,\eta_m\, N_0 R_{dc}\ \omega\ *\ (\beta'/k)\, f_m$$

$$= (1/2)\,\eta_m\, N_0\,(\omega/\sigma)\ *\ (\beta'/k)\, f_m\,. \qquad \text{// } f_m \text{ depends on k through } \beta' \qquad (7.2.9)$$

Canceling the common factor $(1/2)\, N_0\ (\omega/\sigma)$ the ratio of interest becomes

$$\frac{E_z(r,m,z)}{E_z(r,0,z)} = \eta_m \frac{\int dk\ C(k,\omega)\, e^{-jkz}\, E_z(r,m;\ \text{single-k})}{\int dk\ C(k,\omega)\, e^{-jkz}\, E_z(r,0;\ \text{single-k})} = \eta_m \frac{\int dk\ C(k,\omega)\, e^{-jkz}\,(\beta'/k)\, f_m}{\int dk\ C(k,\omega)\, e^{-jkz}\,(\beta'/k)\, f_0} \qquad (7.2.10)$$

where $\beta' = \sqrt{\beta^2 - k^2}$. Taking $\omega \to 0$ causes $\beta \to 0$ since $\beta^2 = j\mu\sigma\omega$, so $\beta' \to jk$. In this limit,

$$\text{Lim}_{\omega\to 0}\frac{E_z(r,m,z)}{E_z(r,0,z)} = \eta_m \frac{\int dk'\ C(k',0)\, e^{-jk'z}\ f_m}{\int dk'\ C(k',0)\, e^{-jk'z}\ f_0} \qquad // \beta' = jk' \text{ inside } f_m \text{ and } f_0 \qquad (7.2.11)$$

where f_m and f_0 are complicated functions of $\beta' = jk'$, r and a. Just looking at the z dependence and ignoring all the rest, one sees that the numerator only vanishes if $C(k',0) \equiv 0$, giving a limit of 0 / 0 which is meaningless. Also, if $C(k',0) = 0$, it must be that $c(z,0) = 0$ and then $n(\theta,z) = 0$ and the transmission line has no surface charge. This is impossible since at DC it is a capacitor.

The point is that the superposition idea does not cause the ratio $J_z(r,m,z) / J_z(r,0,z)$ to vanish as $\omega \to 0$, and therefore even with a superposition solution (were it even sensible) one cannot eliminate the paradoxical result that J_z is non-uniform across the round wire cross section at DC.

Taking $C(k',0) = \delta(k'-k)$ reproduces the single-k ratio appearing earlier in (7.2.2).

An interesting superposition is to try $C(k',0) = A(k')\delta(k'-k) + B(k')\delta(k'+k)$ which is then a sum of oppositely directed waves on the transmission line. One can construct a reflection scenario where the transmission line voltage vanishes at some z = L where we assume the presence of a shorting bar. In this reflection scenario, it is possible to have a finite current I in the transmission line, but the J_z asymmetry still persists as $\omega \to 0$, as it does for any superposed solution. But we know that for two parallel round wires carrying a finite current I and -I at DC, the current density J_z should be uniform over the cross section.

7.3 A proof that J_z must be uniform at DC

Most readers would agree that the current in any uniform wire is evenly spread out over the cross section of the wire at DC. This seems a natural result, but still it is not totally obvious. For example, the electrons in our transmission line's round wire experience the magnetic field of both the round wire and the other conductor. This magnetic field causes an initial transverse deflection of the flowing conduction electrons due to the Lorentz force acting on them. But this deflection ceases when tiny charges accumulate on the wire surfaces which create a transverse electric field which then neutralizes the deflection. There is then "no reason" for the current to be non-uniform. This DC Hall effect is described in much detail in Appendix N.

Perhaps a better explanation is in terms of the eddy current analysis of Appendix P. In that Appendix, all J_z asymmetry in a round wire (skin effect and proximity effect) is associated with eddy currents, and eddy currents vanish at DC.

Since we have found anomalous J_z behavior of our Appendix D theory as $\omega \to 0$, we want to examine the Appendix D method where we *start off* at $\omega = 0$ instead of obtaining a result for $\omega > 0$ and then taking the limit $\omega \to 0$. It should be useful to see exactly what happens in this approach.

Since $\beta^2 = -j\omega\mu\sigma$ from (1.5.1d), the vector Helmholtz equation $(\nabla^2+\beta^2)\mathbf{E} = 0$ becomes a vector Laplace equation $\nabla^2\mathbf{E} = 0$. At $\omega = 0$ we have

$$k(\omega) = -j\sqrt{(R+j\omega L)(G+j\omega C)} \rightarrow -j\sqrt{RG} \quad \Rightarrow \quad k = -j\sqrt{RG} \tag{7.3.1}$$

$$Z_0(\omega) = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \rightarrow \sqrt{R/G} \quad \Rightarrow \quad Z_0 = \sqrt{R/G} \ . \tag{7.3.2}$$

The current I is then finite,

$$I = V/Z_0 = V\sqrt{G/R} \ . \tag{7.3.3}$$

At this point, we imagine that G is extremely small but non-zero (perhaps $G = 10^{-18}$ mho/m) , so the current I is then very small but non-zero. Since $k = -j\sqrt{RG} \approx 0$ (R is likely also small), we take $\partial_z \rightarrow -jk \approx 0$ wherever it appears. Since $k = k(\omega) \approx 0$, the ansatz factor $e^{-jkz} \approx 1$, and there is no z-dependence in the problem. Yes, moving a great distance down the line there is a very slight exponential decay in all fields due to $Im(k) = -\sqrt{RG}$, but we shall just ignore this slight decay. Since $\partial_z \rightarrow 0$ wherever it appears, we will replace ∇^2 by its 2D version ∇^2_{2D} in the vector Laplace equation examined below.

The two Appendix D boundary conditions (D.2.24) and (D.2.27) in θ space become, since ω = 0,

$E_r(r=a,\theta) = 0$ no charge pumping at DC since n(θ) is constant
$E_\theta(r=a,\theta) = 0$ surface charge adjusts to make this be so (7.3.4)

and in m space,

$E_r(r=a,m) = 0$
$E_\theta(r=a,m) = 0$. (7.3.5)

The entire Appendix D problem (at ω=0) can then be represented by this system of equations:

$$\nabla^2_{2D}\mathbf{E}(r,\theta) = 0 \qquad \mathrm{div}_{2D}\,\mathbf{E}(r,\theta) = 0 \qquad E_r(r=a,\theta) = 0 \quad E_\theta(r=a,\theta) = 0 \tag{7.3.6}$$

which we restate in m space using $\partial_\theta \rightarrow jm$,

$$\nabla^2_{2D}\mathbf{E}(r,m) = 0 \qquad \mathrm{div}_{2D}\,\mathbf{E}(r,m) = 0 \qquad E_r(r=a,m) = 0 \quad E_\theta(r=a,m) = 0 \ . \tag{7.3.7}$$

As our first step in solving this system, we confiscate the equation set (D.1.10) setting β = 0 and k = 0 to obtain these simplified results:

The Three Helmholtz Equations and the div E = 0 equation (in partial waves) *(D.1.20)*

$[\nabla^2\mathbf{E}]_z = 0$:

$$[r^2\partial_r^2 + r\,\partial_r - m^2\,]\,E_z(r,m) = 0 \tag{D.1.15}$$

$[\nabla^2 \mathbf{E}]_r = 0$:

$$[r^2\partial_r^2 + r\partial_r - (m^2+1)\ ]\ E_r(r,m) - 2jm\ E_\theta(r,m) = 0 \qquad \textit{(D.1.17)}$$

$[\nabla^2 \mathbf{E}]_\theta = 0$:

$$[r^2\partial_r^2 + r\partial_r - (m^2+1)]\ E_\theta(r,m)\ + 2jmE_r(r,m) = 0 \qquad \textit{(D.1.18)}$$

div $\mathbf{E} = 0$:

$$\partial_r\ [r\ E_r(r,m)] + jmE_\theta(r,m) = 0 \qquad \textit{(D.1.19)}$$

The E_z equation

Since $E_z(r,m) = E_z(r,-m)$ according to the first equation in the box, we assume $m \geq 0$ for simplicity. Since the first equation is the radial equation of the 2D Laplace equation, we know the solutions are $A_m r^m + B_m r^{-m}$ for $m > 0$, and $C + D\ln r$ for $m = 0$. Since E_z is finite at $r = 0$, we reject r^{-m} and also $\ln(r)$. Thus, for $m \geq 0$ we have $E_z(r,m) = A_{zm} r^m$ where A_{zm} are constants to be determined :

$$\begin{aligned} E_z(r,m) &= A_{zm}\ r^m \qquad & m > 0 \\ E_z(r,0) &= A_{z0} & m = 0 \end{aligned} \qquad (7.3.8)$$

Maple is happy to verify the claimed general solutions:

```
f := r^2*Diff(Ez,r,r) + r*Diff(Ez,r) - m^2*Ez;
```

$$f := r^2\left(\frac{\partial^2}{\partial r^2} Ez\right) + r\left(\frac{\partial}{\partial r} Ez\right) - m^2\, Ez$$

```
Ez := A*r^m + B*r^(-m);
```

$$Ez := A\, r^m + B\, r^{(-m)}$$

```
value(f): simplify(%);
```

$$0$$

```
m := 0;
```

$$m := 0$$

```
Ez := C + D*ln(r);
```

$$Ez := C + D\ln(r)$$

```
value(f): simplify(%);
```

$$0$$

The E_r equation

For $m = 0$ this equation reads

$$[r^2\partial_r^2 + r\partial_r - 1]\ E_r(r,0)\ = 0\ .$$

Maple gives the two solutions as

```
eq := r^2*Diff(Ez(r),r,r) + r*Diff(Ez(r),r) - (1)*Ez(r);
```

$$eq := r^2\left(\frac{\partial^2}{\partial r^2}\mathrm{Ez}(r)\right) + r\left(\frac{\partial}{\partial r}\mathrm{Ez}(r)\right) - \mathrm{Ez}(r)$$

```
dsolve(eq);
```

$$\mathrm{Ez}(r) = \frac{_C1\,(r^2-1)}{r} + \frac{_C2\,(r^2+1)}{r}$$

But r ± 1/r both diverge at r = 0, so neither solution works, and we conclude that $E_r(r,0) = 0$.

For m > 0 we solve the div **E** = 0 equation to get $jmE_\theta(r,m) = -\partial_r [r\,E_r(r,m)]$. Just as in Appendix D, we insert this into the E_r equation to eliminate E_θ with this result

$$[r^2\partial_r^2 + r\partial_r - (m^2+1)]\,E_r(r,m) - 2jm\,E_\theta(r,m) = 0$$

$$[r^2\partial_r^2 + r\partial_r - (m^2+1)]\,E_r(r,m) - 2\{-\partial_r [r\,E_r(r,m)]\} = 0$$

$$[r^2\partial_r^2 + r\partial_r - (m^2+1)]\,E_r(r,m) + 2\{[r\partial_r + 1]\,E_r(r,m)\} = 0$$

$$[r^2\partial_r^2 + 3r\partial_r - (m^2-1)]\,E_r(r,m) = 0$$

Since this equation contains only m^2, we know that $E_r(r,-m) = E_r(r,m)$ so again assume $m \geq 0$.
Maple gives the solution as

```
eq := r^2*Diff(Ez(r),r,r) + 3*r*Diff(Ez(r),r) - (m^2-1)*Ez(r);
```

$$eq := r^2\left(\frac{\partial^2}{\partial r^2}\mathrm{Ez}(r)\right) + 3r\left(\frac{\partial}{\partial r}\mathrm{Ez}(r)\right) - (m^2-1)\,\mathrm{Ez}(r)$$

```
dsolve(eq);
```

$$\mathrm{Ez}(r) = _C1\,r^{(-1+m)} + _C2\,r^{(-1-m)}$$

Since m > 0, we reject r^{-1-m} and conclude that

$$E_r(r,m) = A_{rm}\,r^{m-1} \qquad m > 0$$
$$E_r(r,0) = 0 \qquad m = 0 \qquad (7.3.9)$$

<u>The E_θ equation</u>

For m = 0 the equation reads,

$$[r^2\partial_r^2 + r\partial_r - (1)]\,E_\theta(r,0) = 0\,.$$

Since this is the same as the $E_r(r,0)$ equation which had no acceptable solutions, we conclude that as well for $E_r(r,0)$.

For m > 0 we use the div**E** = 0 equation to find that

$$jmE_\theta(r,m) = -\partial_r [r\, E_r(r,m)] = -\partial_r [\, r\, A_{rm}\, r^{m-1}\,] = -A_{rm}\partial_r[r^m] = -A_{rm} m\, r^{m-1}$$

or

$$jE_\theta(r,m) = -A_{rm}\, r^{m-1}\ .$$

Thus

$$\begin{aligned} &E_\theta(r,m) = j\, A_{rm} r^{m-1} && m > 0 \\ &E_\theta(r,0) = 0 && m = 0\ . \end{aligned} \qquad (7.3.10)$$

Alternatively, we could write the E_θ equation as

$$[r^2\partial_r{}^2 + r\partial_r - (m^2+1)]\, E_\theta(r,m) + 2jmE_r(r,m) = 0$$

$$\begin{aligned} &[r^2\partial_r{}^2 + r\partial_r - (m^2+1)]\, E_\theta(r,m) + 2jm\, A_r\, r^{m-1} = 0 && m > 0 \\ &[r^2\partial_r{}^2 + r\partial_r - \ (1)\ \ ]\, E_\theta(r,0) = 0 && m = 0 \end{aligned}$$

The m = 0 equation we already dealt with (no non-trivial solution). Maple then verifies that $E_\theta(r,m) = j\, A_r r^{m-1}$ satisfies the m>0 equation,

```
f := r^2*Diff(Eth(r),r,r) + r*Diff(Eth(r),r) - (m^2+1)*Eth(r) +2*I*m*Arm*r^(m-1);
```

$$f := r^2\left(\frac{\partial^2}{\partial r^2}\mathrm{Eth}(r)\right) + r\left(\frac{\partial}{\partial r}\mathrm{Eth}(r)\right) - (m^2+1)\,\mathrm{Eth}(r) + 2\,I\,m\,Arm\,r^{(m-1)}$$

```
Eth := r -> I*Arm*r^(m-1);
```

$$Eth := r \to I\,Arm\,r^{(m-1)}$$

```
value(f): simplify(%);
```

$$0$$

and we end up with the same results as above,

$$\begin{aligned} &E_\theta(r,m) = j\, A_{rm}\, r^{m-1} && m > 0 \\ &E_\theta(r,0) = 0 && m = 0\ . \end{aligned}$$

We now summarize our partial wave E field solutions at this point:

$$\begin{aligned} &E_z(r,m) = A_{zm} r^m && m > 0 \\ &E_z(r,0) = A_{z0} && m = 0 \end{aligned}$$

$$\begin{aligned} &E_r(r,m) = A_{rm}\, r^{m-1} && m > 0 \\ &E_r(r,0) = 0 && m = 0 \end{aligned}$$

$$\begin{aligned} &E_\theta(r,m) = j\, A_{rm}\, r^{m-1} && m > 0 \\ &E_\theta(r,0) = 0 && m = 0\ . \end{aligned} \qquad (7.3.11)$$

Finally we apply the two boundary conditions $E_r(r=a,m) = 0$ and $E_\theta(r=a,m) = 0$. They require that $A_{rm} = 0$ for m>0, so our final solution set is then

$$\begin{aligned} E_z(r,m) &= A_{zm} r^m && m > 0 \\ E_z(r,0) &= A_{z0} && m = 0 \\ E_r(r,m) &= 0 && m \geq 0 \\ E_\theta(r,m) &= 0 && m \geq 0 \end{aligned} \qquad (7.3.12)$$

The current I must be the cross-section integral of $J_z(r,0) = \sigma E_z(r,0) = \sigma A_{z0}$, so

$$I = \sigma A_{z0} * \pi a^2 \quad \Rightarrow \quad A_{z0} = I/(\pi a^2 \sigma) = I R \, . \qquad (7.3.13)$$

The fields are then,

$$\begin{aligned} E_z(r,m) &= A_{zm} r^m && m > 0 \\ E_z(r,0) &= IR && m = 0 \\ E_r(r,m) &= 0 && m \geq 0 \\ E_\theta(r,m) &= 0 && m \geq 0 \end{aligned} \qquad (7.3.14)$$

We now arrive at an interesting fact: the vector Helmholtz equation (which became a vector Laplace equation), the div**E** = 0 condition and the two boundary conditions are insufficient to determine the coefficients A_{zm} for m > 0. The system is underspecified! This situation did not arise in Appendix D as presented for general ω.

To nail down these coefficients -- which of course are critical to our proof that J_z is uniform in this DC problem -- we must call upon the Maxwell curl E equation (1.1.2) with $\partial_t \rightarrow j\omega$,

$$\text{curl } \mathbf{E} = - j\omega \mathbf{B} = 0 \qquad \text{at } \omega = 0 \, . \qquad (7.3.15)$$

We write this as

$$\text{curl } \mathbf{E}(r,\theta,z) = \hat{\mathbf{r}} \, [\, r^{-1} \partial_\theta E_z - \partial_z E_\theta] + \hat{\boldsymbol{\theta}} \, [\partial_z E_r - \partial_r E_z] + \hat{\mathbf{z}} \, [\, r^{-1} \partial_r (r E_\theta) - r^{-1} \partial_\theta E_r \,] \qquad (7.3.16)$$

or

$$\text{curl } \mathbf{E}(r,m,z) = \hat{\mathbf{r}} \, [\, r^{-1} jm E_z + jk E_\theta] + \hat{\boldsymbol{\theta}} \, [-jk E_r - \partial_r E_z] + \hat{\mathbf{z}} \, [\, r^{-1} \partial_r (r E_\theta) - r^{-1} jm E_r \,] \, . \qquad (7.3.17)$$

Setting $k \approx 0$ as above this becomes

$$\text{curl } \mathbf{E}(r,m,z) = \hat{\mathbf{r}} \, [\, r^{-1} jm E_z(r,m)] + \hat{\boldsymbol{\theta}} \, [- \partial_r E_z(r,m)] + \hat{\mathbf{z}} \, [\, r^{-1} \partial_r (r E_\theta(r,m)) - r^{-1} jm E_r(r,m) \,] \, .$$

But only E_z is non-vanishing, so this simplifies to

$$\text{curl } \mathbf{E}(r,m,z) = \hat{\mathbf{r}} \, [\, r^{-1} jm E_z(r,m)] + \hat{\boldsymbol{\theta}} \, [- \partial_r E_z(r,m)] \qquad (7.3.18)$$

so we must have

$$r^{-1}jmE_z(r,m) = 0$$
$$-\partial_r E_z(r,m) = 0 \ . \qquad (7.3.19)$$

We evaluate these first for m = 0,

$$0 = 0\ E_z(r,0) \qquad \text{condition met}$$
$$\partial_r E_z(r,0) = 0 \qquad \text{condition met since } E_z(r,0) = IR = \text{constant.} \qquad (7.3.20)$$

For m > 0 the conditions say that

$$r^{-1}jm\ A_{zm} r^m = 0$$
$$\partial_r\ A_{zm} r^m = 0 \qquad \text{or} \qquad A_{zm}\, m\, r^{m-1} = 0 \qquad (7.3.21)$$

Both conditions are satisfied only if $A_{zm} = 0$ for m > 0. We then arrive at a final field set:

$$E_z(r,m) = 0 \qquad m > 0$$
$$E_z(r,0) = IR \qquad m = 0$$
$$E_r(r,m) = 0 \qquad m \geq 0$$
$$E_\theta(r,m) = 0 \qquad m \geq 0 \qquad (7.3.22)$$

which then tells us that $J_z = \sigma E_z$ is in fact uniform over the round wire cross section at DC.

7.4 Second Sign of Trouble: Infinite B fields as ω→0

Recall from (D.11.7) the E fields for small ω (with the assumption G = 0),

$$E_z(r,m) = (1/2)\,\eta_m\, I\, R_{dc}\ (r/a)^{|m|}\ (|m|+1)$$
$$E_r(r,m) = (j/4)\,\eta_m\, I\, R_{dc}\ (ak)\ [(r/a)^{|m|+1} + (r/a)^{|m|-1}] \qquad \textit{(D.11.7)}$$
$$E_\theta(r,m) = (1/4)\,\eta_m\, I\, R_{dc}\ (ak)\ [(r/a)^{|m|+1} - (r/a)^{|m|-1}]$$

$$E_z(r,0) = I\, R_{dc}$$
$$E_r(r,0) = (j/2)\, I\, R_{dc}\ (ak)\ (r/a)$$
$$E_\theta(r,0) = 0 \qquad \text{// low } \omega \text{ E fields}$$

We now replace I by CV (ω/k) [see (D.2.31c) or the end of Sec 7.1 above] to get,

$$E_z(r,m) = (1/2)\,\eta_m\, CV(\omega/k)\ R_{dc}\ (r/a)^{|m|}\ (|m|+1) \qquad (7.4.1)$$
$$E_r(r,m) = (j/4)\,\eta_m\, CV(\omega)\ R_{dc}\ (a)\ [(r/a)^{|m|+1} + (r/a)^{|m|-1}]$$
$$E_\theta(r,m) = (1/4)\,\eta_m\, CV(\omega)\ R_{dc}\ (a)\ [(r/a)^{|m|+1} - (r/a)^{|m|-1}]$$

$$E_z(r,0) = CV(\omega/k)\ R_{dc}$$

$E_r(r,0) = (j/2)\, CV(\omega)\, R_{dc}\, r$
$E_\theta(r,0) = 0$ // low ω E fields

Using (D.4.7),

$B_r(r,m) = (j/\omega)\,[\text{curl } E]_r = (j/\omega)\,[r^{-1} jmE_z + jkE_\theta]$
$B_\theta(r,m) = (j/\omega)\,[\text{curl } E]_\theta = (j/\omega)[-jkE_r - \partial_r E_z]$
$B_z(r,m) = (j/\omega)\,[\text{curl } E]_z = (j/\omega)\,[r^{-1}\partial_r(rE_\theta) - r^{-1} jmE_r]$, *(D.4.7)*

we have Maple compute the corresponding B fields.

Enter the Bi expressions

```
Br := (j/omega)*((j*m/r)*Ez + j*k*Eth);
```

$$Br := \frac{j\left(\frac{j\,m\,Ez}{r} + j\,k\,Eth\right)}{\omega}$$

```
Bth := (j/omega)*(-j*k*Er - Diff(Ez,r));
```

$$Bth := \frac{j\left(-j\,k\,Er - \left(\frac{\partial}{\partial r} Ez\right)\right)}{\omega}$$

```
Bz := (j/omega)*(1/r)*( Diff(r*Eth,r) - j*m*Er);
```

$$Bz := \frac{j\left(\left(\frac{\partial}{\partial r} r\,Eth\right) - j\,m\,Er\right)}{\omega\, r}$$

Momentarily omitting the common factor $\eta_m CVR_{dc}$, we enter the E fields for m > 0 from (D.11.7) above,

Enter the Ei expressions for m > 0

```
Ez := (1/2)*(omega/k)*(r/a)^m*(m+1);
```

$$Ez := \frac{1}{2}\,\frac{\omega\left(\frac{r}{a}\right)^m (m+1)}{k}$$

```
Er := (j/4)*(omega*a)*( (r/a)^(m+1) + (r/a)^(m-1));
```

$$Er := \frac{1}{4} j\,\omega\, a\left(\left(\frac{r}{a}\right)^{(m+1)} + \left(\frac{r}{a}\right)^{(m-1)}\right)$$

```
Eth := (1/4)*(omega*a)*( (r/a)^(m+1) - (r/a)^(m-1));
```

$$Eth := \frac{1}{4}\,\omega\, a\left(\left(\frac{r}{a}\right)^{(m+1)} - \left(\frac{r}{a}\right)^{(m-1)}\right)$$

Maple then computes the resulting B fields:

```
Br: simplify(%);
```

$$\frac{1}{4}\frac{\left(\frac{r}{a}\right)^m (-2m^2-2m-k^2r^2+k^2a^2)}{rk}$$

```
value(Bth): simplify(%);
```

$$\frac{1}{4}\frac{j\left(\frac{r}{a}\right)^m (k^2r^2+k^2a^2-2m^2-2m)}{rk}$$

```
value(Bz): simplify(%);
```

$$\frac{1}{2}j\left(\frac{r}{a}\right)^m (m+1)$$

Using $(r/a)^m/r = (r/a)^m (a/r)(1/a) = (r/a)^{m-1}(1/a)$, the Maple results for m > 0 are,

$B_r = (1/4)\,(r/a)^{m-1}\,(1/ak)[\,-2m(m+1) + k^2(a^2-r^2)]$
$B_\theta = (j/4)\,(r/a)^{m-1}\,(1/ak)\,[\,-2m(m+1) + k^2(r^2+a^2)]$
$B_z = (j/2)\,(r/a)^m\,(m+1)$

We repeat the effort for m = 0,

Enter the Ei expressions for m = 0

```
Ez := (omega/k);
```

$$Ez := \frac{\omega}{k}$$

```
Er := (j/2)*omega*r;
```

$$Er := \frac{1}{2}j\,\omega\,r$$

```
Eth := 0;
```

$$Eth := 0$$

to get

```
m := 0;
```

$$m := 0$$

```
Br: simplify(%);
```

$$0$$

```
value(Bth): simplify(%);
```

$$\frac{1}{2}j\,k\,r$$

```
value(Bz): simplify(%);
```

$$0$$

Thus, restoring the $\eta_m CVR_{dc}$ factor, for small ω the B fields inside the round wire are given by

B fields in round wire for small ω for G = 0 (vacuum dielectric) (7.4.2)

$$B_r(r,m) = (1/4)\,\eta_m C V R_{dc}\,(r/a)^{m-1}\,(1/ak)[\,-2m(m+1) + k^2(a^2-r^2)]$$
$$B_\theta(r,m) = (j/4)\,\eta_m C V R_{dc}\,(r/a)^{m-1}\,(1/ak)\,[\,-2m(m+1) + k^2(r^2+a^2)]$$
$$B_z(r,m) = (j/2)\,\eta_m C V R_{dc}\,(r/a)^m\,(m+1) \qquad m > 0$$

$$B_r(r,0) = 0$$
$$B_\theta(r,0) = (j/2)\,CV\,R_{dc}\,kr$$
$$B_z(r,0) = 0 \qquad m = 0$$

According to (Q.4.9), and assuming that $k = -j\sqrt{(R+j\omega L)(G+j\omega C)}$, we see that

$$k \to \sqrt{R_{dc2}C/2}\;\sqrt{\omega}\;(1-j) = \sqrt{R_{dc2}C}\;\sqrt{\omega}\;e^{-j\pi/4} \qquad \text{as } \omega \to 0 \qquad (7.4.3)$$

where we continue to assume $G_{dc} = 0$ so $\omega_d = 0$. We are now using R_{dc} as the resistance per length of our round wire conductor, while R_{dc2} is the resistance of both transmission line conductors per length. Note that $k \to 0$ as $\sqrt{\omega}$. The above fields then become

$$B_r(r,m) = (1/4)\,\eta_m C V R_{dc}\,(r/a)^{m-1}\,(1/a)[\,-2m(m+1)\,] \,/\, [\,\sqrt{R_{dc2}C}\;\sqrt{\omega}\;e^{-j\pi/4}]$$
$$B_\theta(r,m) = (j/4)\,\eta_m C V R_{dc}\,(r/a)^{m-1}\,(1/a)\,[\,-2m(m+1)\,] \,/\, [\,\sqrt{R_{dc2}C}\;\sqrt{\omega}\;e^{-j\pi/4}]$$
$$B_z(r,m) = (j/2)\,\eta_m C V R_{dc}\,(r/a)^m\,(m+1) \qquad m > 0$$

$$B_r(r,0) = 0$$
$$B_\theta(r,0) = (j/2)\,CV\,R_{dc}\,r\;[\,\sqrt{R_{dc2}C}\;\sqrt{\omega}\;e^{-j\pi/4}]$$
$$B_z(r,0) = 0 \qquad m = 0 \qquad (7.4.4)$$

As ω→0, the three m = 0 fields go to zero. This is as expected since the current vanishes:

$$I = CV\,(\omega/k) = CV\,\omega \,/\, [\sqrt{R_{dc2}C}\;\sqrt{\omega}\;e^{-j\pi/4}] = CV\,\sqrt{\omega}\;/\,[\sqrt{R_{dc2}C}\,e^{-j\pi/4}] \to 0 \qquad (7.4.5)$$

However, as ω→0, the m> 0 fields all diverge as $1/\sqrt{\omega}$!!

For completeness, we now treat the case with non-zero dielectric conductance G > 0. As discussed above (D.9.25), for this case we must generalize the Appendix D charge pump boundary condition and the net result is that one adds a factor (ξ_d/ε_d) to all the E fields in (7.4.1). This factor then propagates into all the B fields in (7.4.2). Now from (1.5.1c) we have

$$\xi_d \equiv \varepsilon_d - j\sigma_d/\omega$$

so

$$(\xi_d/\varepsilon_d) \;=\; \to \;(\varepsilon_d - j\sigma_d/\omega)/\varepsilon_d \;\to\; -j(\sigma_d/\varepsilon_d)\,(1/\omega) \;=\; (G_{dc}/C)\,(1/j\omega) \qquad \text{as } \omega\to 0 \qquad (7.4.6)$$

where we use $\sigma_d/\varepsilon_d = G_{dc}/C$ from (4.11.34). Thus,

B fields in round wire for small ω for G > 0 (7.4 7)

$B_r(r,m) = (1/4)\,\eta_m CVR_{dc}\,(r/a)^{m-1}\,(1/ak)[\,-2m(m+1) + k^2(a^2-r^2)]\,(G_{dc}/C)\,(1/j\omega)$
$B_\theta(r,m) = (j/4)\,\eta_m CVR_{dc}\,(r/a)^{m-1}\,(1/ak)\,[\,-2m(m+1) + k^2(r^2+a^2)]\,(G_{dc}/C)\,(1/j\omega)$
$B_z(r,m) = (j/2)\,\eta_m CVR_{dc}\,(r/a)^{m}\,(m+1)$ $\quad m > 0$

$B_r(r,0) = 0$
$B_\theta(r,0) = (j/2)\,CV\,R_{dc}\,kr\,(G_{dc}/C)\,(1/j\omega)$
$B_z(r,0) = 0$ $\quad m = 0$

According to (Q.4.6), and assuming that $k = -j\sqrt{(R+j\omega L)(G+j\omega C)}$, we see that

$k \rightarrow -\sqrt{R_{dc}G_{dc}} \equiv k_1$ as $\omega \rightarrow 0$ // a very small constant value (7.4.8)

Using this for k in the above box we get

$B_r(r,m) = (1/4)\,\eta_m CVR_{dc}\,(r/a)^{m-1}\,(1/ak_1)[\,-2m(m+1) + k^2(a^2-r^2)]\,(G_{dc}/C)\,(1/j\omega)$ // $\rightarrow 1/\omega$
$B_\theta(r,m) = (j/4)\,\eta_m CVR_{dc}\,(r/a)^{m-1}\,(1/ak_1)\,[\,-2m(m+1) + k^2(r^2+a^2)]\,(G_{dc}/C)\,(1/j\omega)$ // $\rightarrow 1/\omega$
$B_z(r,m) = (j/2)\,\eta_m CVR_{dc}\,(r/a)^{m}\,(m+1)$ $\quad m > 0$ // $\rightarrow$ constant

$B_r(r,0) = 0$
$B_\theta(r,0) = (j/2)\,CV\,R_{dc}\,k_1 r\,(G_{dc}/C)\,(1/j\omega)$ // $\rightarrow 1/\omega$
$B_z(r,0) = 0$ $\quad m = 0$ (7.4.9)

Now we find that for G > 0, most of the B field components diverge as 1/ω as ω→0 !! This is even worse than that G = 0 case where divergence was $1/\sqrt{\omega}$.

This then is our second sign of trouble: B fields are going infinite as ω→ 0. It seems clear that the physical B field of a transmission line operating at DC should be finite since there are no infinite currents anywhere. Thus, the theory of Appendix D combined with the idea that $k = -j\sqrt{(R+j\omega L)(G+j\omega C)}$ is *invalid* as ω→ 0.

7.5 What is the cause of the Trouble as ω→ 0 ?

In Appendix M it is argued that $|\mathbf{A}_t| < 10^{-4}\,|A_z|$ for from 0 to 500 GHz. However, this relative smallness of the transverse vector potential for a transmission line does not seem to translate into the smallness of transverse *derivatives* of the potential.

In Step 1 (3.7.5) it was shown indirectly that one can approximate div **A** as $\partial_z A_z$ only when ω is large. In other words, $(\partial_x A_x + \partial_y A_y)$ cannot be neglected relative to $\partial_z A_z$ when ω is small. Since we have never studied the small transverse potentials much, it is not clear just how small ω has to be for this neglect to

be unjustified, but one might vaguely assume for a round conductor of radius a that $a = \delta = \sqrt{2/(\omega\mu\sigma)}$ might provide a ballpark value below which $(\partial_x A_x + \partial_y A_y)$ should not be neglected. Certainly then in the limit $\omega \rightarrow 0$, one should not be neglecting $(\partial_x A_x + \partial_y A_y)$.

A more generic argument comparing $(\partial_x A_x + \partial_y A_y)$ to $(\partial_z A_z)$ is this. If we were to assume that A_z behaves as e^{-jkz} , we would then be comparing $| \partial_x A_x + \partial_y A_y |$ with $| k A_z |$. We assumed in Chapter 4 that in fact $| \partial_x A_x + \partial_y A_y | << | k A_z |$ so that then div $\mathbf{A} \approx \partial_z A_z$. However, as $\omega \rightarrow 0$, the wavelength λ becomes very long and $k = 2\pi/\lambda$ becomes very small, so at some point $| \partial_x A_x + \partial_y A_y | << | k A_z |$ will become unjustified.

In Chapter it was assumed that $| \partial_x A_x + \partial_y A_y | << | k A_z |$. This assumption was made in going from (4.12.2) to (4.12.3), which we justified there based on $|\mathbf{A}_t| < 10^{-4} |A_z|$. If we maintain the transverse derivatives $(\partial_x A_x + \partial_y A_y)$ and carry through the analysis of Chapter 4, we arrive at these modified second-order transmission line equations (see Appendix S),

$$\frac{d^2V(z)}{dz^2} - \mathbf{zy}\, V(z) = (-\mathbf{z}/L_e)T(z) \qquad \frac{d^2i(z)}{dz^2} - \mathbf{zy}\, i(z) = (1/L_e)\,\partial_z T(z) \ . \qquad \textit{(4.12.17)} \qquad \textit{(S.29)}$$

In Appendix S we implement an averaging procedure where V(z), i(z), T(z) and L_e are certain double averages over the perimeters of the two conductors, though the averaging can be ignored for widely spaced conductors. The quantity T(z) is given by

$$T(z) \equiv \langle T(\mathbf{x}_1,\mathbf{x}_2)\rangle_{C1,C2} \ . \qquad \textit{(S.19)}$$

where

$$T(\mathbf{x}_1,\mathbf{x}_2) \equiv (\partial_x A_{x12}(\mathbf{x}_1) - \partial_x A_{x12}(\mathbf{x}_2)) + (\partial_y A_{y12}(\mathbf{x}_1) - \partial_y A_{y12}(\mathbf{x}_2)) \ . \quad \text{// dim(T) = tesla} \qquad \textit{(S.17)}$$

We don't know how to compute the function T because we don't know the transverse potential functions, but the argument above suggests that at low ω we cannot just set T(z) = 0.

With $T(z) \neq 0$, the two second order transmission line equations are inhomogeneous, meaning they have driving terms on the right side. The presence of driving terms means that the solutions of these equations no longer have the simple form e^{-jkz} where $k = -j\sqrt{\mathbf{zy}}$ as we found in (5.3.3) and following equations.

Recall now the simple traveling wave ansatz assumed at the start of Appendix D for the electric field inside a round wire as one of two transmission line conductors,

$$\mathbf{E}(r,\theta,z,t) = e^{j(\omega t - kz)}\, \mathbf{E}(r,\theta) \ . \qquad \textit{(D.1.1)}$$

This led to all quantities in Appendix D having this same $e^{j(\omega t - kz)}$ dependence on z, and matching at the conductor boundary then required that fields in the dielectric have this same $e^{j(\omega t - kz)}$ form. This in turn results in the potential V(z) having this same e^{-jkz} dependence. But we have just argued above that for small ω, this z dependence of V(z) is unjustified since T(z) cannot be neglected.

The upshot is that since the e^{-jkz} ansatz made at the very start of Appendix D is invalid for low ω, any conclusions of Appendix D are invalid at low ω. Two such conclusions were noted in Section 7.2 and 7.4 above, that J_z is asymmetric as ω→0, and that certain magnetic fields are infinite as ω→ 0.

There are various other hints of trouble at low ω floating around in Chapters 3 and 4, where we often had to assume the strong or extreme skin effect limits, meaning large ω. A dramatic inconsistency at very low ω was shown in Figure 3.6a which is a plot of magnetic fields lines as DC for two round wires. Since the B fields are clearly not tangent to the conductor surfaces, we know that A_z is not constant on the these surfaces, yet that was assumed true, as shown for example in (5.3.11).

We reach the conclusion that the basic theory presented in Chapters 4 and 5 (and Appendix D for round wires) is adequate at "high frequency" which means in the skin effect regime, though results might be valid at somewhat lower ω. On the other hand, the approximations of eddy current theory are valid at "low frequency", as presented qualitatively in Appendix P. More analytic eddy current analyses do exist, such as that of Rodrígues and Valli.

To find a complete analytic solution valid for all ω for the infinite transmission line one must face up to a full-bore boundary value problem for the two (or more) conductors, where all the conductor interior problems and the overall dielectric exterior problem are treated simultaneously with Maxwell's Equations, with fulfillment of boundary conditions on all **E** and **B** field components on all conductor surfaces. We have not attempted such a solution in this document. Since the vector potential is not constant on conductor surfaces at all ω, it seems unlikely that the King potential approach of Chapter 4 would be useful.